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GENERALIZED STABILITY OF ISOMETRIES ON REAL BANACH SPACES

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ABSTRACT. Let X and Y be real Banach spaces and $\varepsilon > 0$, $p > 1$. Let $f : X \to Y$ be a bijective mapping with $f(0) = 0$ satisfying

$$\Big| \|f(x) - f(y)\| - \|x - y\| \Big| \leq \varepsilon \|x - y\|^p$$

for all $x \in X$ and, let $f^{-1} : Y \to X$ be uniformly continuous. Then there exist a constant $\delta > 0$ and $N(\varepsilon, p)$ such that $\lim_{\varepsilon \to 0} N(\varepsilon, p) = 0$ and a unique surjective isometry $I : X \to Y$ satisfying $\|f(x) - I(x)\| \leq N(\varepsilon, p)\|x\|^p$ for all $x \in X$ with $\|x\| \leq \delta$.

1. Introduction

Throughout this paper X and Y denote real Banach spaces. It is a well-known classical result of Mazur and Ulam [5] that an isometry f from X onto Y for which $f(0) = 0$ is automatically linear. A mapping $f : X \to Y$ is called an (ε, p)-isometry if

$$\Big| \|f(x) - f(y)\| - \|x - y\| \Big| \leq \varepsilon \|x - y\|^p$$

for all $x, y \in X$.

In 1983 J. Gevirtz [2] showed that if $T : X \to Y$ is a surjective $(\varepsilon, 0)$-isometry, then there exists a unique isometry $I : X \to Y$ such that

$$\|T(x) - I(x)\| \leq 20(\sqrt{2} - 1)^{-1}((\varepsilon\|x\|)^{\frac{1}{2}} + 40\varepsilon)$$

for all $x \in X$. Using a result of P. M. Gruber [3], there exists a unique surjective isometry $I : X \to Y$ for which $\|T(x) - I(x)\| \leq 5\varepsilon$ for all $x \in X$.

In 1995 M. Omladič and P. Šemrl [6] showed that for any an $(\varepsilon, 0)$-isometry $f : X \to Y$ there exists a unique surjective linear isometry $I : X \to Y$ such that $\|f(x) - I(x)\| \leq 2\varepsilon$ for all $x \in X$.

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J. Lindenstrauss and A. Szankowski [4] studied a wider concept of approximate isometries. For given surjective mapping f from a real Banach space X onto a real Banach space Y they considered the function

$$\Phi_f(t) = \sup\Big\{\Big|\|f(x) - f(y)\| - \|x - y\|\Big| : \|x - y\| \le t \text{ or } \|f(x) - f(y)\| \le t\Big\}$$

for $t \ge 0$. They proved that if

$$\int_1^\infty \frac{\Phi_f(t)}{t^2} dt < \infty,$$

then there is an isometry I from X onto Y, defined by

$$I(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

such that $\|f(x) - I(x)\| = 0(\|x\|)$ as $\|x\| \to \infty$.

Using the result of J. Lindenstrauss and A. Szankowski [4], G. Dolinar [1] showed that if $f : X \to Y$ is a surjective (ε, p)-isometry with $f(0) = 0$ and $0 \le p < 1$, then there exist a constant $N(p)$ and a surjective isometry $I : X \to Y$ such that

$$\|f(x) - I(x)\| \le \varepsilon N(p)\|x\|^p$$

for all $x \in X$.

Also he showed that for an (ε, p)-isometry f from a real Banach space X into a real Hilbert space H with $f(0) = 0$ and $0 < p < 1$, there exists a linear isometry $I : X \to H$ such that

$$\|f(x) - I(x)\| \le C(\varepsilon, p)\max\{\|x\|^p, \|x\|^{\frac{1+p}{2}}\}$$

for all $x \in X$, where $C(\varepsilon, p) \to 0$ as $\varepsilon \to 0$.

