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BALANCEDNESS AND CONCAVITY OF FRACTIONAL DOMINATION GAMES

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Abstract. In this paper, we introduce a fractional domination game arising from fractional domination problems on graphs and focus on its balancedness and concavity. We first characterize the core of the fractional domination game and show that its core is always non-empty taking use of dual theory of linear programming. Furthermore we study concavity of this game.

1. Introduction

In this paper we investigate a kind of cooperative cost game that arises from fractional domination problems on graphs. Domination problems are widely studied in graph theory. A comprehensive overview on domination problem is provide by [5], [6].

Given a graph $G = (V, E; \omega)$ with vertex set V, edge set E and weight function $\omega : V \to R_+$, a function $f : V \to [0, 1]$ is called a dominating function of G if for each vertex $v \in V$, $\sum_{u \in N[v]} f(u) \geq 1$, where $N[v] = \{v\} \cup \{u \in V : (u, v) \in E\}$ is the closed neighborhood of v in graph G. The fractional domination problem is to find a dominating function f which minimizes the total weight $\sum_{v \in V} f(v)\omega(v)$.

The fractional domination game problem has many practical applications. For example, let $G = (V, E)$ be a graph in which vertices represent cities and edges represent pairs of cities they are neighbors. Suppose each city has a service station, such as blood bank and gas station, which may be used to store certain amount of resources. Suppose those stations in different cities have different cost for the storage of each unit of resources. Then the problem is to decide the storage amount of

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resources which is placed with minimum total cost in each station, such that each city can be supported or rescued by sufficient amount of resources from the stations in its own city and its neighboring cities when necessary. This example can be formulated as a fractional domination set problem on the connection graph G among the cities.

On the other hand, a natural question arising from the above example is how to allocate the total cost of the resource storage among all the participating cities. In this paper, we introduce a closely related cooperative cost game, the fractional domination game, to model the cost allocation problem and focus on the balancedness and concavity of this game model.

Various fairness and rationality requirements proposed for the allocations of total cost derive many solution concepts in cooperative game theory. Among all the solution concepts, the core is the most important one which has been extensively studied in many game models. The main technique used in this work is linear program duality characterization of the core of our fractional domination game. This technique has offered much for cooperative games. Shapley and Shubik [10] formulated a two-sided market as the assignment game, and showed that the core is exactly the set of optimal solutions of a linear program dual to the optimal assignment problem. This approach is further exploited in the study of linear production game [2, 7], partition game [4], packing and covering games [3], and recently dominating set games [11]. Velzen [11] introduced three kinds of cooperative games that arise from the weighted minimum dominating set problem on graphs. It was shown that the core of each game is non-empty if and only if the corresponding linear program relaxation of the weighted minimum dominating set problem has an integer optimal solution, and in this case, an element in the core can be found in polynomial time.

This paper is organized as follows. In section 2, we give some notions from cooperative game theory and introduce a cooperative game that models the cost allocation problem arising from fractional domination problems on graphs. In section 3, we present a characterization of the core elements of the fractional domination game and show that the core equals the set of the optimal solutions to the dual linear program of fractional domination problem. It follows that finding a core element can be carried out in polynomial time. In the final section, we study the concavity of the fractional domination game.

2. Definition of fractional domination game

In this section, we introduce a kind of cooperative cost games that models the cost allocation problem arising from fractional domination problems on graphs. We begin with some concepts and notions in cooperative game theory.

2.1. Cooperative game theory

A cooperative game (in characteristic function form) $\Gamma = (N, c)$ consists of a player set N and a characteristic function $c : 2^N \rightarrow R$ with $c(\emptyset) = 0$. For each coalition $S \subseteq N$, $c(S)$ represents the revenue or cost achieved by the players in S together. The main issue is how to fairly distribute the total revenue or cost $c(N)$ among all the players. We present the definition here only for cost games, with the understanding that symmetric statement also holds for revenue games.

