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ON PERIODICIZING FUNCTIONS

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ABSTRACT. In this paper we introduce a new concept, a "periodicizing" function for the linear differential equation with the periodic forcing function. Moreover, we construct this function, which is closely related with the solution of a difference equation and an indefinite sum. Using this function, we can obtain a representation of solutions from which we see immediately the asymptotic behavior of the solutions.

1. Introduction

We denote by $\mathbb{R}$ and by $\mathbb{C}$ the set of real numbers and the set of complex numbers, respectively.

Let us consider a linear differential equation of the form

$$\frac{dx}{dt} = Ax + f(t), \quad x(0) = w \in \mathbb{C}^d, \tag{1}$$

where $A \in M_d(\mathbb{C})$, the set of all complex $d \times d$ matrices, and $f : \mathbb{R} \to \mathbb{C}^d$ is a nontrivial continuous τ-periodic function.

The purpose of this paper is to find a periodicizing function for Equation (1).

It is well known that the solution of the above equation is expressed as

$$x(t) := x(t; 0, w) = e^{At}w + \int_0^t e^{A(t-s)} f(s)ds. \tag{2}$$

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However, from this representation it is not easy to see asymptotic behaviors of solutions of Equation (1). In the paper [2], we gave a new representation of the solution of Equation (1), from which asymptotic behaviors of solutions are seen.

Its representation is essentially related to a periodicizing function for Equation (1), as stated below. We take a continuous function $z(t)$ such that the function

$$h(t) := z(t) + \int_0^t e^{A(t-s)} f(s)ds, \quad t \in \mathbb{R},$$

becomes a nonconstant continuous τ-periodic function. Then the solution $x(t)$ of Equation (1) is rewritten as

$$x(t) = (e^{At}w - z(t)) + h(t).$$

The first term of the right hand side is well known. We call such a function $z(t)$ a "periodicizing" function (for Equation (1)). Therefore, to find a periodicizing function is very important in obtaining the representation of solutions and in studying asymptotic behaviors of solutions for Equation (1). In this paper we will construct a periodicizing function $z(t)$, which is closely related to the solution of a difference equation and an indefinite sum.

2. Discrete linear difference equations and the indefinite sum

2.1. Discrete linear difference equations

Let $\sigma(A)$ be the set of all eigenvalues of A and m the index of $\lambda \in \sigma(A)$. Let $M_\lambda = \mathcal{N}((A - \lambda E)^m)$ be the generalized eigenspace of $\lambda \in \sigma(A)$, where $E \in M_d(\mathbb{C})$ stands for the unite matrix. Then we have the direct sum decomposition

$$\mathbb{C}^d = \bigoplus_{\lambda \in \sigma(A)} M_\lambda.$$

Let P_λ be the projection on $\mathbb{C}^d$ to M_λ induced from this decomposition. Set $\mathbb{N} := \{1, 2, 3, \ldots\}$.

Now, we solve the discrete linear difference equation of the form

$$x_{n+1} = e^{\tau A} x_n + b, \quad x_0 = w, \tag{3}$$

where $n \in \mathbb{N} \cup \{0\}$. For the simplicity of the description, we set

$$\varepsilon(z) = \frac{1}{e^z - 1}, \quad \varepsilon^{(i)}(z) = \frac{d^i}{dz^i}\frac{1}{e^z - 1}.$$

Moreover, we define $X_\lambda(A)$ and $Y_\lambda(A)$ for $\lambda \in \sigma(A)$ as

$$X_\lambda(A) = \sum_{i=0}^{m-1} \varepsilon^{(i)}(\tau\lambda)\frac{\tau^i}{i!}(A - \lambda E)^i, \quad \text{if } e^{\tau\lambda} \neq 1,$$

and

$$Y_\lambda(A) = \sum_{i=0}^{m-1} B_i \frac{\tau^i}{i!}(A - \lambda E)^i, \quad \text{if } e^{\tau\lambda} = 1,$$

where $B_i, i \in \mathbb{N} \cup \{0\}$, stand for Bernoulli's numbers, refer to [3].

The following result can be found in [2].

