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TOPOLOGICAL ASPECTS OF FILTERS IN LATTICE IMPLICATION ALGEBRAS

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ABSTRACT. A condition for a lattice filter to be a filter is given. Using the prime filter theorem which is discussed in [2], topological aspects on filters are discussed.

1. Introduction

In order to research the logical system whose propositional value is given in a lattice, Xu [3] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [4] introduced the notions of filter and implicative filter in a lattice implication algebra, and investigated their properties. The present author [1] gave a characterization of a filter, and provided some equivalent conditions for a filter to be an implicative filter in a lattice implication algebra. In this paper, we first give a condition for a lattice filter to be a filter in a lattice implication algebra. Using the prime filter theorem which is discussed in [2], we discuss topological aspects on filters.

2. Preliminaries

DEFINITION 2.1. (Xu [3]) By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\rightarrow$ " satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

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(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$
for all $x, y, z \in L$.

Note that the conditions (L1) and (L2) are equivalent to the conditions
(L3) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(L4) $x \to (y \vee z) = (x \to y) \vee (x \to z)$, respectively.
We can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \to y = 1$.

EXAMPLE 2.2. (Xu and Qin [4]) Let $L := \{0, a, b, c, 1\}$. Define the partially ordered relation on L as $0 < a < b < c < 1$, and define

$$x \wedge y := \min\{x, y\},\ x \vee y := \max\{x, y\}$$

for all $x, y \in L$ and "$'$" and "$\to$" as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra.

In the sequel the binary operation "$\to$" will be denoted by juxtaposition. We can define a partial ordering "$\leq$" on a lattice implication algebra L by $x \leq y$ if and only if $xy = 1$.
In a lattice implication algebra L, the following hold (see [3]):
(P1) $0x = 1$, $1x = x$ and $x1 = 1$.
(P2) $xy \leq (yz)(xz)$.
(P3) $x \leq y$ implies $yz \leq xz$ and $zx \leq zy$.
(P4) $x' = x0$.
(P5) $x \vee y = (xy)y$.
(P6) $((yx)y')' = x \wedge y = ((xy)x')'$.
(P7) $x \leq (xy)y$.
A *lattice filter* is a nonempty subset F of a lattice $(L, \wedge, \vee)$ satisfying
(F1) $x \leq y$ and $x \in F$ imply $y \in F$,
(F2) $x, y \in F$ implies $x \wedge y \in F$.

A nonempty subset F of a lattice implication algebra L is called a *filter* of L if it satisfies for all $x, y \in L$,
(F3) $1 \in F$,

(F4) $x \in F$ and $xy \in F$ imply $y \in F$.

A proper filter P of a lattice implication algebra L is said to be *prime* if whenever $x \vee y \in P$ then $x \in P$ or $y \in P$.

PROPOSITION 2.3. (Jun [1, Proposition 3.2]) *Every filter F of a lattice implication algebra L has the following property:*

$$x \leq y \text{ and } x \in F \text{ imply } y \in F.$$

3. Topological aspects

Let L be a lattice implication algebra. We say that a relation $\sim$ is a *congruence relation* on L if it is an equivalence relation on L with the property that if $x \sim y$ then $xz \sim yz$ and $zx \sim zy$ for all $z \in L$. We shall denote a general congruence by θ. Clearly, $\theta[1] := \{x \in L \mid x \sim 1\}$ is a filter of L. On the other hand, given a filter F of L, we can define a congruence relation $\theta(F)$ on L by $x \sim y \bmod \theta(F)$ if and only if $xy \in F$ and $yx \in F$. It is easily verified that $\theta(F)$ is a congruence relation on L. The set of all filters of L is a lattice, and so is the set of all congruences on L.

THEOREM 3.1. *Let L be a lattice implication algebra. Then the lattice of congruence relations on L is isomorphic to the lattice of all filters of L.*

Proof. Define a mapping

$$\Phi : \{\text{lattice of congruence relations on } L\} \rightarrow \{\text{lattice of filters of } L\}$$

by $\Phi(\theta) = \theta[1]$. Clearly, if $\theta \subset \phi$, then $\Phi(\theta) \subset \Phi(\phi)$. Conversely, suppose that $\Phi(\theta) \subset \Phi(\phi)$. We claim that $\theta \subset \phi$. Let $x \sim y \bmod \theta$. Then $xy \sim 1 \bmod \theta$ and $yx \sim 1 \bmod \theta$. Thus

$$xy,\, yx \in \theta[1] = \Phi(\theta) \subset \Phi(\phi) = \phi[1],$$

and so $xy \sim 1 \bmod \phi$ and $yx \sim 1 \bmod \phi$. Therefore $x \vee y \sim y \bmod \phi$ and $x \vee y \sim x \bmod \phi$ which imply that $x \sim y \bmod \phi$. Hence Φ is 1-1. We now claim that Φ is onto. Let G be a filter of L and consider the corresponding congruence relation $\theta(G)$ on L. Then

$$\begin{aligned} \Phi(\theta(G)) &= \theta(G)[1] = \{x \in L \mid x \sim 1 \bmod \theta(G)\} \\ &= \{x \in L \mid x \in G\} = G. \end{aligned}$$

