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AN $MMAP[3]/PH/1$ QUEUE WITH NEGATIVE CUSTOMERS AND DISASTERS

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ABSTRACT. We consider a single-server queue with service time distribution of phase type where positive customers, negative customers and disasters arrive according to a Markovian arrival process with marked transitions (MMAP). We derive simple formulae for the stationary queue length distributions. The Laplace-Stieltjes transforms (LST's) of the sojourn time distributions under the combinations of removal policies and service disciplines are also obtained by using the absorption time distribution of a Markov chain.

1. Introduction

Computer systems without viruses have been modelled and analyzed by using conventional queueing system. However, conventional queueing model is not appropriate to the system with viruses since the effects of viruses to the system are different from those of ordinary jobs. Some viruses just infect one or more files and the infected files may not be recoverable and should be deleted. Some viruses may be critical to the system and destroy all the files in the system and the system is sent for repair. A virus may originate from outside the system e.g., floppy disk, or may come from another system e.g., by an electronic mail. The more files may induce more infected ones. Thus the arrivals of ordinary jobs and viruses may not be independent. From this example, we consider a single-server queue with three types of arrivals; positive customers, negative customers and disasters whose arrival processes are correlated. Positive customers are ordinary ones who form a queue but the negative

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customers and disasters do not form a queue. Negative customer is a signal to delete a positive customer in the system if any presents, and disappeared immediately. A disaster removes all the customers in the system upon its arrival and causes the system to start a repair period. For the reflection of the correlation among the three we use a Markovian arrival process with marked transitions (MMAP) introduced in He and Neuts [9] as an arrival process.

Queues with negative arrivals or disasters have been much developed since its introduction by Gelenbe [5], e. g. see [8, 7, 10, 13, 15, 14]. For a survey and comprehensive references for queues with negative customers or disasters, see Artalejo [1] and for a general treatments of the queueing networks with negative customers, see Chao et al. [2]. Recently a queue with correlated arrivals of customers and negative customers and/or disasters was analyzed in [15, 14]. The model considered in this paper is simpler than that in Shin [14] except introducing negative customers. However, our model and results have some different features from those in Shin [14]. The negative arrival deletes a customer in the system and the negative customers may be considered as a supplementary server. Thus our model can be considered as a queue in which the arrival process and the service times are not independent. The queue considered in this paper provides a Markov chain with block structured transition rates that is neither $M/G/1$ type nor $GI/M/1$ type considered in Neuts [11, 12] but it is a block matrix version of the Markov chain in Chen and Renshaw [3]. We derive simple formulae for the queue length distributions by using the fundamental matrix of the transient quasi-birth-and-death (QBD) process. The Laplace-Stieltjes transforms (LST's) of sojourn time distributions under the combinations of two removal policies, Removal of Customers at the End (RCE) and Removal of Customer in the Head (RCH) and two service disciplines, First-Come-First-Served (FCFS) and Last-Come-First-Served (LCFS) preemptive repeat with resampling are obtained by using the absorption time distribution of a Markov chain with absorbing states.

This paper is organized as follows. We describe the model in detail in Section 2. In Section 3, we investigate the queue length process. The stationary distributions for the queue length process at an arbitrary time and at embedded points are presented in Sections 4 and 5. In Section 6, the LST's of sojourn time distributions are derived.

Throughout the paper, we denote by $\mathbf{1}_n$ and I_n the column n-vector whose components are all 1 and identity matrix of size n, respectively and $\mathbf{e}_{n,k}$ represents the $n \times 1$ vector whose kth component is 1 and others

are all 0. If it is clear in the context, the size n of vectors and identity matrix may be omitted.

2. The model

We consider a single-server queue with three types of arrivals, positive customers, negative customers and disasters. We assume that the service time is a phase type distribution with representation $PH(\boldsymbol{\beta}, S)$ of order ν with $S^0 = -S\mathbf{1} \geq \mathbf{0}$ and $S^0 \neq \mathbf{0}$. We consider two service disciplines, FCFS and LCFS with preemptive repeat with resampling. An arrival of a negative customer causes a positive customer to leave the system. Two removal strategies, RCE and RCH are considered. When a disaster occurs, it removes all the customers in the system and damages the system, which requires repair time of phase type distribution with representation $PH(\boldsymbol{\gamma}, L)$ of order r with $L^0 = -L\mathbf{1} \geq \mathbf{0}$ and $L^0 \neq \mathbf{0}$ for the system to be operated normally again. Even though the system might be empty upon a disaster arrival, the system still needs to be repaired. During the repair time, any types of arrivals are not allowed to enter the system. For an arrival process of positive customers, negative customers and disasters, consider an MMAP with representation $(D_0, D_{-1}, D_{-2}, D_1)$, where D_k, $k = -1, -2, 1$ are nonnegative $m \times m$ matrices and the matrix D_0 of size m has strictly negative diagonal elements and nonnegative off-diagonal elements. The matrices D_{-2}, D_{-1} and D_1 correspond to arrival rates of disasters, negative customers and positive customers, respectively. The first arrival of positive customers after a repair period occurs in batch of size k with rate $\alpha_k D_1$, $k \geq 1$ and $\bar{\alpha} = \sum_{k=1}^{\infty} k\alpha_k < \infty$. We assume that $D = D_0 + D_{-1} + D_{-2} + D_1$ is an irreducible infinitesimal generator with $(D_{-1} + D_{-2})\mathbf{1} \neq \mathbf{0}$ and $D_1\mathbf{1} \neq \mathbf{0}$. Note from the construction that the counting processes for the arrivals of positive customers, negative customers and disasters are MAP's with representations $(D - D_1, D_1)$, $(D - D_{-1}, D_{-1})$ and $(D - D_{-2}, D_{-2})$, respectively. It can be seen from He and Neuts [9] that the covariance between any two counting processes among the three is not zero and hence they are not independent.