In case $p = 1$, from the result of J. Lindenstrauss and A. Szankowski [4] we obtain an $(\varepsilon, 1)$-isometry T from l_2 onto itself which satisfies for every linear operator L on l_2 there is an $x \in l_2$ so that

$$\|T(x) - L(x)\| \ge \|x\|.$$

Thus we cannot obtain an isometry $I : l_2 \to l_2$ such that

$$\|T(x) - I(x)\| \le C(\varepsilon)\|x\|$$

for all $x \in l_2$ where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

In this paper, given $p > 1$ and $\varepsilon > 0$, let $f : X \to Y$ be a bijective (ε, p)-isometry with $f(0) = 0$ and, let $f^{-1} : Y \to X$ be uniformly continuous. Then there exist a constant $\delta > 0$ and $N(\varepsilon, p)$ such that $\lim_{\varepsilon \to 0} N(\varepsilon, p) = 0$ and a unique surjective isometry $I : X \to Y$ satisfying

$$\|f(x) - I(x)\| \le N(\varepsilon, p)\|x\|^p$$

for all $x \in X$ with $\|x\| \le \delta$.

2. Results

The following theorem is a generalized proposition of J. Lindenstrauss and A. Szankowski [4].

THEOREM 1. *Let f be a bijective mapping from a real Banach space X onto a real Banach space Y. Let $\alpha \ge 0$ and ϕ be monotone increasing functions such that $\phi(2t) \le 2^\alpha \phi(t)$ for real number $t \ge 0$ and*

$$\Big| \|f(x) - f(y)\| - \|x - y\| \Big| \le \phi(\|x - y\|)$$

for all $x, y \in X$. For given $\delta > 0$, let ψ be a mapping satisfying

$$\Big| \|f(x) - f(y)\| - \|x - y\| \Big| \le \psi(\|f(x) - f(y)\|)$$

for all $x, y \in X$ with $\|f(x) - f(y)\| \le \delta$.

Let $d = \|x - y\| \le \delta$ and n be a positive integer such that $2^n \ge \frac{d}{4(\phi(d)+\psi(d))}$. Then we obtain

$$\begin{aligned} &\|f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2}\| \\ \le\ & 8\psi(d) + 8\phi(d) + \frac{1}{2}\sum_{k=-1}^{n-1} \phi(2^{\frac{1-k}{2}} \max(1, 2^{\alpha-1})d) \end{aligned}$$

Before proving the theorem, we define some notations and prove the following lemmas. Given $x, y \in X$ with $d = \|x - y\| \le \delta$, let $u = \frac{x+y}{2}$ and $v = \frac{f(x)+f(y)}{2}$. We define inductively two sequences $\{u_j\}_{j=-\infty}^{\infty} \subset X$ and $\{v_j\}_{j=-\infty}^{\infty} \subset Y$ by putting $u_0 = u$, $v_0 = v$

$$\begin{aligned} (1) \qquad u_{j+1} &= f^{-1}(v_{-j}),\ v_{j+1} = f(u_{-j}),\ j = 0, 1, 2, \ldots \\ u_{-j} &= au_j, v_{-j} = bv_j, \end{aligned}$$

where a and b are the isometries of X and Y satisfying

$$ax' = 2u - x',\ \text{by}' = 2v - y'$$

for all $x' \in X$ and $y' \in Y$. We define the following numbers

$$(2) \qquad \begin{aligned} \delta_{2j} &= \max\{ \ \|u_{2k} - u_{2l}\| \mid |k-l| = j, -j \le k, l \le j \ \} \\ \triangle_{2j} &= \max_{0 \le i \le j}\{ \ \|u_{2i} - x\|, \|u_{2i} - y\| \ \} \end{aligned}$$

for $j = 0, 1, 2, \ldots$.

LEMMA 1.

$$\delta_{2^{m+1}} \ge 2\delta_{2^m} - 2^m \phi(\delta_{2^m})$$

$m = 1, 2, 3, \ldots$.

