Bull. Korean Math. Soc. **43** (2006), No. 2, pp. 319–331

# ON SOLVABILITY AND ALGORITHM OF GENERAL STRONGLY NONLINEAR VARIATIONAL-LIKE INEQUALITIES

ZEQING LIU, JUHE SUN, SOO HAK SHIM, AND SHIN MIN KANG

ABSTRACT. In this paper, a new class of general strongly nonlinear variational-like inequalities was introduced and studied. The existence and uniqueness of solutions and a new iterative algorithm for the general strongly nonlinear variational-like inequality are established and suggested, respectively. The convergence criteria of the iterative sequence generated by the iterative algorithm are also given.

## 1. Introduction

It is known that an important and useful generalization of variational inequalities is variational-like inequality [1]–[33]. Recently, by using the Berge maximum theorem, Tian [26] and Yao [32] studied some mixed variational-like inequalities, Huang and Deng [8] extended the auxiliary principle technique to study a class of generalized strongly nonlinear mixed variational-like inequalities in Hilbert spaces. Ding [4], [5] and others introduced and studied some classes of nonlinear variational-like inequalities in reflexive Banach spaces. Verma [27]–[31] discussed some classes of variational inequalities involving various nonlinear monotone operators in Hilbert spaces.

In this paper, we introduce and study a new class of general strongly nonlinear variational-like inequalities. The existence and uniqueness of solutions and a new iterative algorithm for the general strongly nonlinear

Received January 11, 2005.

2000 Mathematics Subject Classification: 47J20, 49J40.

Key words and phrases: general strongly nonlinear variational-like inequality, existence and uniqueness, contraction mapping.

This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (2004C063) and Korea Research Foundation Grant (KRF-2003-005-C00013).

variational-like inequality are proved and suggested, respectively. The convergence criteria of the sequence generated by the iterative algorithm are given.

## 2. Preliminaries

Let $H$ be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $K$ be a nonempty closed convex subset of $H$, $A, C, F : K \to H$, $N, M : H \times H \to H$ and $\eta : K \times K \to H$ be mappings, and $f : K \to (-\infty, \infty]$ be a real functional. Suppose that $a : H \times H \to (-\infty, \infty)$ is a coercive continuous bilinear form, that is, there exist positive constants $c$ and $d$ such that

(C1) $a(v, v) \geq c\|v\|^2$, $\forall v \in H$;

(C2) $a(u, v) \leq d\|u\|\|v\|$, $\forall u, v \in H$.

Clearly, $c \leq d$.

We consider the following general strongly nonlinear variational-like inequality problem:

Find $u \in K$ such that

$$\begin{aligned} & a(u, v-u) + f(v) - f(u) \\ & \geq \langle N(Au, Cu) - M(Eu, Fu), \eta(v, u) \rangle, \quad \forall v \in K. \end{aligned} \tag{2.1}$$

Special Cases

(A) If $N(Au, Cu) = Au - Cu$, $a(u, v) = 0$ and $M(Eu, Fu) = 0$ for all $u, v \in K$, then the general strongly nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Cu - Au, \eta(v, u) \rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.2}$$

which was introduced and studied by Ding [4].

(B) If $N(Au, Cu) = Au - Cu$, $a(u, v) = 0$, $M(Eu, Fu) = 0$ and $\eta(u, v) = gu - gv$ for all $u, v \in K$, then the general strongly nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Cu - Au, gv - gu \rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.3}$$

which was studied by Yao [32].

Definition 2.1. Let $A : K \to H$, $N : H \times H \to H$ and $\eta : K \times K \to H$ be mappings.

(1) $A$ is said to be *Lipschitz continuous* with constant $\alpha$ if there exists a constant $\alpha > 0$ such that

$$\|Au - Av\| \leq \alpha\|u - v\|, \quad \forall u, v \in K.$$

(2) $N$ is said to be *Lipschitz continuous* with constant $\beta$ in the first argument if there exists a constant $\beta > 0$ such that

$$\|N(u, w) - N(v, w)\| \leq \beta\|u - v\|, \quad \forall u, v, w \in H.$$

(3) $N$ is said to be *strongly monotone* with constant $\gamma$ with respect to $A$ in the second argument if

$$\langle N(w, Au) - N(w, Av), u - v\rangle \geq \gamma\|u - v\|^2, \quad \forall u, v \in K, \ w \in H.$$