A vector $z = \{z_1, z_2, \ldots, z_n\}$ is called an *imputation* if and only if $\sum_{i \in N} z_i = c(N)$ and $z_i \leq c(\{i\})$. The *core* of a game $\Gamma = (N, c)$ is defined by

$$\text{Core}(\Gamma) = \{z \in R^n : z(N) = c(N) \text{ and } z(S) \leq c(S), \ \forall S \subseteq N\},$$

where $z(S) = \sum_{i \in S} z_i$ for $S \subseteq N$. The constraints imposed on Core(Γ), which is called group rationality, ensure that no coalition would have an incentive to split from the grand coalition N, and do better on its own.

The study of the core is closely associated with another important concept, the balanced set. The collection $\mathcal{B}$ of subsets of N is balanced if there exists a set of positive numbers β_S ($S \in \mathcal{B}$), such that for each $i \in N$, we have $\sum_{i \in S \in \mathcal{B}} \beta_S = 1$. A game (N, c) is called *balanced* if $\sum_{S \in \mathcal{B}} \beta_S c(S) \leq c(N)$ holds for every balanced collection $\mathcal{B}$ with weights $\{\beta_S : S \in \mathcal{B}\}$. With techniques essentially the same as linear programming duality, Bondareva [1] and Shapley [9] proved that a game has non-empty core if and only if it is balanced.

A cooperative game $\Gamma = (N, c)$ is called concave if it holds that

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T) \quad \text{for all } S, T \in 2^N.$$

It is easy to check that $\Gamma = (N, c)$ is concave if and only if for all $i, j \in N$ with $i \neq j$ and $S \subset N \setminus \{i, j\}$, it hold that

$$c(S \cup \{i\}) - c(S) \geq c(S \cup \{i, j\}) - c(S \cup \{j\}).$$

That is to say, for a concave game the marginal contribution of a player to any coalition is at most his marginal contribution to a smaller coalition. Shaply [8] showed that the core of a concave game is always nonempty.

2.2. Fractional domination games

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Two distinct vertices $u, v \in V$ are called adjacent if $(u, v) \in E$. For any non-empty set $V' \subseteq V$, the induced subgraph by V', denoted by $G[V']$, is a subgraph of G whose vertex set is V' and whose edge set is the set of edges having both endpoints in V'. The closed neighborhood of vertex $v \in V$ is $N[v] = \{u \in V : (u, v) \in E\} \cup \{v\}$. For any subset $S \subseteq V$, we define the closed neighboring set of S to be the union of the closed neighborhoods of all vertices in S, denoted by $N[S] = \bigcup_{v \in S} N[v]$.

Given a graph $G = (V, E; \omega)$ with vertex weight function $\omega : V \to R_+$, a function $f : V \to [0, 1]$ is a fractional dominating function of G if for every vertex $v \in V$, $\sum_{u \in N[v]} f(u) \geq 1$. Thus, if S is a dominating set of graph G and we define the function f where $f(v) = 1$ if $v \in S$ and $f(v) = 0$ if $v \notin S$, then f is a dominating function of G. In the rest of this paper, for convenience, we denote $\sum_{u \in S} f(u)$ and $f(i)$ by $f(S)$ and f_i, respectively.

The fractional domination problem is to find a fractional dominating function f which minimizes the total weight $\sum_{v \in V} f(v)\omega(v)$. The minimum weighted domination number $\gamma^*(G)$ is defined as the minimum value $\sum_{v \in V} f(v)\omega(v)$ among all fractional dominating function f of graph G. Now, we introduce a cooperative game that models the cost allocation problem arising from fractional domination function problem. Given a weighted graph $G = (V, E; \omega)$ with vertex weight function $\omega : V \to R_+$, the *fractional domination game* $\Gamma = (N, c)$ corresponding to graph G is defined as follows:

1. The player set is $N = V = \{1, 2, \ldots, n\}$;
2. For each coalition $S \subseteq N$,

$$c(S) = \min\{ \sum_{i \in S} f(i)\omega(i) \mid f : V \to [0, 1] \text{ and } \sum_{i \in N[j] \cap S} f(i) \geq 1, \ \forall j \in S\}.$$

That is, the cost $c(S)$ incurred by the coalition S is the minimum weighted domination number of the induced subgraph $G[S]$. Our purpose is to discuss the allocations of the total cost $c(N)$ among the participating players based on the core of this game model.

