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COMPLETION FOR TIGHT SIGN-CENTRAL MATRICES

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ABSTRACT. A real matrix A is called a sign-central matrix if for, every matrix $\tilde{A}$ with the same sign pattern as A, the convex hull of columns of $\tilde{A}$ contains the zero vector. A sign-central matrix A is called a tight sign-central matrix if the Hadamard (entrywise) product of any two columns of A contains a negative component. A real vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ is called stable if $|x_1| \leq |x_2| \leq \cdots \leq |x_n|$. A tight sign-central matrix is called a *tight* sign-central matrix* if each of its columns is stable.

In this paper, for a matrix B, we characterize those matrices C such that $[B, C]$ is tight (tight*) sign-central. We also construct the matrix C with smallest number of columns among all matrices C such that $[B, C]$ is tight* sign-central.

1. Introduction

Throughout this paper all the matrices and vectors are assumed to be real matrices and real vectors.

The sign of a real number a, sign(a), is defined by

$$\text{sign}(a) = \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{if } a = 0, \\ -1, & \text{if } a < 0. \end{cases}$$

The *sign pattern* of a real matrix A is the $(0, 1, -1)$-matrix obtained from A by replacing each of the entries by its sign. For a real matrix A, let $\mathcal{Q}(A)$ denote the set of all real matrices with the same sign pattern as A. A real matrix A is called a *sign-central matrix* if, for every $\tilde{A} \in \mathcal{Q}(A)$,

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the convex hull of columns of $\tilde{A}$ contains the zero vector $\mathbf{0}$ [1]. A sign-central matrix A is called a *tight sign-central* matrix if the Hadamard (entrywise) product of any two columns of A contains a negative entry [3]. A sign-central matrix A is called a *minimal sign-central* matrix if each of the submatrices obtained from A by deleting a column is not sign-central. In [3], Hwang et. al. proved that every tight sign-central matrix is minimal sign-central and characterized the tight sign-central matrices. They also determined the lower bound of the number of columns of a tight sign-central matrix in terms of the number of rows and the number of zero entries of the matrix.

In [4], Hwang et. al. determined the maximum and minimum numbers of zero entries of a tight sign-central matrix and the maximum ratio of the number of zero entries to the total number of entries of a tight sign-central matrix.

Consider a partial matrix $A = [B, \square]$ in which the columns of B are specified and $\square$ is unspecified part. We are interested in filling up the rest of the columns so that A has certain property related to the sign-centrality. For example let

$$A_1 = \begin{bmatrix} 0 & 1 & ? \\ 1 & 0 & ? \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & ? & ? & ? \\ -1 & ? & ? & ? \end{bmatrix}.$$

Fill up unspecified entries of A_1 and A_2 respectively as follows,

$$A_1' = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}, \ A_2' = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

A_1' is sign-central, A_2' is tight sign-central. We say that A_1 is completed to a sign-central matrix and A_2 is completed to a tight sign-central matrix. But there is no way to complete A_1 to a tight sign-central matrix.

A real vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ is called stable if $|x_1| \le |x_2| \le \cdots \le |x_n|$. A matrix is called *stable* if each of its columns is stable. A stable tight sign-central matrix is called a *tight* sign-central matrix.*

In this paper, for a matrix B, we characterize those matrices C such that $[B, C]$ is tight (tight*) sign-central. We also construct the matrix C with smallest number of columns among all matrices C such that $[B, C]$ is tight* sign-central.

2. Preliminaries

Since the sign-centrality of a matrix depends only on the sign pattern of the matrix, we assume that all the matrices which appear in the sequel are sign pattern matrices.

The following lemma is a basic tool to test the sign-centrality.

LEMMA 2.1. ([1], [2]). *An $m \times n$ matrix A is sign-central if and only if, for every nonsingular $(0, 1, -1)$-diagonal matrix D, DA has a nonnegative column.*

For a vector $\mathbf{x}$, let $\omega(\mathbf{x}) = 2^{\sigma(\mathbf{x})} - 1$ where $\sigma(\mathbf{x})$ denotes the number of zero components of $\mathbf{x}$. For a matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$, let $\omega(A) = \sum_{i=1}^{n} \omega(\mathbf{a}_i)$. $\omega(A)$ is called the *weight* of A [3].

Let $E_1 = [1, -1]$, and for $k \geq 2$, let E_k denote the $k \times 2^k$ matrix defined by

$$E_k = \begin{bmatrix} \mathbf{e}^T & -\mathbf{e}^T \\ E_{k-1} & E_{k-1} \end{bmatrix} \tag{2.1}$$

where and in the sequel $\mathbf{e}$ denotes the all 1's vector with suitable number of components.