3. Queue length process

The evolution of the number of customers in the system, the arrival phase, the repair phase and the service phase are represented by the

continuous time Markov chain. Let $\boldsymbol{k} = \{(k,i,j) : 1 \le i \le m,\ 1 \le j \le \nu\}$, $k = 1,2,\ldots$ with (k,i,j) corresponding to the state that there are k customers in the system and the arrival phase is i and the service phase is j. Because the first arrival of positive customers after a repair period has different features from ordinary ones, we distinguish the states of empty system into the three cases, repair period, the time interval from the end of repair period to the first arrival of positive customers and the period from the first visit to the set of states $\{\boldsymbol{k}, k \ge 1\}$ to the disaster arrival and we respectively denote the states by $\mathbf{0}^* = \{(0^*,i,l) : 1 \le i \le m,\ 1 \le l \le r\}$, $\mathbf{0}' = \{(0',i) : 1 \le i \le m\}$ and $\mathbf{0} = \{(0,i) : 1 \le i \le m\}$. Then the state of the Markov chain is given by $\mathcal{S} = \{\mathbf{0}^*, \mathbf{0}', \mathbf{0}, \mathbf{1}, \ldots\}$. Labelling the states in the lexicographic order, the generator Q of the Markov chain is given by

$$
(3.1) \qquad Q = \begin{array}{c} \\ \mathbf{0}^* \\ \mathbf{0}' \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccccc} \mathbf{0}^* & \mathbf{0}' & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots \end{array} \\
\begin{pmatrix}
B_{0^*} & B & O & O & O & O & \cdots \\
C' & \tilde{A}_1 & O & \alpha_1\tilde{A}_0 & \alpha_2\tilde{A}_0 & \alpha_3\tilde{A}_0 & \cdots \\
C' & O & \tilde{A}_1 & \tilde{A}_0 & & & \\
C & O & \tilde{A}_2 & A_1 & A_0 & & \\
C & O & & A_2 & A_1 & A_0 & \\
C & O & & & A_2 & A_1 & \cdots \\
\vdots & \vdots & & & & \vdots &
\end{pmatrix}
\end{array},
$$

where

$$
\begin{array}{lll}
B_{0^*} = D \oplus L, & B = I_m \otimes L^0, & C' = D_{-2} \otimes \boldsymbol{\gamma}, \\
C = D_{-2} \otimes \mathbf{1}_\nu \cdot \boldsymbol{\gamma}, & \tilde{A}_0 = D_1 \otimes \boldsymbol{\beta}, & \tilde{A}_1 = D_0 + D_{-1}, \\
A_1 = D_0 \oplus S, & \tilde{A}_2 = D_{-1} \otimes \mathbf{1}_\nu + I_m \otimes S^0 &
\end{array}
$$

and A_0 and A_2 depend on the service disciplines and the removal strategies, respectively as follows

$$
(3.2) \qquad A_0 = \begin{cases} D_1 \otimes I_\nu, & \text{FCFS} \\ D_1 \otimes (\mathbf{1}_\nu \cdot \boldsymbol{\beta}), & \text{LCFS}, \end{cases}
$$

$$
(3.3) \qquad A_2 = \begin{cases} D_{-1} \otimes (\mathbf{1}_\nu \cdot \boldsymbol{\beta}) + I_m \otimes (S^0 \cdot \boldsymbol{\beta}), & \text{RCH} \\ D_{-1} \oplus (S^0 \cdot \boldsymbol{\beta}), & \text{RCE}, \end{cases}
$$

where $\otimes$ and $\oplus$ denote the Kronecker product and Kronecker sum [6], respectively.

Since the matrix $A = A_0 + A_1 + A_2$ is not conservative, that is, $A\mathbf{1}_{m\nu} \le 0$ and $A\mathbf{1}_{m\nu} \ne 0$, the Markov chain Q is positive recurrent [16].

4. Stationary distribution

Let $\boldsymbol{\pi}$ be the stationary distribution of Q, that is, $\boldsymbol{\pi}Q = 0$ and $\boldsymbol{\pi}\mathbf{1} = 1$. Write the vector $\boldsymbol{\pi}$ in the block partitioned form $\boldsymbol{\pi} = (\boldsymbol{\pi_{0^*}}, \boldsymbol{\pi_{0'}}, \boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \ldots)$ with

$$\begin{aligned} \boldsymbol{\pi_{0^*}} &= (\pi_{0^*ij}, 1 \le i \le m, 1 \le j \le r),\ \boldsymbol{\pi_{0'}} = (\pi_{0'i}, 1 \le i \le m), \\ \boldsymbol{\pi_0} &= (\pi_{0i}, 1 \le i \le m),\ \boldsymbol{\pi_k} = (\pi_{kij}, 1 \le i \le m, 1 \le j \le \nu),\ k \ge 1. \end{aligned}$$

LEMMA 4.1. *Let*

$$\boldsymbol{\pi_{1^*}} = \boldsymbol{\pi_{0'}} + \boldsymbol{\pi_0} + \sum_{k=1}^{\infty} \sum_{j=1}^{\nu} \tilde{\boldsymbol{\pi}}_{kj},$$

where $\tilde{\boldsymbol{\pi}}_{kj} = (\pi_{kij}, 1 \le i \le m)$ *is an* $1 \times m$ *vector. Then* $(\boldsymbol{\pi_{0^*}}, \boldsymbol{\pi_{1^*}})$ *is the solution of the equation*

$$(\boldsymbol{\pi_{0^*}}\ \boldsymbol{\pi_{1^*}}) \begin{pmatrix} D \oplus L & I_m \otimes L^0 \\ D_{-2} \otimes \boldsymbol{\gamma} & D - D_{-2} \end{pmatrix} = 0 \tag{4.1}$$

with the normalizing condition

$$\boldsymbol{\pi_{0^*}}\mathbf{1}_{mr} + \boldsymbol{\pi_{1^*}}\mathbf{1}_m = 1. \tag{4.2}$$

