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ON CHARACTERIZATIONS OF REAL HYPERSURFACES WITH η-PARALLEL RICCI OPERATORS IN A COMPLEX SPACE FORM

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ABSTRACT. We shall give a characterization of a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$, whose Ricci operator and structure tensor commute each other on the holomorphic distribution of M, and the Ricci operator is η-parallel.

0. Introduction

A complex n-dimensional Kaeherian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

R. Takagi ([7]) classified all homogeneous real hypersurfaces in $P_n(\mathbb{C})$ into six model spaces A_1, A_2, B, C, D and E (see also [8]). J. Berndt ([2]) has completed the classification of homogeneous real hypersurfaces with principal structure vector fields in $H_n(\mathbb{C})$, which are divided into the model spaces A_0, A_1, A_2 and B. A real hypersurface of type A_1 or A_2 in $P_n(\mathbb{C})$ or that of A_0, A_1 or A_2 in $H_n(\mathbb{C})$ is said to be *of type A* for simplicity.

We shall denote the induced almost contact metric structure of the real hypersurface M in $M_n(c)$ by $(\phi, \langle , \rangle, \xi, \eta)$. The Ricci operator of M will be denoted by S, and the shape operator or the second fundamental tensor field of M by A. The *holomorphic distribution* T_0 of a real hypersurface M in $M_n(c)$ is defined by

$$T_0(p) = \{X \in T_p(M) \mid \langle X, \xi \rangle_p = 0\},$$

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where $T_p(M)$ is the tangent space of M at $p \in M$. A (1,1) type tensor field K of M is said to be *η-parallel* if $\langle(\nabla_X K)Y, Z\rangle = 0$ for any vector fields X, Y and Z in T_0.

Many authors have occupied themselves with the study of geometrical properties of real hypersurfaces with η-parallel Ricci operators (see [1], [3], [4], [5], [6] and [9]). Recently, Baikoussis ([1]) studied real hypersurfaces in $M_n(c)$ with certain conditions related to the Ricci operator and the structure tensor field ϕ. With conditions on the η-parallel Ricci operator, Kimura and Maeda ([3]) and Suh ([6]) proved the following.

THEOREM A ([3], [6]). *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then the Ricci operator of M is η-parallel and the structure vector field ξ is principal if and only if M is locally congruent to one of the model spaces of type A or type B.*

The purpose of this paper is to give some characterizations of real hypersurfaces with a special η-parallel Ricci operator by applying Theorem A. Namely, we shall prove the followings.

THEOREM 1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies*

$$\langle(S\phi - \phi S)X, Y\rangle = 0, \tag{0.1}$$

$$(\nabla_X S)Y = \mu\langle\phi X, Y\rangle\xi \tag{0.2}$$

for any X and Y in T_0, where μ is a scalar function on M, then M is locally congruent to one of the model spaces of type A or type B.

THEOREM 2. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies (0.1) and*

$$(\nabla_X S)Y = \nu\langle\phi AX, Y\rangle\xi \tag{0.3}$$

for any X and Y in T_0, where ν is a scalar function on M, then M is locally congruent to one of the model spaces of type A or type B.

1. Preliminaries

Let M be a real hypersurface immersed in a complex space form $(M_n(c), \langle,\rangle, J)$ of constant holomorphic sectional curvature c, and let N

be a unit normal vector field on an open neighborhood in M. For a local tangent vector field X on the neighborhood, the images of X and N under the almost complex structure J of $M_n(c)$ can be expressed by

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where ϕ defines a linear transformation on the tangent space $T_p(M)$ of M at any point $p \in M$, and η and ξ denote a 1-form and a unit tangent vector field on the neighborhood respectively. Then, denoting the Riemannian metric on M induced from the metric on $M_n(c)$ by the same symbol $\langle , \rangle$, it is easy to see that

$$\langle \phi X, Y\rangle + \langle \phi Y, X\rangle = 0, \qquad \langle \xi, X\rangle = \eta(X)$$

for any tangent vector field X and Y on M. The collection $(\phi, \langle , \rangle, \xi, \eta)$ is called an *almost contact metric structure* on M, and satisfies

$$\text{(1.1)} \quad \begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1, \\ &\langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y). \end{aligned}$$