(4) $N$ is said to be *$\eta$-antimonotone* with respect to $A$ in the first argument if

$$\langle N(Au, w) - N(Av, w), \eta(u, v)\rangle \leq 0, \quad \forall u, v \in K, \ w \in H.$$

(5) $N$ is said to be *$\eta$-strongly monotone* with constant $\xi$ with respect to $A$ in the first argument if there exists a constant $\xi > 0$ such that

$$\langle N(Au, w,) - N(Av, w), \eta(u, v)\rangle \geq \xi\|u - v\|^2, \quad \forall u, v \in K, \ w \in H.$$

(6) $N$ is said to be *$\eta$-relaxed Lipschitz* with constant $\zeta$ with respect to $A$ in the second argument if there exists a constant $\zeta > 0$ such that

$$\langle N(w, Au) - N(w, Av), \eta(u, v)\rangle \leq -\zeta\|u - v\|^2, \quad \forall u, v \in K, \ w \in H.$$

(7) $\eta$ is said to be *Lipschitz continuous* with constant $\delta$ if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta\|u - v\|, \quad \forall u, v \in K.$$

(8) $\eta$ is said to be *strongly monotone* with constant $\omega$ if there exists a constant $\omega > 0$ such that

$$\langle u - v, \eta(u, v)\rangle \geq \omega\|u - v\|^2, \quad \forall u, v \in K.$$

Similarly, we can define the Lipschitz continuity of $N$ in the second argument.

LEMMA 2.1. ([1], [2]) *Let $X$ be a nonempty closed convex subset of a Hausdorff linear topological space $E$, and $\phi, \psi : X \times X \to R$ be mappings satisfying the following conditions:*

(a) $\psi(x,y) \le \phi(x,y)$, $\forall x,y \in X$, *and* $\psi(x,x) \ge 0$, $\forall x \in X$;

(b) *for each* $x \in X$, $\phi(x,y)$ *is upper semicontinuous with respect to* $y$;

(c) *for each* $y \in X$, *the set* $\{x \in X : \psi(x,y) < 0\}$ *is a convex set;*

(d) *there exists a nonempty compact set* $K \subset X$ *and* $x_0 \in K$ *such that* $\psi(x_0, y) < 0$, $\forall y \in X \setminus K$;

*Then there exists* $\hat{y} \in K$ *such that* $\phi(x, \hat{y}) \ge 0$, $\forall x \in X$.

## 3. Existence theorems

In this section, we give two existence theorems of solutions for the general strongly nonlinear variational-like inequality (2.1).

THEOREM 3.1. *Let $a : H \times H \to (-\infty, \infty)$ be a coercive continuous bilinear form with (C1) and (C2) and $f : K \to (-\infty, \infty]$ be a proper convex lower semicontinuous functional with $int(dom f) \cap K \neq \emptyset$. Suppose that $A, C, E : K \to H$ and $N, M : H \times H \to H$ are continuous mappings, $\eta : K \times K \to H$ is Lipschitz continuous with constant $\delta$, for each $v \in K$, $\eta(\cdot, v)$ is continuous and $\eta(v,u) = -\eta(u,v)$ for all $u, v \in K$. Assume that $N$ is $\eta$-antimonotone with respect to $A$ in the first argument and $\eta$-relaxed Lipschitz with constant $\xi$ with respect to $C$ in the second argument. Let $M$ be $\eta$-strongly monotone with constant $\varrho$ with respect to $E$ in the first argument and Lipschitz continuous with constant $\vartheta$ in the second argument. Let $F : K \to H$ be Lipschitz continuous with constant $l$. Suppose that for given $x, y \in H$ and $v \in K$, the mappings $u \mapsto \langle N(x,y), \eta(u,v)\rangle$ and $u \mapsto \langle M(x,y), \eta(v,u)\rangle$ be concave and upper semicontinuous. If $\frac{\delta\vartheta l}{c+\xi+\varrho} < 1$, then the general strongly nonlinear variational-like inequality (2.1) has a unique solution in $K$.*

*Proof.* First of all we show that for each fixed $\hat{u} \in K$, there exists a unique $\hat{w} \in K$ such that

$$
\begin{aligned}
&a(\hat{w}, v - \hat{w}) + f(v) - f(\hat{w}) \\
&\ge \langle N(A\hat{w}, C\hat{w}) - M(E\hat{w}, F\hat{u}), \eta(v, \hat{w})\rangle, \quad \forall v \in K.
\end{aligned}
\tag{3.1}
$$

Let $\hat{u}$ be in $K$. Define the functionals $\phi$ and $\psi : K \times K \to R$ by

$$
\begin{aligned}
\phi(v,w) = &\ a(v, v-w) + f(v) - f(w) \\
&- \langle N(Av, Cv) - M(Ev, F\hat{u}), \eta(v,w)\rangle
\end{aligned}
$$

and
$$\begin{aligned}\psi(v,w) &= a(w, v-w) + f(v) - f(w)\\ &\quad - \langle N(Aw, Cw) - M(Ew, F\hat{u}), \eta(v,w)\rangle\end{aligned}$$
for all $v, w \in K$.