Let $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ be an $m \times n$ matrix such that $\mathbf{a}_i$ has k_i zero components, $(i = 1, 2, \ldots, n)$. For each $i = 1, 2, \ldots, n$, let A_i be the $m \times 2^{k_i}$ matrix obtained from the $m \times 2^{k_i}$ matrix $\mathbf{a}_i\mathbf{e}^T$ by replacing the $k_i \times 2^{k_i}$ zero submatrix by E_{k_i} (as defined by (2.1)), and let A^c denotes the matrix obtained from A by replacing the column i by A_i, $i = 1, 2, \ldots, n$. For example, if

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}, \tag{2.2}$$

then

$$A^c = \left[\begin{array}{cccc|cc|c} 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ \hline 1 & 1 & 1 & 1 & 1 & -1 & -1 \end{array}\right].$$

Note that A^c is uniquely determined by A.

Let $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] = [a_{ij}]$. Suppose that $a_{st} = 0$. Let A' be the matrix obtained from A by replacing $\mathbf{a}_t$ with $[\mathbf{a}_t', \mathbf{a}_t'']$ where

$$\mathbf{a}' = [a_{1t}, \ldots, a_{s-1,t}, 1, a_{s+1,t}, \ldots, a_{mt}]^T,$$
$$\mathbf{a}'' = [a_{1t}, \ldots, a_{s-1,t}, -1, a_{s+1,t}, \ldots, a_{mt}]^T.$$

The replacement of $\mathbf{a}_t$ by $[\mathbf{a}_t^{'}, \mathbf{a}_t^{''}]$ is called the *splitting of* $\mathbf{a}_t$ *at* a_{st}.

Notice also that A^c can be obtained from A by a sequence of splitting columns at $a_{st} = 0$ for the largest (s,t) in the lexicographical order. For example, for the matrix A in (2.2) A^c can be obtained as follows.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc|cc} 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cc|c} 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ \hline 1 & 1 & 1 & 1 & 1 & -1 & -1 \end{array}\right].$$

The last matrix equals A^c upto permutation of columns.

LEMMA 2.2. ([3]) *For an $m \times n$ matrix A, the following are equivalent.*

(*a*) *A is tight sign-central.*
(*b*) *A is sign-central and $\omega(A) + n = 2^m$.*
(*c*) *$A^c = E_m$ upto permutation of columns.*

LEMMA 2.3. *For a matrix A, let $\hat{A}$ be the matrix obtained from A by splitting a column. Then*

(*a*) *A is sign-central if and only if $\hat{A}$ is sign-central.*
(*b*) *A is tight sign-central if and only if $\hat{A}$ is tight sign-central.*

Proof. Write $A = [\mathbf{a}_1, B]$. We may assume that $\hat{A} = [\mathbf{a}_1^{'}, \mathbf{a}_1^{''}, B]$ is obtained from A by splitting the first column at $a_{i1} = 0$ for some i.

(*a*) It is clear that, for every nonsingular $(0, 1, -1)$-diagonal matrix D, DA has a nonnegative column if and only if $D\hat{A}$ has a nonnegative column, and (*a*) is proved.

(*b*) Suppose that $\mathbf{a}_1$ has exactly k zero components. Then each of $\mathbf{a}_1^{'}$ and $\mathbf{a}_2^{''}$ has $k-1$ zero components. Note that $\omega(\hat{A}) = 2(2^{k-1}-1)+\omega(B) = 2^k - 2 + \omega(B) = \omega(A) - 1$. Let n, $\hat{n}$ be the number of columns of A and $\hat{A}$ respectively. Then $n + 1 = \hat{n}$. Thus $\omega(\hat{A}) + \hat{n} = \omega(A) + n$. Now (*b*) follows from (*a*) and Lemma 2.2. □

LEMMA 2.4. ([3]). *Let A be a stable matrix of the form*

$$A = \begin{bmatrix} \mathbf{0}^T & \mathbf{e}^T & -\mathbf{e}^T \\ X & Y & Z \end{bmatrix}.$$

Then A is tight sign-central if and only if $[X, Y]$ is tight sign-central and $Y = Z$ upto permutation of columns.

3. Main results

Let $A = [B, \Box]$ be an $m \times n$ partial matrix in which the columns of B are specified. Can we complete A to a sign-central (tight sign-central, tight* sign-central) matrix? The answer to this question may depend on the specified columns and the number of columns to be filled.