Proof. Note that the component π_{0^*ij} of $\boldsymbol{\pi_{0^*}}$ denotes the probability that the system is in repair period with the arrival phase of i and repair phase of j in stationary state. Similarly, π_{1^*i} of $\boldsymbol{\pi_{1^*}}$ is the probability of that the system is operating normally with arrival phase of i. The Lemma is proved from the observation that the system is under repair and operating state, alternately and the disaster occurs according an MAP with representation $(D - D_{-2}, D_{-2})$. □

Partitioning the state space into $\tilde{\mathbf{0}} = (\mathbf{0^*}, \mathbf{0'}, \mathbf{0})$ and $\tilde{\mathbf{1}} = (\mathbf{1}, \mathbf{2}, \ldots)$, we rewrite the matrix Q in the following block partitioned form

$$Q = \begin{array}{c} \\ \tilde{\mathbf{0}} \\ \tilde{\mathbf{1}} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{\mathbf{0}} & \tilde{\mathbf{1}} \end{array} \\ \begin{pmatrix} B_{\tilde{0}} & B^* \\ C^* & Q^* \end{pmatrix} \end{array}$$

and let $\boldsymbol{\pi}_{\tilde{0}} = (\boldsymbol{\pi_{0^*}}, \boldsymbol{\pi_{0'}}, \boldsymbol{\pi_0})$, $\boldsymbol{\pi}_{\tilde{1}} = (\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \ldots)$. We have from $\boldsymbol{\pi}Q = 0$ that

$$\boldsymbol{\pi}_{\tilde{0}} B_{\tilde{0}} + \boldsymbol{\pi}_{\tilde{1}} C^* = 0, \quad \boldsymbol{\pi}_{\tilde{0}} B^* + \boldsymbol{\pi}_{\tilde{1}} Q^* = 0$$

and hence

$$\boldsymbol{\pi}_{\tilde{0}}(B_{\tilde{0}} + B^*(-Q^*)^{-1}C^*) = 0, \tag{4.3}$$

$$\boldsymbol{\pi}_{\tilde{1}} = \boldsymbol{\pi}_{\tilde{0}} B^*(-Q^*)^{-1}, \tag{4.4}$$

where $(-Q^*)^{-1}$ is the fundamental matrix of Q^*. Note that $\boldsymbol{\pi}_{\tilde{\mathbf{0}}}$ is a stationary vector of the censored Markov chain of Q with censoring set $\tilde{\mathbf{0}}$.

After correcting typos for $V'(1,k)$ and $V'(2,k)$ in Theorem 10 of Choi et al. [4] and rewriting the results, the block matrix components $X(i,j)$ of $(-Q^*)^{-1}$ are given as follows.

LEMMA 4.2 (Choi et al. [4]). *Let G_1, G_2 and R_1, R_2 be the minimal nonnegative solutions of the following equations*

$$A_2 + A_1G_1 + A_0G_1^2 = 0, \qquad A_2G_2^2 + A_1G_2 + A_0 = 0,$$
$$R_1^2A_2 + R_1A_1 + A_0 = 0, \qquad A_2 + R_2A_1 + R_2^2A_0 = 0.$$

The (i,j) block matrix components $X(i,j)$, $i,j = 1,2,\ldots$ of the fundamental matrix $(-Q^*)^{-1} = (X(i,j))$ of Q^* are given as follows.

(1) *Column blocks:*

$$X(i,1) = G_1^{i-1}[-(A_1 + A_0G_1)]^{-1},\ i \geq 1,$$
$$X(i,j) = \begin{cases} G_1^{i-1}U_j + G_2^{j-i}U_1, & 1 \leq i \leq j, \\ G_1^{i-1}U_j + G_1^{i-j}U_1, & i \geq j+1, \end{cases} \quad j \geq 2,$$

where

$$U_1 = [-(A_2G_2 + A_1 + A_0G_1)]^{-1},$$
$$U_j = (A_1 + A_0G_1)^{-1}A_2G_2^jU_1,\ j \geq 2.$$

(2) *Row blocks:*

$$X(1,j) = [-(A_1 + R_1A_2)]^{-1}R_1^{j-1},\ j \geq 1,$$
$$X(i,j) = \begin{cases} V_iR_1^{j-1} + V_1R_2^{i-j}, & 1 \leq j \leq i, \\ V_iR_1^{j-1} + V_1R_1^{j-i}, & j \geq i+1, \end{cases} \quad i \geq 2,$$

where

$$V_1 = [-(R_2A_0 + A_1 + R_1A_2)]^{-1},$$
$$V_i = (A_1 + R_1A_2)^{-1}R_2^iA_0V_1,\ i \geq 2.$$