Let ∇ be the Riemannian connection with respect to the metric $\langle , \rangle$ on M, and A be the shape operator in the direction of N on M. Then we have

$$\text{(1.2)} \qquad \nabla_X \xi = \phi A X, \qquad (\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle \xi.$$

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are given by

$$\text{(1.3)} \quad \begin{aligned} R(X,Y)Z = &\frac{c}{4}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle \phi Y, Z\rangle \phi X \\ &- \langle \phi X, Z\rangle \phi Y - 2\langle \phi X, Y\rangle \phi Z\} \\ &+ \langle AY, Z\rangle AX - \langle AX, Z\rangle AY, \end{aligned}$$

$$\text{(1.4)} \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y\rangle \xi\}$$

for any tangent vector fields X, Y and Z on M, where R is the Riemannian curvature tensor of M. Then it is easily seen from (1.3) that the Ricci operator S of M is expressed by

$$\text{(1.5)} \qquad SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + mAX - A^2 X,$$

where $m = traceA$ is the mean curvature of M, and the covariant derivative of (1.5) is given by

$$(1.6)\qquad \begin{aligned}(\nabla_X S)Y = &-\frac{3c}{4}\{\langle\phi AX, Y\rangle\xi + \eta(Y)\phi AX\} + (Xm)AY \\ &+ m(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y.\end{aligned}$$

If the vector field $\phi\nabla_\xi\xi$ does not vanish, that is, the length β of $\phi\nabla_\xi\xi$ is not equal to zero, then it is easily seen from (1.1) and (1.2) that

$$(1.7)\qquad A\xi = \alpha\xi + \beta U,$$

where $\alpha = \langle A\xi, \xi\rangle$ and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi$. Therefore U is a unit tangent vector field on M and $U \in T_0$. If the vector field U can not be defined, then we may consider $\beta = 0$ identically. Therefore $A\xi$ is always expressed as in (1.7).

2. η-parallel Ricci operators

In this section, we assume that a real hypersurface M in $M_n(c)$, $c \neq 0$, $n \geq 3$, satisfies (0.1) and has η-parallel Ricci operator S, that is,

$$(2.1)\qquad \langle(\nabla_X S)Y, Z\rangle = 0$$

for any X, Y and Z in T_0. We also assume that β given in (1.7) does not vanish on M. Then it is easy to see from (1.5) and (1.7) that (0.1) is equivalent to

$$(2.2)\qquad (A^2\phi - \phi A^2)X - m(A\phi - \phi A)X = \beta\langle(\alpha - m)U + AU, \phi X\rangle\xi$$

for any X in T_0. If we differentiate (0.1) covariantly and take account of (0.1), (1.1), (1.2), (1.5), (1.7) and (2.1), then we obtain

$$(2.3)\qquad \begin{aligned}&(m-\alpha)\{\langle U, Y\rangle\langle AX, Z\rangle + \langle U, Z\rangle\langle AX, Y\rangle \\ &+ \langle\phi U, Y\rangle\langle AX, \phi Z\rangle + \langle\phi U, Z\rangle\langle AX, \phi Y\rangle\} \\ &- \langle AU, Y\rangle\langle AX, Z\rangle - \langle AU, Z\rangle\langle AX, Y\rangle \\ &+ \langle AU, \phi Y\rangle\langle AX, \phi Z\rangle + \langle AU, \phi Z\rangle\langle AX, \phi Y\rangle \\ = &\ 0\end{aligned}$$

for any X, Y and Z in T_0, where we have used the equation

$$\begin{aligned}\nabla_X Y = & \langle \nabla_X Y, \xi\rangle\xi + (\nabla_X Y)_0 \\ = & -\langle \phi AX, Y\rangle\xi + (\nabla_X Y)_0, \quad (\nabla_X Y)_0 \in T_0.\end{aligned}$$