We check that the functionals $\phi$ and $\psi$ satisfy all the conditions of Lemma 2.1 in the weak topology. It is easy to see for all $v, w \in K$,
$$\begin{aligned}&\phi(v,w) - \psi(v,w)\\ &= a(v-w, v-w) - \langle N(Av, Cv) - N(Aw, Cv), \eta(v,w)\rangle\\ &\quad - \langle N(Aw, Cv) - N(Aw, Cw), \eta(v,w)\rangle\\ &\quad + \langle M(Ev, F\hat{u}) - M(Ew, F\hat{u}), \eta(v,w)\rangle\\ &\geq (c + \xi + \varrho)\|v - w\|^2 \geq 0,\end{aligned}$$
which yields that $\phi$ and $\psi$ satisfy the condition (a) of Lemma 2.1. Note that $f$ is a convex lower semicontinuous functional and for given $x, y \in H$, $v \in K$, the mappings $u \mapsto \langle N(x,y),\ \eta(u,v)\rangle$ and $u \mapsto \langle M(x,y), \eta(v,u)\rangle$ are concave and upper semicontinuous. It follows that $\phi(v,w)$ is weakly upper semicontinuous with respect to $w$ and the set $\{v \in K : \psi(v,w) < 0\}$ is convex for each $w \in K$. Therefore the conditions (b) and (c) of Lemma 2.1 hold. Since $f$ is proper convex lower semicontinuous, for each $v \in int(domf)$, $\partial f(v) \neq \emptyset$, see Ekeland and Teman [7]. Let $v^*$ be in $int(domf) \cap K$. It follows that
$$f(u) \geq f(v^*) + \langle r, u - v^*\rangle, \quad \forall r \in \partial f(v^*), u \in K.$$
Put
$$D = (c + \xi + \varrho)^{-1}(\|r\| + \delta\|N(Av^*, Cv^*)\| + \delta\|M(Ev^*, F\hat{u})\|)$$
and
$$T = \{w \in K : \|w - v^*\| \leq D\}.$$
Clearly, $T$ is a weakly compact subset of $K$ and for any $w \in K \setminus T$
$$\begin{aligned}\psi(v^*, w) &= a(w - v^*, v^* - w) + f(v^*) - f(w)\\ &\quad - \langle N(Aw, Cw) - M(Ew, F\hat{u}), \eta(v^*, w)\rangle\\ &\leq - a(w - v^*, w - v^*) - \langle r, w - v^*\rangle\\ &\quad + \langle N(Aw, Cw) - N(Av^*, Cw), \eta(w, v^*)\rangle + \langle N(Av^*, Cw)\\ &\quad - N(Av^*, Cv^*), \eta(w, v^*)\rangle + \langle N(Av^*, Cv^*), \eta(w, v^*)\rangle\\ &\quad - \langle M(Ew, F\hat{u}) - M(Ev^*, F\hat{u}), \eta(w, v^*)\rangle\\ &\quad - \langle M(Ev^*, F\hat{u}), \eta(w, v^*)\rangle\end{aligned}$$

$$\begin{aligned}&\leq -\|w-v^*\|[(c+\xi+\varrho)\|w-v^*\|-\|r\|-\delta\|N(Av^*,Cv^*)\|\\&\quad -\delta\|M(Ev^*,F\hat{u})\|]\\&<0,\end{aligned}$$

which means that the condition (d) of Lemma 2.1 holds. Thus Lemma 2.1 ensures that there exists a $\hat{w}\in K$ such that $\phi(v,\hat{w})\geq 0$ for all $v\in K$, that is,

$$\begin{aligned}&a(v,v-\hat{w})+f(v)-f(\hat{w})\\&\geq\langle N(Av,Cv)-M(Ev,F\hat{u}),\eta(v,\hat{w})\rangle,\quad \forall v\in K.\end{aligned}\tag{3.2}$$