PROPOSITION 4.3. *The stationary distribution $\boldsymbol{\pi}$ is given by*

$$\boldsymbol{\pi}_{\mathbf{0}'} = \boldsymbol{\pi}_{\mathbf{0}^*}B(-\tilde{A}_1)^{-1}, \tag{4.5}$$
$$\boldsymbol{\pi}_{\mathbf{0}} = \boldsymbol{\pi}_{\mathbf{0}'}(\tilde{A}_0Y(1)\tilde{A}_2)[-(\tilde{A}_1 + \tilde{A}_0X(1,1)\tilde{A}_2)]^{-1}, \tag{4.6}$$
$$\boldsymbol{\pi}_{\boldsymbol{k}} = \boldsymbol{\pi}_{\mathbf{0}'}\tilde{A}_0Y(k) + \boldsymbol{\pi}_{\mathbf{0}}\tilde{A}_0X(1,k),\ k \geq 1, \tag{4.7}$$

where

$$Y(k) = \sum_{i=1}^{\infty} \alpha_i X(i,k), \ k = 1, 2, \ldots$$

and $\boldsymbol{\pi}_{0^*}$ *is given in lemma 4.1.*

Proof. Straightforward computation yields that

(4.8)

$$B_{\tilde{0}} + B^*(-Q^*)^{-1}C^* = \begin{pmatrix} B_{0^*} & B & O \\ C' + \tilde{A}_0 Y C & \tilde{A}_1 & \tilde{A}_0 Y(1)\tilde{A}_2 \\ C' + \tilde{A}_0 X(1) C & O & \tilde{A}_1 + \tilde{A}_0 X(1,1)\tilde{A}_2 \end{pmatrix},$$

where $Y = \sum_{k=1}^{\infty} Y(k)$ and $X(1) = \sum_{k=1}^{\infty} X(1,k)$. Thus we have from (4.3) and (4.8) that (4.5) and (4.6). Similarly, (4.7) is obtained from (4.4). □

5. Queue length distribution at embedded points

In this section, we derive distributions for the system states at embedded points such as the epoch of a customer arrival, the epoch of a disaster arrival.

5.1. At an arrival epoch of a positive arrival

Let $\boldsymbol{x}_n = (x_{nij}, 1 \le i \le m, 1 \le j \le \nu)$ $(n \ge 1)$ be an $m\nu$-vector whose component x_{nij} is the conditional probability that given a customer is about to arrive at the system, there are n customers in the system immediately before a positive arrival, and the phases of the arrival process and the service distribution are i and j right after the arrival of the customer. Analogous descriptions hold for $\boldsymbol{x}_{0'} = (x_{0'i}, 1 \le i \le m)$ and $\boldsymbol{x}_0 = (x_{0i}, 1 \le i \le m)$.

Note that the probability that when the system is operating, there is a positive arrival with the phase transition of arrival process and the service time from (i,j) to (i',j') in $(t,\ t+\Delta t)$ is $[D_1]_{ii'}\delta_{jj'}\Delta t + o(\Delta t)$, where $\delta_{jj'}$ is 1 for $j = j'$ and 0 otherwise and if under the FCFS discipline and is $[D_1]_{ii'}\beta_{j'}\Delta t + o(\Delta t)$ under the LCFS discipline. Thus when the system is in state $\boldsymbol{n}$, $n \ge 1$, the probability that there is an arrival of a customer with the phase transitions of arrival process and the service time in $(t,\ t+\Delta t)$ is $\boldsymbol{\pi}_n A_0 \Delta t + o(\Delta t)$. It can be easily seen that the probability of a positive arrival in $(t,\ t+\Delta t)$ is $\lambda \Delta t + o(\Delta t)$, where

$$\lambda = (\bar{\alpha}\boldsymbol{\pi}_{0'} + \boldsymbol{\pi}_0) D_1 \mathbf{1}_m + \Pi_1[(D_1 \mathbf{1}_m) \otimes \mathbf{1}_\nu]$$

is the arrival rate of customers and $\Pi_1 = \sum_{k=1}^{\infty} \boldsymbol{\pi_k}$ is given by

$$\begin{aligned} \Pi_1 &= \boldsymbol{\pi}_{0'}\tilde{A}_0 Y + \boldsymbol{\pi}_0 \tilde{A}_0[-(A_1 + R_1 A_2)]^{-1}(I - R_1)^{-1} \\ &= \boldsymbol{\pi}_{0'}\tilde{A}_0 Y + \boldsymbol{\pi}_0 \tilde{A}_0[-(A_0 + A_1 + R_1 A_2)]^{-1}. \end{aligned}$$

Thus the probability vector that the system is in state $\boldsymbol{n}$, $n \geq 1$ at the epoch of a customer arrival is

$$\boldsymbol{x}_n = \lim_{\Delta t \to 0} \frac{\boldsymbol{\pi}_n A_0 \Delta t + o(\Delta t)}{\lambda \Delta t + o(\Delta t)} = \frac{1}{\lambda} \boldsymbol{\pi}_n A_0, \ n \geq 1.$$

Similarly, the probabilities that the customer finds the system is in states $\mathbf{0}'$ and $\mathbf{0}$ upon its arrival are respectively given by

$$\boldsymbol{x}_{0'} = \frac{1}{\lambda} \boldsymbol{\pi}_{0'} \tilde{A}_0, \quad \boldsymbol{x}_0 = \frac{1}{\lambda} \boldsymbol{\pi}_0 \tilde{A}_0.$$

For the state of the system immediately after an arrival of randomly chosen (positive) customers, called tagged customer, we first assume the FCFS discipline. Let $\boldsymbol{y}_{n,k} = (y_{n,k,ij}, 1 \leq i \leq m, 1 \leq j \leq \nu)$ $(n \geq 0, k \geq 0)$ be an $m\nu$-vector whose component $y_{n,k,ij}$ is the conditional probability that given a positive customer is about to arrive at the system, there are n and k customers ahead and behind of the customer, respectively with arrival phase of i and service phase of j immediately after the arrival. Using the similar procedure of $\boldsymbol{x}_n$, $\boldsymbol{y}_{n,k}$, $n \geq 0$, $k \geq 0$ are given by

$$\begin{aligned} \boldsymbol{y}_{0,0} &= \frac{1}{\lambda}\left[\frac{\alpha_1}{\bar{\alpha}} \boldsymbol{\pi_{0'}} + \boldsymbol{\pi_0}\right] \tilde{A}_0, \\ \boldsymbol{y}_{n,0} &= \frac{1}{\lambda}\left[\frac{\alpha_{n+1}}{\bar{\alpha}} \boldsymbol{\pi_{0'}} \tilde{A}_0 + \boldsymbol{\pi_n} A_0\right], \ n \geq 1 \\ \boldsymbol{y}_{n,k} &= \frac{1}{\lambda} \frac{\alpha_{n+k+1}}{\bar{\alpha}} \boldsymbol{\pi_{0'}} \tilde{A}_0, \ n \geq 0, \ k \geq 1. \end{aligned}$$