Proof. Let $-j+1 \le k,\ l \le j$ with $|k-l| = j$. Then we have

$$(3) \qquad \begin{aligned} & \|f(u_{-2k+2}) - f(u_{-2l+2})\| \\ = \ & \|v_{2k-1} - v_{2l-1}\| = \|bv_{2k-1} - bv_{2l-1}\| \\ = \ & \|v_{-2k+1} - v_{-2l+1}\| = \|f(u_{2k}) - f(u_{2l})\|. \end{aligned}$$

Thus (1), (2) and (3) imply

$$(4) \qquad \begin{aligned} & \Big| \|u_{2k-2} - u_{2l-2}\| - \|u_{2k} - u_{2l}\| \Big| \\ = \ & \Big| \|au_{-2k+2} - au_{-2l+2}\| - \|u_{2k} - u_{2l}\| \Big| \\ = \ & \Big| \|u_{-2k+2} - u_{-2l+2}\| - \|u_{2k} - u_{2l}\| \Big| \\ \le \ & \Big| \|u_{-2k+2} - u_{-2l+2}\| - \|f(u_{-2k+2}) - f(u_{-2l+2})\| \Big| \\ & + \Big| \|f(u_{-2k+2}) - f(u_{-2l+2})\| - \|f(u_{2k}) - f(u_{2l})\| \Big| \\ & + \Big| \|f(u_{2k}) - f(u_{2l})\| - \|u_{2k} - u_{2l}\| \Big| \\ \le \ & \phi(\|u_{-2k+2} - u_{-2l+2}\|) + \phi(\|u_{2k} - u_{2l}\|) \\ \le \ & 2\phi(\delta_{2j}). \end{aligned}$$

Let k, l be such that $-2^{m-1} \le k,\ l \le 2^{m-1}$ with $|k-l| = 2^{m-1}$ and

$$(5) \qquad \delta_{2^m} = \|u_{2k} - u_{2l}\|.$$

Since $\|u_{2k} - u_{2l}\| = \|u_{-2k} - u_{-2l}\|$, $2^{m-2} \le |k| \le 2m - 1$ or $2^{m-2} \le |l| \le 2m - 1$, there is no loss of generality to assume that $k \ge 2^{m-2}$.

Applying (4), we have

$$\begin{aligned}
&\Big|\|u_{2k}-u_{2l}\|-\|u_{2^m}-u_0\|\Big|\\
\le\ &\Big|\|u_{2k}-u_{2l}\|-\|u_{2k+2}-u_{2l+2}\|\Big|+\cdots+\cdots\\
&+\Big|\|u_{2^m-2}-u_{-2}\|-\|u_{2^m}-u_0\|\Big|\\
\le\ &(2^{m-1}-k)\phi(\delta_{2^m})\\
\le\ &2^{m-1}\phi(\delta_{2^m}).
\end{aligned}$$

Hence we get

$$\|u_{2k}-u_{2l}\|-2^{m-1}\phi(\delta_{2^m})\le\|u_{2^m}-u_0\|. \tag{6}$$

Thus (5) and (6) imply

$$\begin{aligned}
\delta_{2^{m+1}} &\ge \|u_{2^{-m}}-u_{2^m}\|=2\|u_{2^m}-u_0\|\\
&\ge 2\delta_{2^m}-2^m\phi(\delta_{2^m})
\end{aligned}$$

for $m=1,2,3,\dots$. □

LEMMA 2. *If* $j\le\frac{d}{4(\phi(d)+\psi(d))}$, *then* $\triangle_{2j}\le\frac{3d}{4}$.

Proof. Without loss of generality we can assume that $\phi(d)\le\frac{d}{4}$ and $\psi(d)\le\frac{d}{4}$. Since $\triangle_0\le\frac{d}{2}\le\frac{3d}{4}$, we assume that $\triangle_{2k}\le\frac{3d}{4}$, for $0<k<j$

$$\begin{aligned}
\|u_{-2k}-y\| &= \|au_{-2k}-ay\|=\|u_{2k}-x\|\\
&\le \triangle_{2k}\le\frac{3d}{4}.
\end{aligned} \tag{7}$$