For example, observe the partial matrices

$$A_1 = \begin{bmatrix} 0 & 1 & ? \\ 1 & 1 & ? \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & ? \\ 1 & ? \end{bmatrix}.$$

A_1 can be completed to a sign-central matrix but it cannot be completed to a tight sign-central matrix. A_2 cannot be completed to a sign-central matrix.

In what follows we deal with the problem of completing a partial matrix to a tight sign-central (tight* sign-central) matrix.

In the sequel, for a matrix A with no two identical columns, we let $E_m \backslash A$ denote the matrix obtained from E_m by deleting every column which appear in A as a column.

Note that in order for a partial matrix $A = [B, \Box]$, where B is the specified part of A, to be completed to a tight sign-central matrix it is necessary that B^c has no two identical columns because A^c must be equal to E_m upto permutation of columns.

THEOREM 3.1. *Let B be a matrix with m rows such that B^c has no two identical columns. Let C be a matrix with m rows. Then $[B, C]$ is tight sign-central if and only if $C^c = E_m \backslash B^c$ upto permutation of columns.*

Proof. If $[B, C]$ is tight sign-central, then, by Lemma 2.2, $[B^c, C^c] = [B, C]^c = E_m$ so that $C^c = E_m \backslash B^c$ upto permutation of columns.

Conversely, if $C^c = E_m \backslash B^c$, then $[B, C]^c = E_m$ upto permutation of columns, and again by Lemma 2.2 $[B, C]$ is tight sign-central. $\Box$

For a matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$, we call a pair $\mathbf{a}_i, \mathbf{a}_j$ of columns a *twin pair* if $\mathbf{a}_i$ and $\mathbf{a}_j$ have the same zero-nonzero pattern and $\mathbf{a}_i \circ \mathbf{a}_j$ has exactly one negative component which is the first nonzero component where $\mathbf{a}_i \circ \mathbf{a}_j$ stands for the Hadamard (entrywise) product of $\mathbf{a}_i$ and $\mathbf{a}_j$. Let A be a nonzero tight* sign-central matrix which may or may not have a zero row. Then, upto permutation of columns, A has the form

$$A = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{e}^T & -\mathbf{e}^T \\ X & Y & Y \end{bmatrix}$$

by Lemma 2.4. Thus a tight* sign-central matrix has a twin pair. Let A be a matrix with a twin pair $\mathbf{a}_i, \mathbf{a}_j$ and let A' be the matrix obtained from A by replacing the pair $\mathbf{a}_i, \mathbf{a}_j$ by $\frac{1}{2}(\mathbf{a}_i + \mathbf{a}_j)$. We call A' a *twin-contraction* of A. For example $\left[\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right]$ is twin contraction of $\left[\begin{smallmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{smallmatrix}\right]$.

For two matrices A, B we say that A is *twin-contractible* to B if there exist a sequence of matrices $A = A_0, A_1, \ldots, A_k = B$ such that A_i is a twin-contraction of A_{i-1}, $(i = 1, 2, \ldots, k)$.

LEMMA 3.2. *Let A be a matrix with a twin pair of columns and let A' be a twin-contraction of A. Then the following hold.*

(*a*) *A is sign-central if and only if A' is sign-central.*
(*b*) *A is tight sign-central if and only if A' is tight sign-central.*
(*c*) *A is tight* sign-central if and only if A' is tight* sign-central.*

Proof. Since A is obtained from A' by splitting a column at a zero entry, $(a), (b)$ follow from Lemma 2.3.

(c) It is clear that A is stable if and only if A' is stable. □

THEOREM 3.3. *A matrix is tight* sign-central if and only if it is twin-contractible to the zero vector* $\mathbf{0}$.

Proof. Suppose that A is a tight* sign-central matrix. We prove that A is twin-contractible to $\mathbf{0}$ by induction on the number n of columns of A.

If $n = 1$. Then $A = \mathbf{0}$ and we are done.

Suppose that $n > 1$. Since A is tight* sign-central, A has a twin pair. Let A' be a twin-contraction of A then A' is tight* sign-central and A' has fewer columns than A. Then, by induction assumption, A' is twin-contractible to $\mathbf{0}$. Thus A is twin-contractible to $\mathbf{0}$.

Conversely, suppose that A is twin-contractible to $\mathbf{0}$. Then there exists a sequence of matrices $A = A_0, A_1, \ldots, A_k = \mathbf{0}$ such that A_i is a twin-contraction of A_{i-1}, $(i = 1, 2, \ldots, k)$. Since $\mathbf{0}$ is tight* sign-central we get that A is tight* sign-central by Lemma 3.2. □

In order for a partial matrix $A = [B, \square]$ to be completed to a tight* sign-central matrix it is necessary that B is stable and B^c has no two identical columns.