Now we put

$$\langle AU, U\rangle = \gamma. \tag{2.4}$$

Then, substituting $Y = Z = U$ and $Y = U$, $Z = \phi U$ into (2.3) and using (1.1), (1.7) and (2.4), we have

$$(m-\alpha-\gamma)AU + \langle AU, \phi U\rangle A\phi U = \beta(m-\alpha-\gamma)\xi, \tag{2.5}$$

$$\langle AU, \phi U\rangle AU - (m-\alpha-\gamma)A\phi U = \beta\langle AU, \phi U\rangle\xi \tag{2.6}$$

respectively. As a similar argument as the above, if we put $X = Y = U$ and $X = \phi U$, $Y = U$ into (2.3), we obtain

$$\begin{aligned}&(m-\alpha-2\gamma)AU - 2\langle AU, \phi U\rangle\phi AU \\ &= \beta(m-\alpha-2\gamma)\xi - \gamma(m-\alpha)U - (m-\alpha)\langle AU, \phi U\rangle\phi U,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\langle AU, \phi U\rangle AU - (m-\alpha-\gamma)A\phi U \\ &+ \langle A\phi U, \phi U\rangle\phi AU + \langle AU, \phi U\rangle\phi A\phi U \\ &= \langle AU, \phi U\rangle(\beta\xi + (m-\alpha)U) + (m-\alpha)\langle A\phi U, \phi U\rangle\phi U\end{aligned} \tag{2.8}$$

respectively.

Next we shall prove some Lemmas.

LEMMA 2.1. *Let M be a real hypersurface with the η-parallel Ricci operator S in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If it satisfies* (0.1) *and β given in* (1.7) *does not vanish on M, then we have*

$$m = \alpha + \gamma \quad \textit{and} \quad \langle AU, \phi U\rangle = 0.$$

Proof. Comparing (2.5) with (2.6), we first have

$$\begin{aligned}&\{(m-\alpha-\gamma)^2 + \langle AU, \phi U\rangle^2\}(AU - \beta\xi) = 0, \\ &\{(m-\alpha-\gamma)^2 + \langle AU, \phi U\rangle^2\}A\phi U = 0.\end{aligned}$$

Assume that there is a point p of M such that $(m - \alpha - \gamma)^2 + \langle AU, \phi U\rangle^2 \neq 0$ at p. Then it follows from the above equations that

$$AU = \beta\xi, \qquad A\phi U = 0 \tag{2.9}$$

on an open neighborhood of p, which implies that

$$\gamma = \langle AU, U\rangle = 0, \quad \langle AU, \phi U\rangle = 0 \quad \text{and} \quad m - \alpha \neq 0. \tag{2.10}$$

Putting $Z = U$ into (2.3) and using (2.9) and (2.10), we obtain

$$\langle AX, Y\rangle = 0$$

for any X and Y in T_0, which together with (2.9) shows that

$$A\xi = \alpha\xi + \beta U, \qquad AX = \beta\langle X, U\rangle\xi \quad \text{for} \quad X \in T_0$$

on an open neighborhood of p. The last two equations imply that $m = traceA = \alpha$, and hence this is a contradiction. □

By virtue of Lemma 2.1, the equations (2.7) and (2.8) are reduced to

$$\gamma(AU - \beta\xi - \gamma U) = 0, \tag{2.11}$$

$$\langle A\phi U, \phi U\rangle(\phi AU - \gamma\phi U) = 0 \tag{2.12}$$

respectively.

LEMMA 2.2. *Under the same assumptions as in Lemma 2.1, we have*

$$AU = \beta\xi + \gamma U.$$

Proof. Assume that there is a point p in M such that $AU \neq \beta\xi + \gamma U$ at p. Then it follows from (2.11) that $\gamma = 0$, and from Lemma 2.1 that $m = \alpha$ on an open neighborhood of p. Since we have $AU \neq \beta\xi$, we see from (1.1) and (2.12) that