Let $t$ be in $(0,1]$ and $v$ be in $K$. Replacing $v$ by $v_t=tv+(1-t)\hat{w}$ in (3.2), we know that

$$\begin{aligned}&a(v_t,t(v-\hat{w}))+f(v_t)-f(\hat{w})\\&\geq\langle N(Av_t,Cv_t)-M(Ev_t,F\hat{u}),\eta(v_t,\hat{w})\rangle,\quad \forall v\in K.\end{aligned}\tag{3.3}$$

Notice that $a$ is bilinear and $f$ is convex. From (3.3) we deduce that

$$\begin{aligned}&t[a(v_t,v-\hat{w})+f(v)-f(\hat{w})]\\&\geq t\langle N(Av_t,Cv_t)-M(Ev_t,F\hat{u}),\eta(v,\hat{w})\rangle,\quad \forall v\in K,\end{aligned}$$

which implies that

$$\begin{aligned}&a(v_t,v-\hat{w})+f(v)-f(\hat{w})\\&\geq\langle N(Av_t,Cv_t)-M(Ev_t,F\hat{u}),\eta(v,\hat{w})\rangle,\quad \forall v\in K.\end{aligned}$$

Letting $t\to 0^+$ in the above inequality, we conclude that

$$\begin{aligned}&a(\hat{w},v-\hat{w})+f(v)-f(\hat{w})\\&\geq\langle N(A\hat{w},C\hat{w})-M(E\hat{w},F\hat{u}),\eta(v,\hat{w})\rangle,\quad \forall v\in K.\end{aligned}$$

That is, $\hat{w}$ is a solution of (3.1). Now we prove the uniqueness. For any two solutions $w_1,w_2\in K$ of (3.1), we see that

$$\begin{aligned}&a(w_1,w_2-w_1)+f(w_2)-f(w_1)\\&\geq\langle N(Aw_1,Cw_1)-M(Ew_1,F\hat{u}),\eta(w_2,w_1)\rangle\end{aligned}$$

and

$$\begin{aligned}&a(w_2,w_1-w_2)+f(w_1)-f(w_2)\\&\geq\langle N(Aw_2,Cw_2)-M(Ew_2,F\hat{u}),\eta(w_1,w_2)\rangle.\end{aligned}$$

Adding these inequalities, we deduce that

$$\begin{aligned}
c\|w_1 - w_2\|^2 &\le a(w_1 - w_2, w_1 - w_2)\\
&\le \langle N(Aw_1, Cw_1) - N(Aw_2, Cw_1), \eta(w_1, w_2)\rangle\\
&\quad + \langle N(Aw_2, Cw_1) - N(Aw_2, Cw_2), \eta(w_1, w_2)\rangle\\
&\quad - \langle M(Ew_1, F\hat{u}) - M(Ew_2, F\hat{u}), \eta(w_1, w_2)\rangle\\
&\le -(\xi + \varrho)\|w_1 - w_2\|^2,
\end{aligned}$$

which yields that $w_1 = w_2$. That is, $\hat{w}$ is the unique solution of (3.1). This means that there exists a mapping $G : K \to K$ satisfying $G(\hat{u}) = \hat{w}$, where $\hat{w}$ is the unique solution of (3.1) for each $\hat{u} \in K$. .

Next we show that $G$ is a contraction mapping. Let $u_1$ and $u_2$ be arbitrary elements in $K$. Using (3.1), we get that

$$\begin{aligned}
&a(Gu_1, Gu_2 - Gu_1) + f(Gu_2) - f(Gu_1)\\
&\ge \langle N(A(Gu_1), C(Gu_1)) - M(E(Gu_1), Fu_1), \eta(Gu_2, Gu_1)\rangle
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&a(Gu_2, Gu_1 - Gu_2) + f(Gu_1) - f(Gu_2)\\
&\ge \langle N(A(Gu_2), C(Gu_2)) - M(E(Gu_2), Fu_2), \eta(Gu_1, Gu_2)\rangle.
\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), we arrive at