3. Balancedness of fractional domination game

In this section we first present a characterization of the core for the fractional domination game. Furthermore, we show that the core of this game is always non-empty. To this purpose, we first introduce two kinds of vertex subsets. Let $G = (V, E)$ be a graph. The closed neighborhood of a vertex $v \in V$ in G is called *v-star*. If $T \subseteq N[v]$ contains v, then T is called a *v-substar*. The set of all v-substars in G is denoted by $\mathcal{T}_v$, i.e., $\mathcal{T}_v = \{T \subseteq V \mid T \text{ is a } v\text{-substar}\}$.

For a dominating set D of graph $G = (V, E)$, it is easy to see that the vertex set V can be covered by disjoint v-substars with $v \in D$, i.e., $V = \bigcup_{v \in D} T_v$, where $T_v \in \mathcal{T}_v$ and $T_u \cap T_v = \emptyset$ if $u \neq v$ in D. In the following lemma, we extend this intuitive result to a dominating function f of graph G. We will show that the vertex set V can be "exactly" covered by a collection of substars induced by the function f. In fact, the proof of the following lemma gives a method to construct such a collection of substars.

LEMMA 3.1. *Let $f : V \to [0,1]$ be a fractional dominating function of graph $G = (V, E)$. Then for each $v \in V$, there exists a set of v-substars induced by f, denoted by $\mathcal{T}_v^f$, and a real valued weight function $\ell_v : \mathcal{T}_v^f \to R_+$ such that*

(i) $\sum_{T \in \mathcal{T}_v^f} \ell_v(T) = f(v)$, *and*

(ii) *for every vertex $u \in V$, the total weight of substars containing vertex u in the collection $\mathcal{T}^f = \{\mathcal{T}_v^f : v \in V\}$ equals 1, i.e.,* $\sum_{v \in V} \sum_{u \in T \in \mathcal{T}_v^f} \ell_v(T) = 1.$

Proof. Since $f : V \to [0,1]$ is a fractional dominating function of graph G, for each vertex $v \in V$ we have $\sum_{u \in N[v]} f(u) \geq 1$. Hence for each vertex $v \in V$, we define the contribution value $\eta^v(u)$ of each vertex u in G being used to ensure vertex v to be "exactly" dominated. That is, the contribution of all the vertices in V to vertex v is a set of nonnegative values $\{\eta^v(u) : u \in V\}$ such that:

$$\begin{cases} \eta^v(v) = f(v) & \\ 0 \leq \eta^v(u) \leq f(u) & \text{for } u \in N[v] \setminus \{v\} \\ \eta^v(u) = 0 & \text{for } u \notin N[v] \end{cases} \tag{3.1}$$

$$\sum_{u\in N[v]} \eta^v(u) = 1. \tag{3.2}$$

Obviously, the set of values $\{\eta^v(u) : u \in V\}$ satisfying formulas (3.1) and (3.2) exists.

For each vertex $v \in V$, we count on the positive values occurring in the set $\{\eta^u(v) : u \in V\}$ which is the contributions of vertex v to be used to ensure all vertices $u \in V$ to be "exactly" dominated, and arrange these values in the increasing order, say, $0 < \eta_1^v < \eta_2^v < \cdots < \eta_s^v = f(v)$. Then we construct a set of v-substars $\mathcal{T}_v^f$ which contains s v-substars and the corresponding weight function $\ell_v : \mathcal{T}_v^f \to R_+$ as follows:

$$\begin{cases} T_{v1} = \{u \in N[v] : \eta^u(v) \geq \eta_1^v\} & \ell_v(T_{v1}) = \eta_1^v \\ T_{v2} = \{u \in N[v] : \eta^u(v) \geq \eta_2^v\} & \ell_v(T_{v2}) = \eta_2^v - \eta_1^v \\ \cdots\cdots & \\ T_{vs} = \{u \in N[v] : \eta^u(v) \geq \eta_s^v\} & \ell_v(T_{vs}) = \eta_s^v - \eta_{s-1}^v. \end{cases}$$

Obviously, the total weight of v-substars in $\mathcal{T}_v^f$ is exactly $f(v)$, and for each vertex $u \in V$ the total weight of v-substars containing vertex u in $\mathcal{T}_v^f$ is $\eta^u(v)$. We put all the set of $\mathcal{T}_v^f$ $(v \in V)$ together, and denote $\mathcal{T}^f = \{\mathcal{T}_v^f : v \in V\}$. Since the contribution value $\eta^u(v)$ $(v \in V)$ satisfies the formula (3.2), we have that $\sum_{v\in V} \sum_{u\in T\in \mathcal{T}_v^f} \ell_v(T) = 1.$ □

Now we provide an efficient description of the core of the fractional domination game in terms of coalitions corresponding to v-stars and v-substars.