Thus Φ is a 1-1 correspondence. Since Φ is isotone, it is the required order isomorphism. This completes the proof. □

THEOREM 3.2. *In a lattice implication algebra, every filter is a lattice filter.*

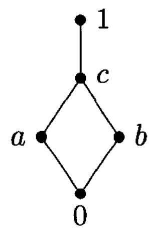
Proof. Let F be a filter of a lattice implication algebra L and let $x, y \in F$. Since

$$y(x \wedge y) = yx \wedge yy = yx \wedge 1 = yx \geq x,$$

it follows from Proposition 2.3 and (F4) that $x \wedge y \in F$. Combining this fact and Proposition 2.3 induces the desired result. □

The converse of Theorem 3.2 may not be true as seen in the following example.

EXAMPLE 3.3. Let $L := \{0, a, b, c, 1\}$ be a set with the following Hasse diagram as a partial ordering. Define a unary operation " $'$ " and a binary operation " $\rightarrow$ " as follows:



x	x'
0	1
a	b
b	a
c	c
1	0

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	c	1	1
b	a	c	1	1	1
c	c	c	c	1	1
1	0	a	b	c	1

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (xy)y \quad \text{and} \quad x \wedge y := ((x'y')y')'$$

for all $x, y \in L$. Then L is a lattice implication algebra. It is easy to see that $F := \{b, c, 1\}$ is a lattice filter of L, but it is not a filter of L since $ac = c \in F$ and $a \notin F$.

We provide a condition for a lattice filter to a filter.

THEOREM 3.4. *In a lattice implication algebra L, every lattice filter is a filter if and only if the identity $x \wedge (xy) = x \wedge y$ holds for all $x, y \in L$.*

Proof. Suppose that every lattice filter is a filter and let $x, y \in L$. Clearly, $x \wedge y \leq x$, xy, that is, $x \wedge y$ is a lower bound of x and xy. Suppose $z \leq x$, xy for some $z \in L$. Then x, $xy \in [z, 1]$, the interval. Note that the interval is a lattice filter, and hence a filter by assumption. It follows from (F4) that $y \in [z, 1]$. Thus we have $z \leq x$, y, and so $z \leq x \wedge y$. Therefore $x \wedge (xy) = x \wedge y$. Conversely, assume that the identity holds. Let F be a lattice filter. Obviously, $1 \in F$. Let $x, y \in L$ be such that $x \in F$ and $xy \in F$. Then, by (F2), $x \wedge (xy) \in F$, that is,

$x \wedge y \in F$. Since $x \wedge y \leq y$, it follows from (F1) that $y \in F$. Hence F is a filter of L. □

For any lattice implication algebra L, let $\mathcal{P}(L)$ and $\mathcal{F}(L)$ denote the set of all prime lattice filters of L and the set of all prime filters of L, respectively. By means of Theorem 3.2, we have $\mathcal{F}(L) \subset \mathcal{P}(L)$. For any $x \in L$, let

$$\bar{x} = \{F \in \mathcal{F}(L) \mid x \notin F\} \text{ and } \hat{x} = \{F \in \mathcal{P}(L) \mid x \notin F\}.$$

Note that $\bar{x} \subset \hat{x}$, and $\{\hat{x} \mid x \in L\}$ forms a basis for topology on $\mathcal{P}(L)$ in which the compact open sets are precisely these basis elements. Obviously $\{\bar{x} \mid x \in L\}$ forms a basis for the subspace topology on $\mathcal{F}(L)$.

LEMMA 3.5. (Liu and Xu [2, Theorem 4]) *Let L be a lattice implication algebra, F a proper filter of L and A a lattice ideal of L. If $F \cap A = \emptyset$, then there exists a prime filter P of L such that $F \subset P$ and $P \cap A = \emptyset$.*

COROLLARY 3.6. *For any $x \neq 1$, there exists $F \in \mathcal{F}(L)$ such that $x \notin F$, that is, $F \in \bar{x}$.*

Proof. Take the filter $\{1\}$ and the lattice ideal $A = [0, x]$. They are disjoint, and so by Lemma 3.5 there exists $F \in \mathcal{F}(L)$ such that $x \notin F$. □

THEOREM 3.7. *Let L be a lattice implication algebra. Then $\mathcal{F}(L)$ is dense in $\mathcal{P}(L)$.*