$$\begin{aligned}
&c\|Gu_1 - Gu_2\|^2\\
&\le a(Gu_1 - Gu_2, Gu_1 - Gu_2)\\
&\le \langle N(A(Gu_1), C(Gu_1)) - N(A(Gu_2), C(Gu_1)), \eta(Gu_1, Gu_2)\rangle\\
&\quad + \langle N(A(Gu_2), C(Gu_1)) - N(A(Gu_2), C(Gu_2)), \eta(Gu_1, Gu_2)\rangle\\
&\quad - \langle M(E(Gu_1), Fu_1) - M(E(Gu_2), Fu_1), \eta(Gu_1, Gu_2)\rangle\\
&\quad - \langle M(E(Gu_2), Fu_1) - M(E(Gu_2), Fu_2), \eta(Gu_1, Gu_2)\rangle\\
&\le\ -(\xi + \varrho)\|Gu_1 - Gu_2\|^2 + \delta\vartheta l\|u_1 - u_2\|\|Gu_1 - Gu_2\|,
\end{aligned}$$

that is,

$$\|Gu_1 - Gu_2\| \le \frac{\delta\vartheta l}{c + \xi + \varrho}\|u_1 - u_2\|,$$

which yields that $G : K \to K$ is a contraction mapping by $\frac{\delta\vartheta l}{c+\xi+\varrho} < 1$ and hence it has a unique fixed point $u \in K$, which is a unique solution of the general strongly nonlinear variational-like inequality (2.1). This completes the proof. □

THEOREM 3.2. *Let $a, f, C, N, M, E, F$ and $\eta$ be as in Theorem 3.1 and $N$ be Lipschitz continuous with constant $\zeta$ in the first argument. Suppose that $A : K \to H$ is Lipschitz continuous with constant $\varepsilon$. If $0 < \frac{\delta\vartheta l}{c+\xi+\varrho-\delta\zeta\varepsilon} < 1$, then the general strongly nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.*

*Proof.* Put

$$D = (c + \xi + \varrho - \delta\zeta\varepsilon)^{-1}(\|r\| + \delta\|N(Av^*, Cv^*)\| + \delta\|M(Ev^*, F\hat{u})\|)$$

and

$$T = \{w \in K : \|w - v^*\| \leq D\}.$$

As in the proof of Theorem 3.1, we conclude that

$$\begin{aligned}\psi(v^*, w) \leq &-a(w - v^*, w - v^*) - \langle r, w - v^*\rangle \\ &+ \langle N(Aw, Cw) - N(Av^*, Cw), \eta(w, v^*)\rangle \\ &+ \langle N(Av^*, Cw) - N(Av^*, Cv^*), \eta(w, v^*)\rangle \\ &+ \langle N(Av^*, Cv^*), \eta(w, v^*)\rangle - \langle M(Ew, F\hat{u}) \\ &- M(Ev^*, F\hat{u}), \eta(w, v^*)\rangle - \langle M(Ev^*, F\hat{u}), \eta(w, v^*)\rangle \\ \leq &-\|w - v^*\|[(c + \xi + \varrho - \delta\zeta\varepsilon)\|w - v^*\| \\ &- \|r\| - \delta\|N(Av^*, Cv^*)\| - \delta\|M(Ev^*, F\hat{u})\|] \\ < &\,0\end{aligned}$$

for any $w \in K \setminus T$. The rest of the argument is now essentially the same as in the proof of Theorem 3.1 and therefore is omitted. □

## 4. Algorithm and convergence theorems

Let's consider the following auxiliary variational-like inequality problem: For any given $u \in K$, find $w \in K$ such that

$$\begin{aligned}&\langle w, \eta(v, w)\rangle \\ &\geq \langle u, \eta(v, w)\rangle + \mu\langle N(Aw, Cw) - M(Ew, Fu), \eta(v, w)\rangle \\ &\quad - \mu a(u, v - w) - \mu f(v) + \mu f(w), \quad \forall v \in K,\end{aligned} \tag{4.1}$$

where $\mu > 0$ is a constant. Clearly, $w = u$ is a solution of the auxiliary variational-like inequality (4.1). Based on this observation, we suggest the following iterative algorithm.

ALGORITHM 4.1. *Let $A, C, E, F : K \to H$, $N, M : H \times H \to H$ and $\eta : K \times K \to H$ be mappings, and $f : K \to (-\infty, \infty]$ be a real functional. For any given $u_0 \in K$, compute sequences $\{u_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ by the iterative schemes*

$$\begin{aligned} &\langle w_n, \eta(v, w_n)\rangle \\ &\geq (1-\alpha_n)\langle u_n, \eta(v, w_n)\rangle \\ &\quad + \alpha_n\langle u_n + \mu N(Aw_n, Cw_n) - \mu M(Ew_n, Fu_n), \eta(v, w_n)\rangle \\ &\quad - \alpha_n\mu a(u_n, v - w_n) - \alpha_n\mu f(v) + \alpha_n\mu f(w_n) \end{aligned} \tag{4.2}$$