THEOREM 3.2. *Let $G = (V, E; \omega)$ be a graph ($|V| = n$) with vertex weight function $\omega : V \to R_+$, and $\Gamma = (N, c)$ be the corresponding fractional domination game. It holds that $x = (x_1, x_2, \ldots, x_n) \in$ Core(Γ) if and only if*

(1) $x(N) = c(N)$;

(2) *for each $j \in N$ and $T_j \in \mathcal{T}_j$ (j-substar set), $x(T_j) \leq \omega_j$.*

Proof. Suppose that $x \in$ Core(Γ). By the definition of the core, we have $x(N) = c(N)$. For each $j \in N$, and each subset j-substar $T_j \in \mathcal{T}_j$, we define the function $f : T_j \to [0, 1]$ such that $f_j = 1$ and $f_i = 0$ for each $i \neq j$. Then f is a dominating function of the induced graph $G[T_j]$, it implies that $x(T_j) \leq c(T_j) \leq \omega_j$.

Now we prove its sufficiency. Let $S \subseteq N$ be an arbitrary coalition, and $f^* : S \to [0, 1]$ be a minimum weight dominating function in the induced graph $G[S]$, that is, $\sum_{j\in S} f_j^* \omega_j = c(S)$.

Followed from Lemma 3.1, for each vertex $j \in S$, there exist a set $\mathcal{T}_j^{f^*}$ of j-substars and a weight function $\ell_j : \mathcal{T}_j^{f^*} \to R_+$ such that $\sum_{T\in\mathcal{T}_j^{f^*}} \ell_j(T) = f_j^*$, and the total weight of j-substars containing vertex $i \in S$ in $\mathcal{T}^{f^*} = \{\mathcal{T}_j^{f^*} : j \in S\}$ is exactly 1. Therefore

$$x(S) = \sum_{j\in S} \sum_{T\in\mathcal{T}_j^{f^*}} \ell_j(T)x(T) \le \sum_{j\in S} \sum_{T\in\mathcal{T}_j^{f^*}} \ell_j(T)\omega_j = \sum_{j\in S} f_j^*\omega_j = c(S).$$

That is, $x \in \mathrm{Core}(\Gamma)$. □

Let $G = (V, E; \omega)$ be a weighted graph with vertex set $V = \{1, 2, \ldots, n\}$. Let $A = (a_{ij})_{n\times n}$ be the closed neighborhood matrix of G, where $a_{ij} = 1$ if vertex i is in the neighborhood $N[j]$, and $a_{ij} = 0$, otherwise. We describe the problem of minimum weight fractional dominating function using the following linear program:

$$\text{(3.3)} \qquad \text{(LP)}: \quad \begin{array}{rl} \gamma^*(G) = \min & \sum_{j=1}^{n} \omega_j x_j \\ s.t. & \begin{cases} Ax \ge 1 \\ x = (x_1, x_2, \ldots, x_n)^t \ge 0 \end{cases} \end{array}$$

We remark that in the linear program (LP), we omit the constraint $x \le 1$ since it is redundant under minimizing the objective function. Consider the dual linear program of (LP):

$$\text{(3.4)} \qquad \text{(DLP)}: \quad \begin{array}{rl} \max & \sum_{i=1}^{n} y_i \\ s.t. & \begin{cases} yA \le \omega \\ y = (y_1, y_2, \ldots, y_n) \ge 0 \end{cases} \end{array}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. In the following, we show that the core of the fractional domination game corresponding to the graph G is the same as the set of the optimal solutions of (DLP).