Let $\boldsymbol{y}_{n,k}(LCFS)$ be the vector corresponding to $\boldsymbol{y}_{n,k}$ under the LCFS discipline. Then it can be seen that

$$\boldsymbol{y}_{0,0}(LCFS) = \boldsymbol{y}_{0,0}, \ \boldsymbol{y}_{n,k}(LCFS) = \boldsymbol{y}_{k,n}, \ n \geq 1, \ k \geq 0$$

and

$$\boldsymbol{y}_{0,k}(LCFS) = \frac{1}{\lambda}\left[\frac{\alpha_{k+1}}{\bar{\alpha}} \boldsymbol{\pi_{0'}} \tilde{A}_0 + \boldsymbol{\pi_k} A_0\right], \ k \geq 1.$$

Note that it seems to $\boldsymbol{y}_{0,k}(LCFS)$ and $\boldsymbol{y}_{k,0}$ have the same formulae, but A_0 depends on the service discipline as given in (3.2).

5.2. Immediately before an arrival of a disaster

Since a disaster can not occur during the repair period, the system is always operating immediately before an occurrence of a disaster. Let $\boldsymbol{z}_n = (z_{nij}, 1 \le i \le m, 1 \le j \le r)$, $n = 0, 1, 2, \ldots$, where z_{nij} is the conditional probability that there are n customers in the system immediately before a disaster arrival and the phases of arrival process and repair time immediately after the arrival of a disaster are i and j, respectively. Following the similar procedure to that of $\boldsymbol{y}_n$, we have that

$$\boldsymbol{z}_n = \begin{cases} \frac{1}{\lambda_d}(\boldsymbol{\pi}_{0'} + \boldsymbol{\pi}_0)C', & n = 0, \\ \frac{1}{\lambda_d}\boldsymbol{\pi}_n C, & n = 1, 2, \ldots, \end{cases}$$

where the arrival rate λ_d of disaster is

$$\lambda_d = (\boldsymbol{\pi}_{0'} + \boldsymbol{\pi}_0)D_{-2}\mathbf{1}_m + \Pi_1[(D_{-2}\mathbf{1}_m) \otimes \mathbf{1}_\nu].$$

6. Sojourn time distributions

Let W denote the time period during which a customer spends in the system from the epoch of arrival to the epoch of his service completion. We assume that W is infinite if the customer is removed from the system before its service completion. We call the monitored customer whose sojourn time distribution is sought for by us the tagged customer. Like all positive customers the tagged customer obeys the specified service discipline and removal strategy after its arrival. We denote by C_n the customer who finds n customers in the system on its arrival. In this section, we derive the LST $W^*(s)$ of the distribution function $W(x) = P(W \le x)$ under combinations of service disciplines FCFS and LCFS with preemptive restart with resampling and removal strategies RCH and RCE. Let $\xi_a(t)$ and $\xi_b(t)$ be the number of customers ahead and behind of the tagged customer at time t, respectively. Let $J(t)$ and $J_s(t)$ be the phases of arrival process and the service time, respectively at time t. By T_d, T_n and T_s we denote the first time that the customer's removal/departure from the system by an arrival of a disaster, a negative customer and a service completion, respectively and let $T = \min(T_d, T_n, T_s)$. Define for $n \ge 0$, $k \ge 0$, $1 \le i \le m$, $1 \le j \le \nu$,

$$W_{n,k,ij}(x) = P(T_s \le x \,|\, \xi_a(0) = n, \xi_b(0) = k, J(0) = i, J_s(0) = j)$$

and let $W_{n,k}(x) = (W_{n,k,ij}(x), 1 \le i \le m, 1 \le j \le \nu)$ be the column $m\nu$-vector which is obtained by listing the $W_{n,k,ij}(x)$ in the lexicographic

order, that is,

$$W_{n,k}(x) = (W_{n,k,11}(x), \dots, W_{n,k,1\nu}(x), \dots, W_{n,k,m1}(x), \dots, W_{n,k,m\nu}(x))^t,$$

where $\boldsymbol{x}^t$ is the transpose of the vector $\boldsymbol{x}$. Then $W^*(s)$ is given by the form

$$W^*(s) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \tilde{\boldsymbol{y}}_{n,k} W^*_{n,k}(s), \tag{6.1}$$

where $\tilde{\boldsymbol{y}}_{n,k}$ is either $\boldsymbol{y}_{n,k}$ or $\boldsymbol{y}_{n,k}(LCFS)$ depending on the service discipline and $W^*_{n,k}(s)$ is the LST of $W_{n,k}(x)$. We shall derive the $W^*_{n,k}(s)$ in terms of the absorption time of a Markov chain reflecting the combinations of removal strategies and service disciplines. For the waiting time distribution, we need the following lemma.