Hence we have

$$\Big|\|f(u_{-2k})-f(y)\|-\|u_{-2k}-y\|\Big|\le\phi(\|u_{-2k}-y\|)\le\phi(d).$$

Thus by (7) and our assumption we have

$$\|f(u_{-2k})-f(y)\|\le\|u_{-2k}-y\|+\phi(d)\le d. \tag{8}$$

So we obtain

$$\begin{aligned}
\|f(u_{2k+2})-f(x)\| &= \|v_{-2k-1}-f(x)\|\\
&= \|bv_{-2k-1}-bf(x)\|\\
&= \|v_{2k+1}-f(y)\|\\
&= \|f(u_{-2k})-f(y)\|.
\end{aligned} \tag{9}$$

From (8) and (9) we have

$$\text{(10)}\qquad \begin{aligned} &\Big|\|u_{2k+2}-x\| - \|f(u_{2k+2})-f(x)\|\Big| \\ \le\ & \psi(\|f(u_{2k+2})-f(x)\|) \le \psi(d). \end{aligned}$$

Thus (9) and (10) imply

$$\text{(11)}\qquad \begin{aligned} &\Big|\|u_{2k+2}-x\| - \|u_{2k}-x\|\Big| \\ =\ & \Big|\|u_{2k+2}-x\| - \|au_{2k}-ax\|\Big| \\ =\ & \Big|\|u_{2k+2}-x\| - \|u_{-2k}-y\|\Big| \\ \le\ & \Big|\|u_{2k+2}-x\| - \|f(u_{2k+2})-f(x)\|\Big| \\ & +\Big|\|f(u_{2k+2})-f(x)\| - \|f(u_{-2k})-y\|\Big| \\ & +\Big|\|f(u_{-2k})-f(y)\| - \|u_{-2k}-y\|\Big| \\ \le\ & \phi(d)+\psi(d). \end{aligned}$$

Replacing x by y in (11), we have

$$\triangle_{2k+2} \le \triangle_{2k} + \phi(d) + \psi(d).$$

Since $\triangle_0 = \frac{d}{2}$, using induction we have

$$\triangle_{2k+2} \le \frac{d}{2} + (k+1)(\phi(d)+\psi(d)).$$

Since $k+1 \le j$, $\triangle_{2k+2} \le \frac{3}{4}d$. □

Proof of the Theorem 1. Suppose there exists the largest non-negative integer m such that $2^m \le \frac{d}{4(\phi(d)+\psi(d))}$.

By Lemma 2, $\triangle_{2^{m+1}} \le d$. Since $\delta_{2^j} \le 2\triangle_{2^j}$ for every j, we have

$$\text{(12)}\qquad \delta_{2^{m-k}} \le \delta_{2^{m+1}} \le 2\triangle_{2^{m+1}} \le 2d$$

for $k = -1, 0, 1, \ldots, m-1$.

By Lemma 1, we get

$$\text{(13)}\qquad \delta_{2^{m-k}} \le \frac{1}{2}\delta_{2^{m-k+1}} + 2^{m-k-1}\phi(\delta_{2^{m-k}}).$$

We will prove that

$$\text{(14)}\qquad \delta_{2^{m-k}} \le 2^{\frac{1-k}{2}}\max(1, 2^{\alpha-1})d$$

for $k = -1, 0, 1, \ldots, m-1$.

If $k = -1$, we have from (12)

$$\delta_{2^{m+1}} \leq 2d \leq 2\max(1, 2^{\alpha-1})d.$$

Suppose (14) holds for $k-1$, $0 \leq k \leq m-1$.
Using (12) and (13),

$$\begin{aligned}
\delta_{2^{m-k}} &\leq \frac{1}{2}\delta_{2^{m-k+1}} + 2^{m-k-1}\phi(\delta_{2^{m-k}}) \\
&\leq \frac{1}{2}\delta_{2^{m-k+1}} + \frac{d}{4(\phi(d)+\psi(d))}2^{-k-1}\phi(2d) \\
&\leq \frac{1}{2}\delta_{2^{m-k+1}} + 2^{-k-3}2^{\alpha}d \\
&\leq 2^{-\frac{k}{2}}[\max(1, 2^{\alpha-1}) + 2^{\frac{-k-6}{2}}2^{\alpha}]d \\
&\leq 2^{\frac{1-k}{2}}\max(1, 2^{\alpha-1})d.
\end{aligned}$$