THEOREM 1. [2] *Let $\lambda \in \sigma(A)$. The component $P_\lambda x_n$ of the solution $x_n, n \in \mathbb{N}$, of Equation* (3) *is given as follows :*

1) *If $e^{\tau\lambda} \neq 1$, then*

$$\begin{aligned} P_\lambda x_n &= e^{n\tau\lambda} \sum_{i=0}^{m-1} n^i \frac{\tau^i}{i!}(A - \lambda E)^i [P_\lambda w + X_\lambda(A)P_\lambda b] - X_\lambda(A)P_\lambda b \\ &= e^{n\tau A}[P_\lambda w + X_\lambda(A)P_\lambda b] - X_\lambda(A)P_\lambda b. \end{aligned}$$

2) *If $e^{\tau\lambda} = 1$, then*

$$P_\lambda x_n = \sum_{i=0}^{m-1} \frac{n^{i+1}}{i+1}\frac{\tau^i}{i!}(A - \lambda E)^i[\tau(A - \lambda E)P_\lambda w + Y_\lambda(A)P_\lambda b] + P_\lambda w.$$

2.2. The indefinite sum

We will give fundamental results on the indefinite sum. Let $\tau > 0$ and $h : [0, \infty) \to \mathbb{C}^d$ be a continuous function.

First, we consider the problem of finding a continuous solution of the following equation

$$\Delta_\tau z(t) := z(t + \tau) - z(t) = h(t), \quad t \in [0, \infty), \tag{4}$$

that is, the indefinite sum $z(t) = \Delta_\tau^{-1} h(t)$. If $z_0(t)$ is one of solutions of Equation (4), then any other solution $z(t)$ is given by

$$z(t) = z_0(t) + c(t)$$

with an arbitrary continuous τ-periodic function $c(t)$(it is called a periodic constant).

LEMMA 2.1.

1) *Let* $\varphi : [0,\tau] \to \mathbb{C}^d$ *be a continuous function such that*

$$\varphi(\tau) = \varphi(0) + h(0). \tag{5}$$

Then a continuous solution $z(t)$ *of Equation* (4) *satisfying the initial condition* $z(s) = \varphi(s), s \in [0,\tau]$, *exists uniquely on* $[0,\infty)$. *Moreover, it is given by*

$$z(s+n\tau) = \varphi(s) + \sum_{i=0}^{n-1} h(s+i\tau), \quad (s \in [0,\tau), \quad n = 1,2,\ldots). \tag{6}$$

2) *Conversely, if a continuous function* $z(t)$ *is a solution of Equation* (4), *then* $\varphi(t) := z(t)$, $t \in [0,\tau]$, *satisfies the condition* (5) *and* $z(t)$ *is given by* (6).

Proof. 1) It follows from Equation (4) that there exists a function $z(t)$, $t \in [0,\infty)$ such that

$$z(s+n\tau) = \varphi(s) + \sum_{i=0}^{n-1} h(s+i\tau), \quad (s \in [0,\tau), \ n \in \mathbb{N}),$$

which is unique, refer to [3]. Then $z(t)$ is continuous on $[0,\infty) \setminus \tau\mathbb{N}$. Claim the continuity of $z(t)$ at points $t = n\tau$, $n \in \mathbb{N}$. Clearly, we have

$$\begin{aligned}\lim_{s\to 0+} z(s+n\tau) &= \lim_{s\to 0+} \varphi(s) + \lim_{s\to 0+} \sum_{k=0}^{n-1} h(s+k\tau)\\ &= \varphi(0) + \sum_{k=0}^{n-1} h(k\tau).\end{aligned}$$

On the other hand, using the condition (5), we have that

$$\begin{aligned}\lim_{s\to 0-} z(s+n\tau) &= \lim_{s\to 0-} \varphi(s+\tau) + \lim_{s\to 0-} \sum_{k=0}^{n-2} h(s+\tau+k\tau)\\ &= \varphi(\tau) + \sum_{k=0}^{n-2} h((1+k)\tau)\\ &= \varphi(0) + h(0) + \sum_{k=1}^{n-1} h(k\tau)\\ &= \varphi(0) + \sum_{k=0}^{n-1} h(k\tau).\end{aligned}$$

Hence

$$\lim_{s\to 0} z(s+n\tau) = z(n\tau) := \varphi(0) + \sum_{k=0}^{n-1} h(k\tau)$$

holds, which means that $z(t)$ is continuous at $t = n\tau, n \in \mathbb{N}$. Therefore, $z : [0,\infty) \to \mathbb{C}^d$ is a continuous solution of Equation (4). 2) is obvious. □

Next, we consider a special case of Equation (4) : that is,

$$z(t+\tau) - z(t) = -B(t)b, \quad t \in [0,\infty), \tag{7}$$

where $b \in \mathbb{C}^d$ and $B(t), t \in [0,\infty)$, is a continuous matrix function such that

$$B(s+k\tau) = B(s)B^k(\tau), \quad k \in \mathbb{N}. \tag{8}$$

In this case, the continuous variable t in Equation (7) is reduced to the discrete variable.