$$\langle A\phi U, \phi U\rangle = 0. \tag{2.13}$$

Putting $X = U$ into (2.2) and using Lemma 2.1, we have

$$A^2\phi U - \phi A^2 U - \alpha(A\phi U - \phi AU) = 0. \tag{2.14}$$

If we multiply (2.14) by ϕU and make use of (2.13) and $\gamma = 0$, then we get

$$\|A\phi U\| = \|AU\| \tag{2.15}$$

on an open neighborhood of p. Multiplying (2.14) by U and using Lemma 2.1, we obtain

$$\langle A\phi U, AU\rangle = 0. \tag{2.16}$$

It is easy to see that $A\phi U \in T_0$. If we put $X = A\phi U$ and $Z = \phi U$ into (2.3) and take account of Lemma 2.1, $\gamma = 0$, $m = \alpha$ and (2.16), then we obtain $\|A\phi U\|^2\langle AU, Y\rangle = 0$ for any Y in T_0, or equivalently

$$\|A\phi U\|^2(AU - \beta\xi) = 0.$$

This shows that $\|A\phi U\| = 0$ and hence (2.15) gives rise to $AU = 0$ on an open neighborhood of p. Therefore we get $\beta = 0$ by (1.7) and hence it is a contradiction. □

By use of (1.7) and Lemmas 2.1 and 2.2, the relation (1.5) gives rise to

$$S\xi = (\frac{n-1}{2}c + \alpha\gamma - \beta^2)\xi. \tag{2.17}$$

We see from (1.1) and (2.17) that $(S\phi - \phi S)\xi = 0$, which together with (0.1) implies that $S\phi = \phi S$ on M, or equivalently

$$A^2\phi - \phi A^2 = (\alpha + \gamma)(A\phi - \phi A) \tag{2.18}$$

on M. Differentiating (2.17) covariantly along X in T_0 and using (1.1), (1.2), (1.5), (2.17) and (2.18), we have

$$(\nabla_X S)\xi = X(\alpha\gamma - \beta^2)\xi + (\alpha\gamma - \beta^2 - \frac{3c}{4})\phi AX - (\alpha + \gamma)\phi A^2 X + \phi A^3 X$$

for any X in T_0. It is easy to see that the above equation and (2.1) imply

$$(\nabla_X S)Y = -\langle A^3X - (\alpha + \gamma)A^2X + (\alpha\gamma - \beta^2 - \frac{3c}{4})AX, \phi Y\rangle\xi \tag{2.19}$$

for any X and Y in T_0.

It follows from (1.7) and Lemma 2.2 that

$$A^2\xi - (\alpha + \gamma)A\xi + (\alpha\gamma - \beta^2)\xi = 0, \tag{2.20}$$

$$A^2U - (\alpha + \gamma)AU + (\alpha\gamma - \beta^2)U = 0. \tag{2.21}$$

It is easily seen from (2.18) and (2.21) that

$$A^2\phi U - (\alpha + \gamma)A\phi U + (\alpha\gamma - \beta^2)\phi U = 0. \tag{2.22}$$

3. Proof of theorems

In this section, we shall prove Theorems 1 and 2. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$.

Proof of Theorem 1. Assume that there is a point p of M such that $\beta \neq 0$ at p. Then there exists an open neighborhood $\mathcal{U}$ of p such that the local unit vector field U is defined on $\mathcal{U}$. Then, since (0.2) shows that the Ricci operator S is η-parallel, we can compare (0.2) with (2.19) and obtain

$$\langle A^3X - (\alpha+\gamma)A^2X + (\alpha\gamma - \beta^2 - \frac{3c}{4})AX - \mu X, Y\rangle = 0$$

for any X and Y in T_0. Using (2.20), the above equation is rewritten as

$$(3.1)\quad A^3X - (\alpha+\gamma)A^2X + (\alpha\gamma - \beta^2 - \frac{3c}{4})AX - \mu X = -\frac{3}{4}c\beta\langle X, U\rangle\xi$$

for any X in T_0.