*and*

$$\begin{aligned} &\langle u_{n+1}, \eta(v, u_{n+1})\rangle \\ &\geq (1-\beta_n)\langle w_n, \eta(v, u_{n+1})\rangle \\ &\quad + \beta_n\langle w_n + \mu N(Au_{n+1}, Cu_{n+1}) \\ &\quad - \mu M(Eu_{n+1}, Fw_n), \eta(v, u_{n+1})\rangle - \beta_n\mu a(w_n, v - u_{n+1}) \\ &\quad - \beta_n\mu f(v) + \beta_n\mu f(u_{n+1}), \end{aligned} \tag{4.3}$$

*for all $v \in K$ and $n \geq 0$, where $\{\alpha_n\}_{n\geq 0}, \{\beta_n\}_{n\geq 0} \subset [0,1]$ with $\sum_{n=0}^{\infty} \beta_n = \infty$.*

THEOREM 4.1. *Let $a, f, F, A, C, E, F, N, M$ and $\eta$ be as in Theorem 3.1. Suppose that $M$ is strongly monotone with constant $\tau$ with respect to $F$ in the second argument and $\eta$ is strongly monotone with constant $\omega$. If $\frac{\delta\vartheta l}{c+\xi+\varrho} < 1$ and there exists a constant $\mu > 0$ such that*

$$\frac{\delta - \omega}{\xi + \varrho} \leq \mu < \min\left\{\frac{\delta}{d}, \frac{2\delta(\delta\tau - d)}{(\delta\vartheta l)^2 - d^2}\right\}, \tag{4.4}$$

*then the general strongly nonlinear variational-like inequality (2.1) possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converges strongly to $u$.*

*Proof.* It follows from Theorem 3.1 that the general strongly nonlinear variational-like inequality (2.1) has a unique solution $u \in K$ such that

$$\begin{aligned} \langle u, \eta(v, u)\rangle \geq{} & (1-\alpha_n)\langle u, \eta(v, u)\rangle \\ & + \alpha_n\langle u + \mu N(Au, Cu) - \mu M(Eu, Fu), \eta(v, u)\rangle \\ & - \alpha_n\mu a(u, v - u) - \alpha_n\mu f(v) + \alpha_n\mu f(u) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\langle u, \eta(v,u)\rangle \geq\ & (1-\beta_n)\langle u, \eta(v,u)\rangle \\ & + \beta_n\langle u + \mu N(Au, Cu) - \mu M(Eu, Fu), \eta(v,u)\rangle \\ & - \beta_n \mu a(u, v-u) - \beta_n \mu f(v) + \beta_n \mu f(u)\end{aligned} \tag{4.6}$$

for all $v \in K$ and $n \geq 0$. Taking $v = u$ in (4.2), $v = w_n$ in (4.5) and adding these inequalities, we get that

$$\begin{aligned}&\omega\|w_n - u\|^2 \\ &\leq (1-\alpha_n)\langle u_n - u, \eta(w_n, u)\rangle \\ &\quad + \alpha_n\mu\langle N(Aw_n, Cw_n) - N(Au, Cw_n), \eta(w_n,u)\rangle \\ &\quad + \alpha_n\mu\langle N(Au, Cw_n) - N(Au, Cu), \eta(w_n,u)\rangle \\ &\quad - \alpha_n\mu\langle M(Ew_n, Fu_n) - M(Eu, Fu_n), \eta(w_n,u)\rangle \\ &\quad + \alpha_n\langle u_n - u - (\mu M(Eu, Fu_n) - \mu M(Eu, Fu)), \eta(w_n,u)\rangle \\ &\quad - \alpha_n a(u_n - u, w_n - u) \\ &\leq \delta\Big[1 - \alpha_n\Big(1 - \mu\frac{d}{\delta} - \sqrt{1 - 2\mu\tau + (\mu\vartheta l)^2}\Big)\Big]\|u_n - u\|\|w_n - u\| \\ &\quad - \mu(\xi + \varrho)\|w_n - u\|^2, \quad \forall n \geq 0,\end{aligned}$$

that is,

$$\begin{aligned}\|w_n - u\| &\leq \theta_1\Big[1 - \alpha_n\Big(1 - \mu\frac{d}{\delta} - \sqrt{1 - 2\mu\tau + (\mu\vartheta l)^2}\Big)\Big]\|u_n - u\| \\ &\leq [1 - \alpha_n(1-\theta_2)]\|u_n - u\| \\ &\leq \|u_n - u\|, \quad \forall n \geq 0,\end{aligned} \tag{4.7}$$