THEOREM 3.4. *Let B be a stable matrix with m rows such that B^c has no two identical columns. Let C be a matrix with m rows. Then $[B, C]$ is tight* sign-central if and only if $E_m \backslash B^c$ is twin-contractible to C.*

Proof. Suppose that $[B, C]$ is tight* sign-central. Then $[B^c, C^c] = [B, C]^c = E_m$. Thus $C^c = E_m \backslash B^c$. Since C^c can be obtained from $C = [c_{ij}]$ by a sequence of splitting columns at $c_{st} = 0$ for the 'largest'(s, t) in the lexicographical order and since splitting a column at the bottom-most zero entry is the reverse operation of twin-contraction, we see that $E_m \backslash B^c$ is twin-contractible to C.

Conversely, suppose that $E_m \backslash B^c$ is twin-contractible to C then, by reversing the twin-contraction sequence, we see that $C^c = E_m \backslash B^c$ upto permutation of columns. Thus $[B, C]$ is tight sign-central by Theorem 3.1. Since C is clearly stable, we see that $[B, C]$ is tight* sign-central. □

Let F be a submatrix of E_m with m rows. Let $F^{\#}$ denote the matrix satisfying the conditions

(i) F is twin-contractible to $F^{\#}$,
(ii) $F^{\#}$ has no twin pair of columns.

Then $F^{\#}$ is uniquely determined by F. For, if F has no twin pair of columns, then $F^{\#} = F$. Suppose that F has twin pairs and that $\mathbf{x}_i, \mathbf{x}_i'$, $(i = 1, \ldots, k)$, are all those. Then $F = [\mathbf{x}_1, \mathbf{x}_1', \mathbf{x}_2, \mathbf{x}_2', \ldots, \mathbf{x}_k, \mathbf{x}_k', G]$ upto permutation of columns where G has no twin pair. Let $\mathbf{y}_i = \frac{1}{2}(\mathbf{x}_i + \mathbf{x}_i')$, $(i = 1, 2, \ldots, k)$, and $H = [Y, G]$ where $Y = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k]$. Since each $\mathbf{y}_i$ has the form $\mathbf{y}_i = [0, \mathbf{z}_i^T]^T$, Y can be written as

$$Y = \begin{bmatrix} \mathbf{0}^T \\ Z \end{bmatrix}.$$

By induction on the number of rows, $Z^{\#}$ is uniquely determined. Since G has no twin pair it follows that

$$F^{\#} = \left[\begin{array}{c|c} \mathbf{0}^T & \\ Z^{\#} & G \end{array}\right]$$

telling us that $F^{\#}$ is uniquely determined.

In the sequel, for a stable matrix B with m rows such that B^c has no two identical columns, let Φ_B denote the matrix $(E_m \backslash B^c)^{\#}$ for brevity.

THEOREM 3.5. *Let B be a stable matrix with m rows such that B^c has no identical columns. Then $[B, \Phi_B]$ is the tight* sign-central matrix which has the smallest number of columns among all tight* sign-central matrices of the form $[B, X]$.*

Proof. By Theorem 3.4, $[B, (E_m \backslash B^c)^{\#}]$ is a tight* sign-central matrix. Suppose that $[B, C]$ is a tight* sign-central matrix. Then $E_m \backslash B^c$ is twin-contractible to C. If C is twin pair free, then $C = (E_m \backslash B^c)^{\#}$ by the uniqueness of $(E_m \backslash B^c)^{\#}$. Otherwise, there is a twin pair free matrix C' to which C is twin-contractible, in which case $C' = (E_m \backslash B^c)^{\#}$. Thus C has at least as many columns as $(E_m \backslash B^c)^{\#}$ has, and the proof is complete. □

Let B be a stable matrix with m rows such that B^c has no two identical columns. Let C be a matrix such that $[B, C]$ is tight* sign-central. One may ask how many columns C can have. For the discussion of this question, for a matrix A, let n_A denote the number of columns of A.

Since $\omega(B) + n_B + \omega(C) + n_C = 2^m$, we see that

$$n_C = 2^m - \omega(B) - n_B - \omega(C) \leq 2^m - \omega(B) - n_B.$$

By Theorem 3.5, $n_C \geq n_{\Phi_B}$. Now from Theorem 3.4 we get the following

COROLLARY 3.6. *Let B be a stable matrix with m rows such that B^c has no two identical columns. Then there is an $m \times k$ matrix C such that $[B, C]$ is tight* sign-central if and only if $n_{\Phi_B} \leq k \leq 2^m - \omega(B) - n_B$.*

EXAMPLE. Let

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then

$$B^c = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \end{bmatrix}$$

so that

$$E_m \backslash B^c = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Hence

$$\Phi_B = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

So

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

is the tight* sign-central matrix with the smallest number of columns among all tight* sign-central matrices with $(0,0,1)^T$, $(0,1,-1)^T$ as the first two columns.