LEMMA 2.2.

1) *Let* $\varphi : [0,\tau] \to \mathbb{C}^d$ *be a continuous function such that*

$$\varphi(\tau) = \varphi(0) - B(0)b. \tag{9}$$

Then a continuous solution $z(t)$ *of Equation* (7) *satisfying the initial condition* $z(t) = \varphi(t), t \in [0,\tau]$ *exists uniquely on* $[0,\infty)$. *Moreover, it is given by*

$$z(s+n\tau) = \varphi(s) - B(s)x_n(0) \quad (s \in [0,\tau),\ n \in \mathbb{N}), \tag{10}$$

where $x_n(0)$ *is the solution of the difference equation of the form*

$$x_{m+1} = B(\tau)x_m + b, \quad x_0 = 0. \tag{11}$$

2) *Conversely, if a continuous function* $z(t)$ *is a solution of Equation* (7), *then* $\varphi(t) := z(t),\ t \in [0,\tau]$, *satisfies the condition* (9) *and* $z(t)$ *is given by* (10).

Proof. 1) If $s \in [0,\tau)$ and $n \in \mathbb{N}$, then from (6) in Lemma 2.1 and (8) it follows that

$$\begin{aligned} z(s+n\tau) &= z(s) - \sum_{i=0}^{n-1} B(s+i\tau)b \\ &= z(s) - B(s)\sum_{i=0}^{n-1} B^i(\tau)b. \end{aligned}$$

Clearly, we have that $\sum_{i=0}^{n-1} B^i(\tau)b = x_n(0)$. 2) is obvious. □

We note that $B(t) = e^{tA}, A \in M_d(\mathbb{C})$, satisfies the condition (8).

3. A periodicizing function

In this section we will construct a periodicizing function for Equation (1) ; that is,

$$\frac{dx}{dt} = Ax(t) + f(t), \;\; x(0) = w \in \mathbb{C}^d.$$

Let $\lambda \in \sigma(A)$. If an M_λ valued function $y(t)$ satisfies the equation

$$\frac{dy}{dt} = Ay(t) + P_\lambda f(t),$$

we say that $y(t)$ is a solution of Equation (1) in M_λ. Clearly, if $x(t)$ is a solution of Equation (1), then $P_\lambda x(t)$ is a solution of Equation (1) in M_λ. Put

$$b_f = \int_0^\tau e^{A(\tau - s)} f(s) ds.$$

To apply our idea for Equation (1), we will translate the solution $x(t) := x(t; 0, w)$ of Equation (1) as follows :

$$x(t) = e^{At} w - z(t) + h(t),$$

where

$$h(t) = z(t) + \int_0^t e^{A(t-s)} f(s) ds. \tag{12}$$

The condition that $h(t)$ is τ-periodic is equivalent to the condition that

$$z(t+\tau) + \int_0^{t+\tau} e^{A(t+\tau-s)} f(s) ds = z(t) + \int_0^t e^{A(t-s)} f(s) ds.$$

Since

$$\int_0^{t+\tau} e^{A(t+\tau-s)} f(s) ds = e^{At} b_f + \int_0^t e^{A(t-s)} f(s) ds,$$

we have

$$\Delta_\tau z(t) := z(t+\tau) - z(t) = -e^{At} b_f. \tag{13}$$

Therefore $z(t)$ is an indefinite sum of $-e^{At} b_f$; that is, $z(t) = \Delta_\tau^{-1}(-e^{At} b_f)$. Summarizing these, we obtain the following result.