LEMMA 6.1. *Consider a Markov chain* $\Xi_0 = \{\Xi_0(t), t \geq 0\}$ *on the state space* $\mathbf{0}_* \cup \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$, *where* $\mathbf{0}_* = \{d, n, s\}$ *and* $\boldsymbol{k} = \{(k,1), \dots, (k,p)\}$, $k \geq 0$ *whose infinitesimal generator* Q_0 *is of the form*

$$Q_0 = \begin{array}{c} \\ \mathbf{0}_* \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} d & n & s & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots \end{array} \\ \begin{pmatrix} O & O & O & O & O & O & O & \cdots \\ \boldsymbol{c}_0 & \boldsymbol{b}_0 & \boldsymbol{a}_0 & A_1^* & A_0^* & & & \\ \boldsymbol{c}_1 & \boldsymbol{b}_1 & \boldsymbol{a}_1 & A_2^* & A_1^* & A_0^* & & \\ \boldsymbol{c}_2 & \boldsymbol{b}_2 & \boldsymbol{a}_2 & & A_2^* & A_1^* & A_0^* & \\ \boldsymbol{c}_3 & \boldsymbol{b}_3 & \boldsymbol{a}_3 & & & A_2^* & A_1^* & \cdots \\ \vdots & \vdots & \vdots & & & & \vdots & \end{pmatrix} \end{array}, \tag{6.2}$$

where A_k^*, $k = 0, 1, 2$ *are* $p \times p$ *matrices and* $\boldsymbol{a}_k$, $\boldsymbol{b}_k$, $\boldsymbol{c}_k$ *are column* p*-vectors with* $\boldsymbol{a}_k + \boldsymbol{b}_k + \boldsymbol{c}_k \neq 0$, $k = 0, 1, 2, \dots$. *Let* T_0 *be the first time that the Markov chain* Ξ_0 *absorbs in* $\mathbf{0}_*$ *and*

$$H_{kj}(x) = P(T_0 \leq x, \Xi_0(T_0) = s \,|\, \Xi(0) = (k,j)), \ k \geq 0, \ 1 \leq j \leq p.$$

Then the LST $H_k^*(s)$ *of the column* p*-vector* $H_k(x) = (H_{kj}(x), j = 1, 2, \dots, p)^t$ *is given as follows:*

If $\boldsymbol{a}_k \neq 0$ *for some* $k \geq 1$, *then* $H_k^*(s)$ *is given by*

$$H_k^*(s) = \sum_{j=0}^{\infty} X_{kj}^*(s) \boldsymbol{a}_j, \ k = 0, 1, 2, \dots, \tag{6.3}$$

where $X_{kj}^*(s)$, $j = 0, 1, 2, \dots$ *are the* k*th row blocks of* $(sI - Q_0^*)^{-1}$ *which can be obtained from Lemma 4.2 by replacing* A_0, A_1 *and* A_2 *with* A_0^*,

$A_1^* - sI$ *and* A_2^*, *respectively and*

$$Q_0^* = \begin{pmatrix} A_1^* & A_0^* & & \\ A_2^* & A_1^* & A_0^* & \\ & A_2^* & A_1^* & \cdots \\ & & \vdots & \end{pmatrix}.$$

In particular, if $\boldsymbol{a}_0 \neq 0$ *and* $\boldsymbol{a}_k = 0$, $k = 1, 2, \ldots$, *then* $H_k^*(s)$ *is given by*

$$H_k^*(s) = [G^*(s)]^k (sI - A_1^* - A_0^* G^*(s))^{-1} \boldsymbol{a}_0, \quad k = 0, 1, 2, \ldots, \tag{6.4}$$

where $G^*(s)$ *is the minimal nonnegative solution of the matrix equation*

$$A_2^* - (sI - A_1^*) G^*(s) + A_0^* (G^*(s))^2 = 0. \tag{6.5}$$

Proof. By using the first step argument, we have that for $k = 0, 1, 2, \ldots$

$$H_k^*(s) = (sI - A_1^*)^{-1} (\boldsymbol{a}_k + A_2^* H_{k-1}^*(s) + A_0^* H_{k+1}^*(s)) \tag{6.6}$$

with $H_{-1}^*(s) = 0$. Writing (6.6) in matrix form, we have that

$$(sI - Q_0^*) H^*(s) = \boldsymbol{a},$$

where

$$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \end{bmatrix}, \quad H^*(s) = \begin{bmatrix} H_0^*(s) \\ H_1^*(s) \\ H_2^*(s) \\ \vdots \end{bmatrix}$$

and hence (6.3) is obtained.

Assume that $\boldsymbol{a}_0 \neq 0$ and $\boldsymbol{a}_k = 0$, $k = 1, 2, \ldots$. Let $G^*(s)$ be the LST of the first passage time to the level $\mathbf{0}$ of the Markov chain Ξ_0 starting the level $\mathbf{1}$. Following the standard arguments in Neuts [12], it can be seen that $G^*(s)$ is the minimal nonnegative solution of the equation

$$G^*(s) = (sI - A_1^*)^{-1} [A_2^* + A_0^* (G^*(s))^2],$$

which is equivalent to (6.5) and

$$H_k^*(s) = [G^*(s)]^k H_0^*(s), \quad k = 1, 2, \ldots. \tag{6.7}$$

The formula (6.4) is obtained by substituting (6.7) into (6.6). □

6.1. RCH with FCFS discipline

Under the RCH removal strategy, upon a negative arrival, a positive customer in service is removed and hence the customers behind the tagged customer do not affect the sojourn time of the tagged customer and hence we can write $W_n(x) = W_{n,k}(x)$ for all $k \geq 0$. Thus $W^*(s)$ is given by

$$W^*(s) = \sum_{n=0}^{\infty} \boldsymbol{y}_n W_n^*(s), \ s \geq 0,$$

where

$$\boldsymbol{y}_n = \sum_{k=0}^{\infty} \boldsymbol{y}_{n,k} = \begin{cases} \frac{1}{\lambda}\left(\frac{1}{\bar{\alpha}}\boldsymbol{\pi}_{0'} + \boldsymbol{\pi}_0\right)\tilde{A}_0, & n = 0, \\ \frac{1}{\lambda}\left(\frac{\bar{\alpha}_{n+1}}{\bar{\alpha}}\boldsymbol{\pi}_{0'}\tilde{A}_0 + \boldsymbol{\pi}_n A_0\right), & n \geq 1, \end{cases}$$

and $\bar{\alpha}_j = \sum_{i=j}^{\infty} \alpha_i$, $j \geq 1$.