Using Lemma 1 repeatedly, if $m \geq 1$, we obtain for $m = 1, 2, 3, \ldots$

$$\delta_{2^{m+1}} \geq 2^m\delta_2 - 2^m\sum_{k=0}^{m-1}\phi(\delta_{2^{m-k}}).$$

Hence we have

$$\delta_2 + \phi(\delta_2) \leq 2^{-m}\delta_{2^{m+1}} + \sum_{k=0}^{m-1}\phi(\delta_{2^{m-k}}) + \phi(\delta_2). \tag{15}$$

Since $\frac{d}{4(\phi(d)+\psi(d))} \leq 2^{m+1}$, $2^{-m} \leq \frac{8(\phi(d)+\psi(d))}{d}$.
Thus (12) and (14) imply

$$\delta_2 + \phi(\delta_2) \leq 16\phi(d) + 16\psi(d) + \sum_{k=-1}^{m-1}\phi(2^{\frac{1-k}{2}}\max(1, 2^{\alpha-1})d) \tag{16}$$

for $m = 1, 2, 3, \ldots$.
If $\frac{d}{4(\phi(d)+\psi(d))} < 2$, then (12) and (14) imply

$$\delta_2 + \phi(\delta_2) \leq 16\phi(d) + 16\psi(d) + \phi(2\max(1, 2^{\alpha-1})d). \tag{17}$$

Since $n > m$, (16) and (17) imply

$$\delta_2 + \phi(\delta_2) \leq 16\phi(d) + 16\psi(d) + \sum_{k=-1}^{n-1}\phi(2^{\frac{1-k}{2}}\max(1, 2^{\alpha-1})d)$$

and

$$\begin{aligned}\|f(u)-v\| &= \|v_0-v_1\| \\ &= \frac{1}{2}\|v_{-1}-v_1\| \\ &= \frac{1}{2}\|f(u_2)-f(u_0)\| \\ &\leq \frac{1}{2}(\delta_2+\phi(\delta_2)).\end{aligned}$$

This completes the proof of the theorem. □

THEOREM 2. *Let X and Y be real Banach spaces and $\varepsilon > 0$, $p > 1$. Let $f : X \to Y$ be a bijective (ε, p)-isometry with $f(0) = 0$ and, let $f^{-1} : Y \to X$ be uniformly continuous. Then there exist a constant $\delta > 0$ and $N(\varepsilon, p)$ such that $\lim_{\varepsilon\to 0} N(\varepsilon,p) = 0$ and a unique surjective isometry $I : X \to Y$ satisfying*

$$\|f(x)-I(x)\| \leq N(\varepsilon,p)\|x\|^p$$

for all $x \in X$ with $\|x\| \leq \delta$.

Proof. Since $t \leq 2(t-\varepsilon t^p)$ for $t \leq (2\varepsilon)^{\frac{1}{1-p}}$, we have

$$\Big|\|f(x)-f(y)\|-\|x-y\|\Big| \leq \varepsilon 2^p\|f(x)-f(y)\|^p$$

for all $x, y \in X$ with $\|x-y\| \leq (2\varepsilon)^{\frac{1}{1-p}}$.

Since f^{-1} is uniformly continuous, there exists $\delta > 0$ such that

$$\|x-y\| \leq (2\varepsilon)^{\frac{1}{1-p}} \quad \text{if} \quad \|f(x)-f(y)\| \leq \delta.$$

Hence we obtain

$$\Big|\|f(x)-f(y)\|-\|x-y\|\Big| \leq \varepsilon 2^p\|f(x)-f(y)\|^p$$

for all $x, y \in X$ with $\|f(x)-f(y)\| \leq \delta$. Let $\|x-y\| = d \leq \delta$ for all $x, y \in X$.