LEMMA 3.1. *A periodicizing function for Equation* (1) *is an indefinite sum of* $-e^{At}b_f$. *Moreover, the solution* $x(t)$ *of Equation* (1) *is expressed as follows :*

$$x(t) = e^{At}w - \Delta_\tau^{-1}(-e^{At}b_f) + h(t),$$

where

$$h(t) = \Delta_\tau^{-1}(-e^{At}b_f) + \int_0^t e^{A(t-s)}f(s)ds$$

is a τ*-periodic function.*

Since $h(t)$ is a τ-periodic function and the second term of the right hand side in (12) is defined on $\mathbb{R}$, the periodicizing function $z(t)$ is well defined on $\mathbb{R}$ provided $z(t)$ is defined on $[0, \infty)$.

Now, we are in a position to state the main theorem in this paper.

THEOREM 2. *Let* $\lambda \in \sigma(A)$.

1) *If* $e^{\tau\lambda} \neq 1$, *then*

$$\Delta_\tau^{-1}(-e^{At}P_\lambda b) = -e^{tA}X_\lambda(A)P_\lambda b + c(t), \quad t \geq 0,$$

where $c(t)$ *is a periodic constant.*

2) *If* $e^{\tau\lambda} = 1$, *then*

$$\Delta_\tau^{-1}(-e^{At}P_\lambda b) = -\frac{e^{\lambda t}}{\tau}\sum_{j=0}^{m-1}\frac{t^{j+1}}{(j+1)!}(A-\lambda E)^j Y_\lambda(A)P_\lambda b + d(t), \quad t \geq 0,$$

where $d(t)$ *is a periodic constant.*

Proof. Let us consider the equation

$$P_\lambda z(t+\tau) - P_\lambda z(t) = -P_\lambda e^{tA}b. \tag{14}$$

It follows from Lemma 2.2 that there exists a continuous solution $P_\lambda z(t)$ of Equation (14), which satisfies the relation

$$P_\lambda z(s+n\tau) = P_\lambda z(s) - P_\lambda e^{sA}x_n(0), \quad (s \in [0,\tau), n = 0,1,2,\ldots), \tag{15}$$

where $x_n(0)$ is the solution of Equation (3) with $w = 0$.

1) Assume that $e^{\lambda\tau} \neq 1$. Put $X = X_\lambda(A)P_\lambda b$. Using Theorem 1 we have

$$P_\lambda x_n(0) = e^{n\tau A}X - X,$$

which yields that

$$\begin{aligned} P_\lambda e^{sA}x_n(0) &= e^{sA}\left(e^{n\tau A}X - X\right) \\ &= -\, e^{sA}X + e^{(s+n\tau)A}X. \end{aligned}$$

Hence the relation (15) is reduced to

$$P_\lambda z(s+n\tau) = \big(P_\lambda z(s) + e^{sA}X\big) - e^{(s+n\tau)A}X.$$

Since

$$P_\lambda z(s+n\tau) + e^{(s+n\tau)A}X = P_\lambda z(s) + e^{sA}X,$$

$c(t) := P_\lambda z(t) + e^{tA}X$ is τ-periodic. Therefore we obtain

$$P_\lambda z(t) = -e^{-tA}X + c(t).$$

2) Assume that $e^{\lambda\tau} = 1$. Put $Y = Y_\lambda(A)P_\lambda b$. Using Theorem 1 again, we have

$$\begin{aligned}
&P_\lambda e^{sA}x_n(0)\\
&= e^{\lambda s}\sum_{k=0}^{m-1}\frac{s^k}{k!}(A-\lambda E)^k\sum_{j=0}^{m-1}\frac{n^{j+1}}{j+1}\frac{\tau^j}{j!}(A-\lambda E)^jY\\
&= e^{\lambda s}\sum_{k=0}^{m-1}\sum_{j=0}^{m-1}\frac{s^k n^{j+1}\tau^j}{k!(j+1)!}(A-\lambda E)^{j+k}Y\\
&= \frac{e^{\lambda s}}{\tau}\sum_{i=0}^{m-1}\sum_{k+j=i}\frac{s^k(n\tau)^{j+1}}{k!(j+1)!}(A-\lambda E)^iY\\
&= \frac{e^{\lambda s}}{\tau}\sum_{i=0}^{m-1}\sum_{k=0}^{i}\frac{s^k(n\tau)^{i-k+1}}{k!(i-k+1)!}(A-\lambda E)^iY\\
&= \frac{e^{\lambda s}}{\tau}\sum_{i=0}^{m-1}\sum_{k=0}^{i+1}\frac{s^k(n\tau)^{i+1-k}}{k!(i+1-k)!}(A-\lambda E)^iY - \frac{e^{\lambda s}}{\tau}\sum_{i=0}^{m-1}\frac{s^{i+1}}{(i+1)!}(A-\lambda E)^iY\\
&= \frac{e^{\lambda(s+n\tau)}}{\tau}\sum_{i=0}^{m-1}\frac{(s+n\tau)^{i+1}}{(i+1)!}(A-\lambda E)^iY - \frac{e^{\lambda s}}{\tau}\sum_{i=0}^{m-1}\frac{s^{i+1}}{(i+1)!}(A-\lambda E)^iY.
\end{aligned}$$