Now we derive $W_n^*(s)$. Consider a Markov chain $\Xi_1 = \{\Xi_1(t), t > 0\}$ on the state space $\mathcal{S} = \mathbf{0}_* \cup \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$, where $\mathbf{0}_* = \{d, n, s\}$ and $\boldsymbol{k} = \{(k, i, j),\ 1 \leq i \leq m, 1 \leq j \leq \nu\}$, $k \geq 0$ with $\Xi_1(t) = (\xi_a(t), J(t), J_s(t))$ for $0 \leq t < T = \min(T_d, T_n, T_s)$ and $\Xi_1(t) \in \mathbf{0}_*$ for $t \geq T$. Labelling the states in the lexicographic order, the generator $Q_{RCH-FCFS}$ of Ξ_1 is of the form (6.2) with $\boldsymbol{a}_k = \boldsymbol{b}_k = 0$ for $k \geq 1$ and

$$\begin{aligned} \boldsymbol{a}_0 &= \mathbf{1}_m \otimes S^0, \\ \boldsymbol{b}_0 &= (D_{-1}\mathbf{1}_m) \otimes \mathbf{1}_\nu, \\ \boldsymbol{c}_k &= (D_{-2}\mathbf{1}_m) \otimes \mathbf{1}_\nu, k \geq 0 \end{aligned}$$

and

$$A_0^* = O, \quad A_1^* = A_1 + A_0, \quad A_2^* = A_2.$$

Note that the sojourn time distribution of the customer C_n is the same as that of the absorption time to the state s of the Markov chain Ξ_1. It follows from Lemma 6.1 that

$$W_n^*(s) = [G^*(s)]^n (sI - A_1 - A_0)^{-1}(\mathbf{1}_m \otimes S^0), \ n = 0, 1, 2, \ldots \tag{6.8}$$

with $G^*(s) = (sI - A_1 - A_0 G^*(s))^{-1} A_2$.

6.2. RCH with LCFS discipline

Using the similar argument for the RCH-FCFS discipline, we see that under the RCH-LCFS discipline, $W_n(x) = W_{n,k}(x)$ and $W^*(s)$ is given by

$$W^*(s) = \sum_{n=0}^{\infty} \boldsymbol{y}_n(LCFS) W_n^*(s), \ s \geq 0,$$

where

$$\boldsymbol{y}_n(LCFS) = \sum_{k=0}^{\infty} \boldsymbol{y}_{n,k}(LCFS) = \begin{cases} \frac{1}{\lambda}\left(\frac{1}{\tilde{\alpha}}\boldsymbol{\pi}_{0'} + \boldsymbol{\pi}_0\right)\tilde{A}_0 + \frac{1}{\lambda}\Pi_1 A_0, & n = 0, \\ \frac{1}{\lambda}\frac{\tilde{\alpha}_{n+1}}{\tilde{\alpha}}\boldsymbol{\pi}_{0'}\tilde{A}_0, & n \geq 1. \end{cases}$$

Under the RCH-LCFS discipline, the generator $Q_{RCH-LCFS}$ of the Markov chain Ξ_1 is given of the form (6.2) with

$$A_0^* = A_0, \quad A_1^* = A_1, \quad A_2^* = A_2$$

and $\boldsymbol{a}_k$, $\boldsymbol{b}_k$ and $\boldsymbol{c}_k$ are the same as those of $Q_{RCH-FCFS}$. Noting that the sojourn time distribution is the same that the distribution of the absorption time to the state s of the Markov chain Ξ_1, we have from Lemma 6.1 that

$$(6.9)\quad W_n^*(s) = [G^*(s)]^n(sI - A_1 - A_0G^*(s))^{-1}(\mathbf{1}_m \otimes S^0),\ n = 0, 1, 2, \ldots,$$

where $G^*(s)$ is the minimal nonnegative solution of the equation

$$A_2 - (sI - A_1)G^*(s) + A_0(G^*(s))^2 = 0.$$

6.3. RCE with FCFS discipline

For a fixed $n \geq 0$, consider a Markov chain $\Xi_2 = \{\Xi_2(t), t > 0\}$ on the state space $\mathcal{S} = \mathbf{0}_* \cup \{\boldsymbol{k}_n^*,\ k = 0, 1, 2, \ldots\}$, where $\boldsymbol{k}_n^* = \{(k, l, i, j), 0 \leq l \leq n, 1 \leq i \leq m, 1 \leq j \leq \nu\}$, $k \geq 0$ with $\Xi_2(t) = (\xi_b(t), \xi_a(t), J(t), J_s(t))$ for $0 \leq t < T$ and $\Xi_2(t) \in \mathbf{0}_*$ for $t \geq T$. Under the RCE-FCFS discipline, the generator $Q_{RCE-FCFS}$ of Ξ_2 can be written by the form (6.2) with

$$\begin{aligned} \boldsymbol{a}_k &= \mathbf{e}_{n+1,1} \otimes (\mathbf{1}_m \otimes S^0), k \geq 0, \\ \boldsymbol{b}_k &= \begin{cases} \mathbf{1}_{n+1} \otimes (D_{-1}\mathbf{1}_m) \otimes \mathbf{1}_\nu, & k = 0, \\ 0, & k \geq 1, \end{cases} \\ \boldsymbol{c}_k &= \mathbf{1}_{n+1} \otimes (D_{-2}\mathbf{1}_m) \otimes \mathbf{1}_\nu, k \geq 0 \end{aligned}$$