where $\theta_1 = \frac{\delta}{\omega + \mu(\xi+\varrho)} \leq 1$ and $\theta_2 = \frac{\mu d}{\delta} + \sqrt{1 - 2\mu\tau + (\mu\vartheta l)^2} < 1$ by (4.4). It follows from (4.3), (4.6) and (4.7) that

$$\begin{aligned}&\omega\|u_{n+1} - u\|^2 \\ &\leq \delta(1-\beta_n)\|w_n - u\|\|u_{n+1} - u\| + \delta\beta_n\theta_2\|w_n - u\|\|u_{n+1} - u\| \\ &\quad - \mu(\xi+\varrho)\|u_{n+1} - u\|^2, \quad \forall n \geq 0,\end{aligned}$$

that is,

$$\begin{aligned}\|u_{n+1} - u\| &\leq [1 - \beta_n(1-\theta_2)]\|u_n - u\| \\ &\leq e^{-(1-\theta_2)\beta_n}\|u_n - u\| \\ &\leq e^{-(1-\theta_2)\sum_{i=0}^{n}\beta_i}\|u_0 - u\|, \quad \forall n \geq 0,\end{aligned}$$

which yields that $\lim_{n\to\infty} \|u_{n+1} - u\| = 0$ by $\sum_{n=0}^{\infty} \beta_n = \infty$. This completes the proof. □

Similarly we have the following result.

THEOREM 4.2. *Let $a, f, F, N, A, C, M, E, F$ and $\eta$ be as in Theorem 3.2 with*

$$\delta < \min\left\{\frac{c+\xi+\varrho}{\vartheta l+\zeta\varepsilon}, \frac{\xi+\varrho}{\zeta\varepsilon}\right\}.$$

*Suppose that $M$ is strongly monotone with constant $\tau$ with respect to $F$ in the second argument and $\eta$ is strongly monotone with constant $\omega$. If there exists a constant $\mu > 0$ satisfying*

$$\frac{\delta-\omega}{\xi+\varrho-\delta\zeta\varepsilon} \le \mu < \min\left\{\frac{\delta}{d}, \frac{2\delta(\delta\tau-d)}{(\delta\vartheta l)^2-d^2}\right\},$$

*then the general strongly nonlinear variational-like inequality (2.1) possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\ge 0}$ generated by Algorithm 4.1 converges strongly to $u$.*

## References


[1] S. S. Chang, *Variational inequalitity and complementarity theory with applications*, Shanghai Sci. Technol., Shanghai (1991).

[2] ———, *On the existence of solutions for a class of quasi-bilinear variational inequalities*, J. Sys. Sci. Math. Scis. **16** (1996), 136–140 [In Chinese].

[3] P. Cubiotti, *Existence of solutions for lower semicontinuous quasi-equilibrium problems*, Comput. Math. Appl. **30** (1995), no. 12, 11–22.

[4] X. P. Ding, *Algorithm of solutions for mixed nonlinear variational-like inequalities in reflexive Banach space*, Appl. Math. Mech. **19** (1998), no. 6, 521–529.

[5] ———, *Existence and algorithm of solutions for nonlinear mixed variational-like inequalities in Banach spaces*, J. Comput. Appl. Math. **157** (2003), no. 2, 419–434.

[6] X. P. Ding and K. K. Tan, *A minimax inequality with applications to existence of equilibrium point and fixed point theorems*, Colloq. Math. **63** (1992), no. 2, 233–247.

[7] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, Holland, 1976.

[8] N. J. Huang and C. X. Deng, *Auxiliary principle and iterative algorithms for generalized set-valued strongly nonlinear mixed variational-like inequalities*, J. Math. Anal. Appl. **256** (2001), no. 2, 345–359.

[9] Z. Liu, J. S. Ume, and S. M. Kang, *Nonlinear variational inequalities on reflexive Banach spaces and topological vector spaces*, Int. J. Math. Math. Sci. **2003** (2003), no. 4, 199–207.