Thus the relation (15) becomes

$$\begin{aligned}
P_\lambda z(s+n\tau) = P_\lambda z(s) &+ \frac{e^{\lambda s}}{\tau}\sum_{j=0}^{m-1}\frac{s^{j+1}}{(j+1)!}(A-\lambda E)^jY\\
&- \frac{e^{\lambda(s+n\tau)}}{\tau}\sum_{j=0}^{m-1}\frac{(s+n\tau)^{j+1}}{(j+1)!}(A-\lambda E)^jY.
\end{aligned}$$

Since

$$d(t) := P_\lambda z(t) + \frac{e^{\lambda t}}{\tau}\sum_{j=0}^{m-1}\frac{t^{j+1}}{(j+1)!}(A-\lambda E)^jY$$

is τ-periodic, we obtain

$$P_\lambda z(t) = -\frac{e^{\lambda t}}{\tau}\sum_{j=0}^{m-1}\frac{t^{j+1}}{(j+1)!}(A-\lambda E)^j Y + d(t).$$

□

Combining Lemma 3.1 and Theorem 2, we can obtain the following result, which is a modification of the one given in [2].

THEOREM 3. *Let $\lambda \in \sigma(A)$ and $x(t) := x(t;0,w)$ be the solution of Equation (1).*

1) *If $e^{\tau\lambda} \neq 1$, then*

$$\begin{aligned} P_\lambda x(t) &= e^{At}[P_\lambda w + X_\lambda(A)P_\lambda b_f] + u_\lambda(t,b_f) \\ &= e^{\lambda t}\sum_{j=0}^{m-1}\frac{t^j}{j!}(A-\lambda E)^j[P_\lambda w + X_\lambda(A)P_\lambda b_f] + u_\lambda(t,b_f), \end{aligned}$$

where

$$u_\lambda(t,b_f) = -e^{At}X_\lambda(A)P_\lambda b_f + \int_0^t e^{(t-s)A}P_\lambda f(s)ds$$

is a τ-periodic solution of Equation (1) in M_λ.

2) *If $e^{\tau\lambda} = 1$, then*

$$\begin{aligned} P_\lambda x(t) = {} & \frac{e^{\lambda t}}{\tau}\sum_{j=0}^{m-1}\frac{t^{j+1}}{(j+1)!}(A-\lambda E)^j[\tau(A-\lambda E)P_\lambda w + Y_\lambda(A)P_\lambda b_f] \\ & + e^{\lambda t}P_\lambda w + v_\lambda(t,b_f), \end{aligned}$$

where $e^{\lambda t}P_\lambda w$ and

$$\begin{aligned} v_\lambda(t,b_f) \ := \ & -\frac{e^{\lambda t}}{\tau}\sum_{j=0}^{m-1}\frac{t^{j+1}}{(j+1)!}(A-\lambda E)^j Y_\lambda(A)P_\lambda b_f \\ & + \int_0^t e^{(t-s)A}P_\lambda f(s)ds \end{aligned}$$

are τ-periodic functions, which are not necessarily solutions of Equation (1) in M_λ.