Putting $X = U$ into (3.1) and using (2.21) and Lemma 2.2, we can get

$$(3.2)\qquad \mu = -\frac{3c}{4}\gamma.$$

If we put $X = \phi U$ into (3.1) and make use of (2.22) and (3.2), then we have

$$(3.3)\qquad A\phi U = \gamma\phi U.$$

By substituting (3.3) into (2.22), we see that $\beta = 0$ on $\mathcal{U}$, and it is a contradiction.

Therefore $\beta = 0$ on the whole M and hence the structure vector field ξ is principal by (1.7). Thus our result follows from Theorem A. □

REMARK. C. Baikoussis proved in [1] that, a real hypersurface M in $M_n(c)$, $c \neq 0$, $n \geq 3$, satisfies (0.1) and (0.2), where μ is a constant, then M is locally congruent to the model spaces of types A, B, C, D and E in the case $M_n(c) = P_n(\mathbb{C})$, and of types A and B in the case $M_n(c) = H_n(\mathbb{C})$. Therefore Theorem 1 is a generalization of this result.

Proof of Theorem 2. Assume that there is a point p of M such that $\beta \neq 0$ at p. Then there exists an open neighborhood $\mathcal{U}$ of p such that $\beta \neq 0$ on $\mathcal{U}$. Then, comparing (0.3) with (2.19) and using (1.1) and (2.20), we have

$$\begin{aligned} &A^3X - (\alpha+\gamma)A^2X + (\alpha\gamma - \beta^2 - \nu - \frac{3c}{4})AX \\ &= -\beta(\nu + \frac{3c}{4})\langle X, U\rangle\xi \end{aligned} \tag{3.4}$$

for any X in T_0. If we put $X = U$ into (3.4) and take account of (2.21) and Lemma 2.2, then we obtain

$$\gamma(\nu + \frac{3c}{4}) = 0 \tag{3.5}$$

on $\mathcal{U}$. Putting $X = \phi U$ into (3.4) and using (2.22), we also get

$$(\nu + \frac{3c}{4})A\phi U = 0. \tag{3.6}$$

Thus we see from (3.5) and (3.6) that $\nu + \frac{3c}{4} = 0$ on $\mathcal{U}$. In fact, if $\nu + \frac{3c}{4} \neq 0$, then we have $A\phi U = 0$ and $\gamma = 0$ by (3.5) and (3.6), and hence $\beta = 0$ by (2.22), which is a contradiction. Therefore the equations (2.20) and (3.4) imply that

$$A^3 - (\alpha+\gamma)A^2 + (\alpha\gamma - \beta^2)A = 0 \tag{3.7}$$

on $\mathcal{U}$. It is easy to see from (3.7) that any principal curvature λ of M is given by

$$\lambda = 0, \qquad \lambda = \frac{\alpha + \gamma \pm \sqrt{(\alpha-\gamma)^2 + 4\beta^2}}{2}. \tag{3.8}$$

Let X and Y be eigenvectors of A at any point $q \in \mathcal{U}$ belonging to the eigenspaces associated with $\lambda = \frac{\alpha+\gamma+\sqrt{(\alpha-\gamma)^2+4\beta^2}}{2}$ and $\lambda = \frac{\alpha+\gamma-\sqrt{(\alpha-\gamma)^2+4\beta^2}}{2}$ respectively. Then X and Y are given by

$$\begin{aligned} X &= 2\beta\xi - (\alpha - \gamma - \sqrt{(\alpha-\gamma)^2 + 4\beta^2})U, \\ Y &= (\alpha - \gamma - \sqrt{(\alpha-\gamma)^2 + 4\beta^2})\xi + 2\beta U. \end{aligned}$$

Since these vector fields show that ϕU is orthogonal to both X and Y, ϕU belongs to the eigenspace associated with $\lambda = 0$ and hence $\alpha\gamma - \beta^2 = 0$ by (2.22). Therefore we see from (3.8) that the nonzero principal curvature of M is given by $\alpha + \gamma$. Since $m = \alpha + \gamma$, we have $rankA \leq 1$ on $\mathcal{U}$ and it is impossible (for instance, see pp. 253 in [5]).

Thus we see that $\beta = 0$ on the whole M, that is, ξ is principal. This theorem follows from Theorem A. □

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