[10] Z. Liu, L. Debnath, S. M. Kang, and J. S. Ume, *Completely generalized multivalued nonlinear quasi-variational inclusions*, Int. J. Math. Math. Sci. **30** (2002), no. 10, 593–604.
[11] ———, *On the generalized nonlinear quasivariational inclusions*, Acta. Math. Inform. Univ. Ostraviensis **11** (2003), no. 1, 81–90.
[12] ———, *Sensitivity analysis for parametric completely generalized nonlinear implicit quasivariational inclusions*, J. Math. Anal. Appl. **277** (2003), no. 1, 142–154.
[13] ———, *Generalized mixed quasivariational inclusions and generalized mixed resolvent equations for fuzzy mappings*, Appl. Math. Comput. **149** (2004), no. 3, 879–891.
[14] Z. Liu, S. M. Kang, and J. S. Ume, *On general variational inclusions with noncompact valued mappings*, Adv. Nonlinear Var. Inequal. **5** (2002), no. 2, 11–25.
[15] ———, *Completely generalized multivalued strongly quasivariational inequalities*, Publ. Math. Debrecen **62** (2003), no. 1-2, 187–204.
[16] ———, *Generalized variational inclusions for fuzzy mappings*, Adv. Nonlinear Var. Inequal. **6** (2003), no. 1, 31–40.
[17] ———, *The solvability of a class of quasivariational inequalities*, Adv. Nonlinear Var. Inequal. **6** (2003), no. 2, 69–78.
[18] Z. Liu and S. M. Kang, *Generalized multivalued nonlinear quasi-variational inclusions*, Math. Nachr. **253** (2003), 45–54.
[19] ———, *Convergence and stability of perturbed three-step iterative algorithm for completely generalized nonlinear quasivariational inequalities*, Appl. Math. Comput. **149** (2004), no. 1, 245–258.
[20] Z. Liu, J. S. Ume, and S. M. Kang, *General strongly nonlinear quasivariational inequalities with relaxed Lipschitz and relaxed monotone mappings*, J. Optim. Theory Appl. **114** (2002), no. 3, 639–656.
[21] ———, *Resolvent equations technique for general variational inclusions*, Proc. Japan Acad., Ser. A Math. Sci. **78** (2002), no. 10, 188–193.
[22] ———, *Nonlinear variational inequalities on reflexive Banach spaces and topological vector spaces*, Int. J. Math. Math. Sci. **2003** (2003), no. 4, 199–207.
[23] ———, *Completely generalized quasivariational inequalities*, Adv. Nonlinear Var. Inequal. **7** (2004), no. 1, 35–46.
[24] P. D. Panagiotopoulos and G. E. Stavroulakis, *New types of variational principles based on the notion of quasidifferentiability*, Acta Mech. **94** (1992), no. 3-4, 171–194.
[25] J. Parida and A. Sen, *A variational-like inequality for multifunctions with applications*, J. Math. Anal. Appl. **124** (1987), no. 1, 73–81.
[26] G. Tian, *Generalized quasi-variational-like inequality problem*, Math. Oper. Res. **18** (1993), no. 3, 752–764.
[27] R. U. Verma, *On generalized variational inequalities involving relaxed Lipschitz and relaxed monotone operators*, J. Math. Anal. Appl. **213** (1997), no. 1, 387–392.
[28] ———, *Generalized variational inequalities and associated nonlinear equations*, Czechoslovak Math. J. **48** (1998), no. 3, 413–418.
[29] ———, *Generalized pseudo-contractions and nonlinear variational inequalities*, Publ. Math. Debrecen **53** (1998), no. 1-2, 23–28.

[30] ———, *The solvability of a class of generalized nonlinear variational inequalities based on an iterative algorithm*, Appl. Math. Lett. **12** (1999), no. 4, 51–53.

[31] ———, *A general iterative algorithm and solvability of nonlinear quasivariational inequalities*, Adv. Nonlinear Var. Inequal. **4** (2001), no. 2, 79–87.

[32] J. C. Yao, *Existence of generalized variational inequalities*, Oper. Res. Lett. **15** (1994), no. 1, 35–40.

[33] ———, *The generalized quasi-variational inequality problem with applications*, J. Math. Anal. Appl. **158** (1991), no. 1, 139–160.

Zeqing Liu and Juhe Sun, Department of Mathematics, Liaoning Normal University, P. O. Box 200, Dalian, Liaoning 116029, P. R. China
*E-mail*: zeqingliu@dl.cn

Soo Hak Shim and Shin Min Kang, Department of Mathematics and Research Institute of Natural Science, Gyeongsang National University, Chinju 660-701, Korea
*E-mail*: math@nongae.gsnu.ac.kr
smkang@nongae.gsnu.ac.kr