Proof. 1) Assume that $e^{\lambda\tau} \neq 1$. Combining Lemma 3.1 and Theorem 2, we have

$$\begin{aligned} P_\lambda x(t) &= e^{At}P_\lambda w - \Delta_\tau^{-1}(-e^{At}P_\lambda b_f) + u_\lambda(t,b_f) \\ &= e^{tA}[P_\lambda w + X_\lambda(A)P_\lambda b_f] + u_\lambda(t,b_f), \end{aligned}$$

where

$$u_\lambda(t, b_f) = -e^{tA}X_\lambda(A)P_\lambda b_f + \int_0^t e^{(t-s)A}P_\lambda f(s)ds.$$

Notice that the periodic constant $c(t)$ is canceled. It is easy to see that $u_\lambda(t, b_f)$ is a τ-periodic solution of Equation (1) in M_λ.

2) Assume that $e^{\lambda\tau} = 1$. In view of Lemma 3.1, we have

$$P_\lambda x(t) = e^{At}P_\lambda w - \Delta_\tau^{-1}(-e^{At}P_\lambda b_f) + v_\lambda(t, b_f),$$

where

$$v_\lambda(t, b_f) = \Delta_\tau^{-1}(-e^{At}P_\lambda b_f) + \int_0^t e^{(t-s)A}P_\lambda f(s)ds.$$

Furthermore, from Theorem 2 we have

$$\begin{aligned}
& e^{tA}P_\lambda w - \Delta_\tau^{-1}(-e^{At}P_\lambda b_f) \\
&= e^{t\lambda}P_\lambda w + e^{t\lambda}\sum_{j=1}^{m-1}\frac{t^j}{j!}(A-\lambda E)^j P_\lambda w \\
&\quad + \frac{e^{\lambda t}}{\tau}\sum_{j=0}^{m-1}\frac{t^{j+1}}{(j+1)!}(A-\lambda E)^j Y_\lambda(A)P_\lambda b_f - d(t) \\
&= e^{t\lambda}P_\lambda w \\
&\quad + \frac{e^{t\lambda}}{\tau}\sum_{j=0}^{m-1}\frac{t^{j+1}}{(j+1)!}(A-\lambda E)^j(\tau(A-\lambda E)P_\lambda w + Y_\lambda(A)P_\lambda b_f) - d(t).
\end{aligned}$$

Therefore

$$\begin{aligned}
P_\lambda x(t) = {} & \frac{e^{t\lambda}}{\tau}\sum_{j=0}^{m-1}\frac{t^{j+1}}{(j+1)!}(A-\lambda E)^j(\tau(A-\lambda E)P_\lambda w + Y_\lambda(A)P_\lambda b_f) \\
& + e^{t\lambda}P_\lambda w + v_\lambda(t, b_f).
\end{aligned}$$

We note that the periodic constant $d(t)$ is canceled. □

Notice that from this result we can easily obtain asymptotic behaviors of solutions of Equation (1), for details, refer to [2].

EXAMPLE. We will explain Theorem 2 and Theorem 3 through a simple one dimensional linear differential equation

$$\frac{dx}{dt} = ax(t) + f(t), \quad x(0) = w \in \mathbb{C}, \tag{16}$$

where $a \in \mathbb{C}$ and f is a continuous τ-periodic scalar function. Then (13) is reduced to

$$\Delta_\tau z(t) := z(t+\tau) - z(t) = -e^{at}b_f.$$

Using Theorem 2 with $B_0 = 1$, we have

$$z(t) := \Delta_\tau^{-1}(-e^{at}b_f) = \begin{cases} \frac{e^{at}}{1-e^{a\tau}}b_f, & (e^{a\tau} \neq 1), \\ -\frac{e^{at}}{\tau}tb_f, & (e^{a\tau} = 1). \end{cases} \tag{17}$$

Therefore, by Theorem 3 the solution $x(t)$ of Equation (16) is expressed as follows.

1) If $e^{a\tau} \neq 1$, then

$$x(t;0,w) = e^{at}\left(w - \frac{1}{1-e^{a\tau}}b_f\right) + u(t,b_f),$$

where

$$u(t,b_f) = e^{at}\frac{1}{1-e^{a\tau}}b_f + \int_0^t e^{a(t-s)}f(s)ds$$

is a τ-periodic solution of Equation (16).

2) If $e^{a\tau} = 1$, then

$$x(t;0,w) = \frac{e^{at}}{\tau}tb_f + e^{at}w + v(t,b_f),$$

where

$$v(t,b_f) = -e^{at}\frac{t}{\tau}b_f + \int_0^t e^{a(t-s)}f(s)ds$$

is a τ-periodic function, however, which is not a solution of Equation (16).

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