and

$$\begin{aligned} A_0^* &= I_{n+1} \otimes (D_1 \otimes I_\nu), \\ A_1^* &= I_{n+1} \otimes (D_0 \oplus S) + J_{n+1} \otimes [I_m \otimes (S^0 \otimes \boldsymbol{\beta})], \\ A_2^* &= I_{n+1} \otimes (D_{-1} \otimes I_\nu), \end{aligned}$$

where J_{n+1} is the $(n+1) \times (n+1)$ matrix of the form

$$J_{n+1} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

We denote $[H_{lk}(x|n)]_{ij}$ by

$$[H_{lk}(x|n)]_{ij} = P(T \le x, \Xi_2(T) = s|\Xi_2(0) = (k,l,i,j)),$$

where $k \ge 0, 0 \le l \le n$ and $1 \le i \le m, 1 \le j \le \nu$ and define the column $(n+1)m\nu$-vector $H_k(x|n)$ and its LST $H_k^*(s|n)$ by

$$H_k(x|n) = \begin{pmatrix} H_{0k}(x|n) \\ H_{1k}(x|n) \\ \vdots \\ H_{nk}(x|n) \end{pmatrix}, \qquad H_k^*(s|n) = \begin{pmatrix} H_{0k}^*(s|n) \\ H_{1k}^*(s|n) \\ \vdots \\ H_{nk}^*(s|n) \end{pmatrix},$$

where $H_{lk}(x|x)$ is the column $m\nu$-vector which is obtained by listing $[H_{lk}(x|x)]_{ij}$, $1 \le i \le m, 1 \le j \le \nu$ in the lexicographic order. It follows from Lemma 6.1 that

$$H_k^*(s|n) = \left[\sum_{j=0}^{\infty} X_{kj}^*(s)\right] [\mathbf{e}_{n+1,1} \otimes (\mathbf{1}_m \otimes S^0)], \tag{6.10}$$

where $X_{kj}^*(s)$, $j = 0,1,2,\ldots$ are the kth row block of $(sI - Q_0^*)^{-1}$. It follows from the construction of the Markov chain Ξ_2 that the LST of the conditional distribution of the sojourn time of the customer given that $\xi_a(0) = n$, $\xi_b(0) = k$, and $J(0) = i$, $J_s(0) = j$ is the (i,j) component of $H_{nk}^*(s|n)$, that is,

$$W_{nk}^*(s) = H_{nk}^*(s|n).$$

6.4. RCE with LCFS discipline

Consider a Markov chain $\Xi_3 = \{\Xi_3(t), t > 0\}$ on the state space $\mathcal{S} = \mathbf{0}_* \cup \{\boldsymbol{k}_n^*,\ k = 0,1,2,\ldots\}$, with $\Xi_3(t) = (\xi_a(t), \xi_b(t), J(t), J_s(t))$ for $0 \le t < T$ and $\Xi_3(t) \in \mathbf{0}_*$ for $t \ge T$. Under the RCE-LCFS discipline, the generator $Q_{RCE-LCFS}$ of Ξ_3 can be written by the form (6.2) with

$$\begin{aligned} \boldsymbol{a}_k &= \begin{cases} \mathbf{1}_{n+1} \otimes (\mathbf{1}_m \otimes S^0), & k = 0, \\ 0, & k \ge 1, \end{cases} \\ \boldsymbol{b}_k &= \mathbf{e}_{n+1,1} \otimes [(D_{-1}\mathbf{1}_m) \otimes \mathbf{1}_\nu], k \ge 0, \\ \boldsymbol{c}_k &= \mathbf{1}_{n+1} \otimes (D_{-2}\mathbf{1}_m) \otimes \mathbf{1}_\nu, k \ge 0 \end{aligned}$$

and

$$\begin{aligned} A_0^* &= I_{n+1} \otimes (D_1 \otimes (\mathbf{1}_m \cdot \boldsymbol{\beta})), \\ A_1^* &= I_{n+1} \otimes (D_0 \oplus S) + J_{n+1} \otimes [I_m \otimes (D_{-1} \otimes I_\nu)], \\ A_2^* &= I_{n+1} \otimes (I_m \otimes (S^0 \cdot \boldsymbol{\beta})). \end{aligned}$$

Similar to the case of RCH-LCFS, we define $[K_{lk}(x|n)]_{ij}$ by

$$[K_{lk}(x|n)]_{ij} = P(T \le x, \Xi_3(T) = s \,|\, \Xi_3(0) = (l,k,i,j)),$$

where $l \ge 0$, $0 \le k \le n$, $1 \le i \le m$, $1 \le j \le \nu$, and denote the column $(n+1)m\nu$-vector by $K_l(x|n)$ whose kth column block of size $m\nu$ is $K_{lk}(x|n)$ and its LST $K_l^*(s|n)$. It follows from Lemma 6.1 that $K_{lk}^*(s|n)$ is given by

$$\begin{aligned} &K_l^*(s|n) \\ &= [G_n^*(s)]^l (sI - A_1^* - A_0^* G_n^*(s))^{-1} [\mathbf{1}_{n+1} \otimes (\mathbf{1}_m \otimes S^0)], \ l = 0,1,2,\ldots, \end{aligned}$$

where $G_n^*(s)$ is the minimal nonnegative solution of the matrix equation

$$A_2^* - (sI - A_1^*)G_n^*(s) + A_0^*(G_n^*(s))^2 = 0.$$

The LST of the conditional distribution of the sojourn time of the customer given that $\xi_a(0) = n$, $\xi_b(0) = k$, and $J(0) = i$, $J_s(0) = j$ is $[K_{nk}^*(s|n)]_{ij}$, that is,

$$W_{nk}^*(s) = K_{nk}^*(s|n).$$

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