THEOREM 3.3. *Let $G = (V, E; \omega)$ be a graph with $|V| = n$ and vertex weight function $\omega : V \to R_+$, $\Gamma = (N, c)$ be the corresponding fractional domination game. Then the core of Γ is always non-empty, and a vector $x = (x_1, x_2, \ldots, x_n)$ is in the core if and only if x is an optimal solution to (DLP). Therefore, $\Gamma = (N, c)$ are blanced.*

Proof. Suppose $x \in \mathrm{Core}(\Gamma)$. Followed from Theorem 3.2 and the duality theorem of linear programming, we have that $x(N) = c(N) = \mathrm{opt(LP)} = \mathrm{opt(DLP)}$ and $x(N[j]) \le \omega_j$ for each $j \in N$, i.e., x satisfies the constrains in (DLP). (Here we use the notation $\mathrm{opt}(Q)$ representing the optimum objective value of the program problem Q.) Thus, x is an optimal solution to (DLP).

On the other hand, let x be an optimal solution of (DLP). We want to prove $x \in \text{Core}(\Gamma)$. By Theorem 3.2, we just need to prove for each $j \in N$ and $T_j \in \mathcal{T}_j$, $x(T_j) \leq \omega_j$.

First, since x is the optimal solution of (DLP), we have

$$x(N) = \text{opt(DLP)} = \text{opt(LP)} = c(N),$$

the second equality holds because of the duality theorem of linear programming. Second, since x is a feasible solution of (DLP), it satisfies that

$$x(N[j]) \leq \omega_j, \quad \text{and} \quad x \geq 0.$$

Hence, for each j-substar T_j, we have $T_j \subseteq N[j]$ and

$$x(T_j) \leq x(N[j]) \leq \omega_j.$$

Followed by Theorem 3.2, $x \in \text{Core}(\Gamma)$. □

COROLLARY 3.4. *Let $G = (V, E; \omega)$ be a graph and $\Gamma = (V, c)$ be the corresponding fractional domination game. Then finding a core element of this game Γ can be carried out in polynomial time.*

4. Concave of fractional domination game

In this section, we consider concavity of the fractional domination game. For general cost functions on the vertices, the fractional domination game will not satisfy concavity. However, there exists an interesting characterization of concavity of the fractional domination game. We know that the fractional domination game is not always concave by the following examples.

EXAMPLE 4.1. (1) Let $F_1 = (V, E; \omega)$ be the graph depicted in figure 1 and $\omega = (1, 1, 1, 1)$. Let $\Gamma = (N, c)$ be the corresponding fractional domination game. Let $S = \{v_1, v_2, v_3\}$ and $T = \{v_2, v_3, v_4\}$. Then it is easy to show that $c(S) = 1, c(T) = 1, c(S \cup T) = 2$ and $c(S \cap T) = 1$. Therefore it is not concave.

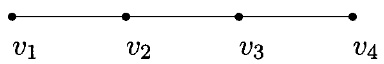

Figure 1. Graph F_1

(2) Let $F_2 = (V, E; \omega)$ be the graph depicted in figure 2 and $\omega = (1, 1, 1, 1)$. Let $\Gamma = (N, c)$ be the corresponding fractional domination game. Then it is easy to check that $f(V) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is an optimal fractional domination function on F_2. Let $S = \{v_1, v_2, v_3\}$ and $T = \{v_1, v_3, v_4\}$. Then $c(S) = 1, c(T) = 1, c(S \cup T) = \frac{4}{3}$ and $c(S \cap T) = 1$. This game also is not concave.

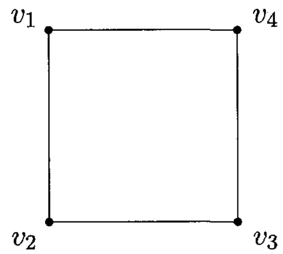

Figure 2. Graph F_2

(3) Let $F_3 = (V, E; \omega)$ be the graph depicted in figure 3 and $\omega = (1, 1, 1, 1)$. Let $\Gamma = (N, c)$ be the corresponding fractional domination game. Let $S = \{v_1, v_2, v_3\}$ and $T = \{v_1, v_3, v_4\}$. Then $c(S) = 1, c(T) = 1, c(S \cup T) = \frac{4}{3}$ and $c(S \cap T) = 2$. This game also is not concave.