Proof. We need to show that the topological closure of $\mathcal{F}(L)$ is $\mathcal{P}(L)$. Let $F \in \mathcal{P}(L)$ and let $\hat{x}$ be any basic neighborhood of F. Then $x \notin F$, and $x \neq 1$. By Corollary 3.6, there exists $G \in \mathcal{F}(L)$ such that $x \notin G$, that is, $G \in \bar{x} \subset \hat{x}$. Hence every basic neighborhood of every element of $\mathcal{P}(L)$ intersects $\mathcal{F}(L)$, that is, $\mathcal{F}(L)$ is dense in $\mathcal{P}(L)$. □

THEOREM 3.8. *Let F be a proper filter of a lattice implication algebra L. Then*

$$F = \bigcap\{G \in \mathcal{F}(L) \mid F \subset G\}.$$

Proof. Since F is proper, we can find $x \notin F$. If we take $A = [0, x]$, we have a lattice ideal which is disjoint from F. Using Lemma 3.5, there exists $G \in \mathcal{F}(L)$ such that $F \subset G$. Clearly

$$F \subset \bigcap\{G \in \mathcal{F}(L) \mid F \subset G\}.$$

Assume that $F \neq \bigcap\{G \in \mathcal{F}(L) \mid F \subset G\}$. Then there exists $x \notin F$ and $x \in \bigcap\{G \in \mathcal{F}(L) \mid F \subset G\}$. Taking the lattice ideal $A = [0, x]$, we can find $G \in \mathcal{F}(L)$ such that $F \subset G$ and $G \cap A = \emptyset$. This contradicts the fact that $x \in G$, and proves the theorem. □

THEOREM 3.9. *Let L be a lattice implication algebra. Then the following statements are equivalent.*

(a) *$\mathcal{F}(L)$ is Hausdorff.*

(b) *For each $x \in L$, $\bar{x}$ is both open and closed in $\mathcal{F}(L)$.*

(c) *$\mathcal{F}(L)$ is a Boolean space.*

(d) *For each $x \in L$, there exists $y \in L$ such that $x \vee y = 1$ and $x \wedge y$ does not belong to any prime filter of L.*

Proof. (a) $\Rightarrow$ (b) Note that for each $x \in L$, $\bar{x}$ is compact in $\mathcal{F}(L)$. Since $\mathcal{F}(L)$ is Hausdorff, it follows that $\bar{x}$ is closed in $\mathcal{F}(L)$. But $\bar{x}$ is open in $\mathcal{F}(L)$. Hence $\bar{x}$ is both open and closed in $\mathcal{F}(L)$.

(b) $\Rightarrow$ (c) We have that $\{\bar{x} \mid x \in L\}$ is a basis for the topology of $\mathcal{F}(L)$ consisting of sets that are both open and closed, and hence $\mathcal{F}(L)$ is Hausdorff and totally disconnected. Since $\mathcal{F}(L) = \bar{0}$ is compact, it follows that $\mathcal{F}(L)$ is a Boolean space.

(c) $\Rightarrow$ (d) Since $\mathcal{F}(L)$ is a Boolean space, it has a basis for the topology consisting of sets that are both open and closed. Hence each member of the basis is compact and open. This means that each member of the basis is of the form $\bar{x}$ for some $x \in L$. Now, for each $a \in L$, $\bar{a}$ is a union of some of these sets $\bar{x}$ in the basis, and since $\bar{a}$ is compact, we can find a finite subcovering. But a finite union of closed sets is closed. Hence $\bar{a}$ is an open and closed subset of $\mathcal{F}(L)$. Thus, for each $x \in L$, $\bar{x}$ is an open and closed subset of $\mathcal{F}(L)$, and so its set theoretic complement $\mathcal{F}(L) \setminus \bar{x}$ is also both open and closed. Since $\mathcal{F}(L)$ is compact, $\mathcal{F}(L) \setminus \bar{x}$ is also compact and open, and thus it is of the form $\bar{y}$ for some $y \in L$. Therefore $\bar{x} \cap \bar{y} = \emptyset$ and $\bar{x} \cup \bar{y} = \mathcal{F}(L)$. This means that $x \vee y = 1$ and $x \wedge y$ does not belong to any prime filter of L.

(d) $\Rightarrow$ (a) Suppose (d) holds. Then $\bar{x} \cap \bar{y} = \emptyset$ and $\bar{x} \cup \bar{y} = \mathcal{F}(L)$. Hence $\bar{x} = \mathcal{F}(L) \setminus \bar{y}$ which is a closed subset of $\mathcal{F}(L)$, and therefore for each $x \in L$, $\bar{x}$ is both open and closed in $\mathcal{F}(L)$. Since $\{\bar{x} \mid x \in L\}$ is a basis for the topology on $\mathcal{F}(L)$, it follows that $\mathcal{F}(L)$ is Hausdorff. □

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