By Theorem 1, we get

$$\begin{aligned}&\|f(\frac{x+y}{2})-\frac{f(x)+f(y)}{2}\| \\ \leq\ & 8\varepsilon d^p + 8\varepsilon 2^p d^p + \frac{1}{2}\sum_{k=-1}^{n-1}\varepsilon(2^{\frac{1-k}{2}}2^{p-1})^p d^p \\ =\ & M(\varepsilon,p)\|x-y\|^p,\end{aligned}$$

where $M(\varepsilon,p) = \varepsilon[8(1+2^p) + \frac{2^{p^2-\frac{p}{2}-1}}{2^{\frac{p}{2}}-1}]$.

Since $f(0) = 0$, we obtain

$$\|f(\frac{x}{2}) - \frac{f(x)}{2}\| \le M(\varepsilon, p)\|x\|^p \tag{18}$$

for all $x \in X$, $\|x\| \le \delta$.

For a given $x \in X$, there is a positive integer n such that $\|2^{-n}x\| \le \delta$. If $m > n$, replacing x by $2^{-m}x$ in (18), we have

$$\|f(2^{-m-1}x) - \frac{1}{2}f(2^{-m}x)\| \le M((\varepsilon, p)\|2^{-m}x\|^p.$$

Hence we obtain

$$\|2^m f(2^{-m}x) - 2^n f(2^{-n}x)\| \le \frac{2M(\varepsilon, p)}{1 - 2^{1-p}} 2^{n(1-p)}\|x\|^p.$$

Thus $\{2^n f(2^{-n}x)\}$ is a Cauchy sequence. So we define $I : X \to Y$ by $I(x) = \lim_{n \to \infty} 2^n f(2^{-n}x)$ for all $x \in X$.

Hence we get

$$\|f(x) - I(x)\| \le N(\varepsilon, p)\|x\|^p \tag{19}$$

for all $x \in X$, $\|x\| \le \delta$, where $N(\varepsilon, p) = \frac{2M(\varepsilon, p)}{1-2^{1-p}}$.

Since f is an (ε, p)-isometry and

$$\left| \|2^n f(2^{-n}x) - 2^n f(2^{-n}y)\| - \|x - y\| \right| \le \varepsilon 2^{(1-p)n}\|x - y\|^p$$

for all $n \in N$, I is an isometry.

Next we will show that I is an unique mapping satisfying (19). Let I' be an another isometry satisfying (19). Since I and I' are linear, a given $x \in X$

$$\begin{aligned} \|I(x) - I'(x)\| &= \|2^n I(2^{-n}x) - 2^n I'(2^{-n}x)\| \\ &\le 2N(\varepsilon, p)2^{n(1-p)}\|x\|^p \end{aligned}$$

for sufficiently large $n \in N$. Thus $I(x) = I'(x)$ for all $x \in X$.

Finally we will show that I is surjective. Assume that there is a non-zero $y \in Y$ so that $\|y - I(x)\| \ge \alpha > 0$ for all $x \in X$. Then there is a sequence $\{x_n\}$ such that $f(x_n) = \frac{1}{n}y$. Since f^{-1} is continuous, $\|x_n\| \to 0$ and

$$\|f(x_n) - I(x_n)\| \ge \frac{1}{n}\alpha \tag{20}$$

for all $n \in N$.

Since f is an (ε, p)-isometry, we have

$$\left| \|x_n\| - \frac{1}{n}\|y\| \right| \le \varepsilon\|x\|^p$$

for all $n \in N$. Thus $\|x_n\| = O(\frac{1}{n})$, and so $\|x_n\|^p = O(\frac{1}{n})$. Since

$$\|f(x_n) - I(x_n)\| \leq N(\varepsilon, p)\|x_n\|^p$$

for sufficiently large $n \in N$, we obtain

$$\|f(x_n) - I(x_n)\| = O(\frac{1}{n}).$$

This contradicts to (20). This completes the proof. □

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