For the matrix

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we see right away that

$$\Phi_B = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

without looking at $E_3 \backslash B^c$, saving great amount of steps of splittings and twin-contractions. Is there some other methods of constructing Φ_B not taking care of whole steps from B to B^c and from $E_m \backslash B^c$ to Φ_B? In what follows we discuss such a construction. We call column $\mathbf{b}_i$ of B with no zero component a *solitary column* if there is no other column $\mathbf{b}_j$ such that $\mathbf{b}_i, \mathbf{b}_j$ form a twin pair.

Suppose that a stable matrix B with the property that B^c has no two identical columns is given.

If B has a zero row so that $B = \left[\begin{smallmatrix} \mathbf{0}^T \\ C \end{smallmatrix}\right]$, then

$$\Phi_B = \begin{bmatrix} \mathbf{0}^T \\ \Phi_C \end{bmatrix}.$$

If B has no zero row, then, upto permutation of columns, B can be written as

$$B = \begin{bmatrix} \mathbf{0}^T & \mathbf{e}^T & -\mathbf{e}^T & \mathbf{a}^T \\ X & Y & Y & Z \end{bmatrix}, \tag{3.1}$$

where $\left[\begin{smallmatrix} \mathbf{a}^T \\ Z \end{smallmatrix}\right]$ consists of the solitary columns of B and $\left[\begin{smallmatrix} \mathbf{0}^T \\ X \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} \mathbf{a}^T \\ Z \end{smallmatrix}\right]$ or both may or may not be vacuous.

THEOREM 3.7. *Let B be a stable matrix without zero rows such that B^c has no two identical columns. Then*

$$\Phi_B = \begin{bmatrix} -\mathbf{a}^T & \mathbf{0}^T \\ Z & \Phi_{[X,Y,Z]} \end{bmatrix}$$

where $\mathbf{a}, X, Y, Z$ are the submatrices of B as appeared in (3.1).

Proof. Let m be the number of rows of B. We prove the theorem by induction on m. The theorem clearly holds for $m = 1$. Let $m > 1$. The matrix $[B, \Phi_B]$ has no solitary column because it is tight* sign-central. So, upto permutation of columns,

$$\Phi_B = \begin{bmatrix} -\mathbf{a}^T & \mathbf{0}^T \\ Z & U \end{bmatrix}$$

for some matrix U, and hence

$$[B,\Phi_B]=\begin{bmatrix}\mathbf{0}^T & \mathbf{e}^T & -\mathbf{e}^T & \mathbf{a}^T & -\mathbf{a}^T & \mathbf{0}^T\\ X & Y & Y & Z & Z & U\end{bmatrix}=\begin{bmatrix}\mathbf{0}^T & \mathbf{e}^T & \mathbf{e}^T & -\mathbf{e}^T & -\mathbf{e}^T & \mathbf{0}^T\\ X & Y & Z & Y & Z & U\end{bmatrix}$$

upto permutation of columns.

Now $[X,Y,Z,U]$ is a tight* sign-central matrix with the smallest number of columns among the tight* sign-central matrices of the form $[X,Y,Z,C]$. Since $[X,Y,Z]$ is stable and $[X,Y,Z]^c$ has no two identical columns, we get $U=\Phi_{[X,Y,Z]}$ by induction, and the proof is complete. □

EXAMPLE. Let

$$B=\begin{bmatrix}0 & 1\\ 0 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}\mathbf{0}^T & \mathbf{e}^T & -\mathbf{e}^T & \mathbf{a}^T\\ X & Y & Y & Z\end{bmatrix}.$$

Then $X=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $Y=\emptyset$, $\mathbf{a}^T=[1]$, $Z=\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$. Therefore

$$\begin{bmatrix}-\mathbf{a}^T\\ Z\end{bmatrix}=\begin{bmatrix}-1\\ 1\\ -1\end{bmatrix}$$

and $[X,Y,Z]=\left[\begin{smallmatrix}0 & 1\\ 1 & -1\end{smallmatrix}\right]$. Since, clearly, $\Phi_{[X,Y,Z]}=\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$, we get

$$\Phi_B=\begin{bmatrix}-1 & 0\\ 1 & -1\\ -1 & -1\end{bmatrix}.$$

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