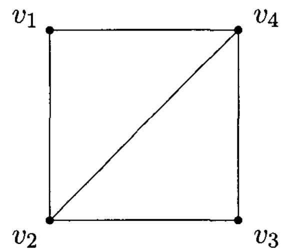

Figure 3. Graph F_3

From above Example 4.1, we see that if graph G contains the three kinds of induced graphs given in Example 4.1, then the corresponding fractional domination game is not concave. So we call the three graphs F_1, F_2 and F_3 given in Example 4.1 the *forbidden graphs.*

To characterize concavity of the fractional domination game, we need to introduce the concept of 2-block graph. A vertex is called a cutvertex of graph G if the subgraph induced by $V \setminus \{v\}$ consists of more components than G. A graph is called 2-connected if it has at least two vertices and contains no cutvertex. A subgraph B is called a block of graph G if it is a maximal 2-connected subgraph. A graph G is a 2-block graph if G

is connected and has at most two blocks which are all complete graphs. It is each to check that

LEMMA 4.2. *A graph G is a 2-block graph if and only if G is connected and has no induced subgraphs as forbidden graphs F_1, F_2 and F_3.*

With the aid of the previous lemma, we now provide a characterization of concave fractional domination game.

THEOREM 4.3. *Let $G = (V, E; \omega)$ be a graph with vertex weight function $\omega_i = 1$ for all $i \in V$, and $\Gamma = (N, c)$ be the corresponding fractional domination game. Then $\Gamma = (N, c)$ is concave if and only if G is a 2-block graph, that is, G has no induced subgraph as the forbidden graphs F_1, F_2 and F_3.*

Proof. Let $\Gamma = (N, c)$ is concave, and suppose G is not a 2-block graph. First, if G has more than two block, then there must be induced a path with length more than 2. From Example 4.1 (1), we see that $\Gamma = (N, c)$ is not concave. This is a contradiction. Second, if one of G's block is not a complete graph, then G must have induced graph like Example 4.1 (1), (2) or (3). Therefore, we see that Γ is not concave. This is also a contradiction. It follows that G must be a 2-block graph.

Conversely, let $G = (V, E; \omega)$ be a 2-block graph, $i, j \in V$ with $i \neq j$ and $S \subset V \setminus \{i, j\}$. We have to show

$$c(S \cup \{i\}) + c(S \cup \{j\}) \geq c(S \cup \{i, j\}) + c(S).$$

Case 1. $G[S]$ has only one component.
If $S \cup \{i\}$ or $S \cup \{j\}$ has two components, then $S \cup \{i\} \cup \{j\}$ has at most two components. Hence,

$$c(S \cup \{i\}) + c(S \cup \{j\}) \geq 2 + 1 \geq c(S \cup \{i\} \cup \{j\}) + c(S).$$

If $S \cup \{i\}$ and $S \cup \{j\}$ both have only one component, then $S \cup \{i\} \cup \{j\}$ also has only one component. Hence,

$$c(S \cup \{i\}) + c(S \cup \{j\}) = 1 + 1 = c(S \cup \{i\} \cup \{j\}) + c(S).$$

Case 2. $G[S]$ has two components.
Each of $\{i\}$ and $\{j\}$ must connect to at least one of the components. So, $S \cup \{i\}$ and $S \cup \{j\}$ has at most two components. If $S \cup \{i\} \cup \{j\}$ has two components, then each of $S \cup \{i\}$ and $S \cup \{j\}$ has also two components. Therefore,

$$c(S \cup \{i\}) + c(S \cup \{j\}) = 2 + 2 = c(S \cup \{i\} \cup \{j\}) + c(S).$$

If $S \cup \{i\} \cup \{j\}$ has only one component, then there must be one cut-vertex of graph G between $\{i\}$ and $\{j\}$. So,

$$c(S \cup \{i\}) + c(S \cup \{j\}) = 1 + 2 = c(S \cup \{i\} \cup \{j\}) + c(S).$$

Therefore, $\Gamma = (N, c)$ is concave. □

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