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DERIVATION OF A PRICE PROCESS FOR MULTITYPE MULTIPLE DEFAULTABLE BONDS

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ABSTRACT

We consider a zero coupon bond that is at the risk of multitype multiple defaults. Assuming defaults occur according to k Cox processes, we find a price process for zero coupon bonds. To derive this process we follow the Lando (1998)'s method which uses conditional expectations instead of the traditional methods.

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1. INTRODUCTION

Let us consider a sovereign bond for some country which is influenced by the defaults of k economically very influential corporations, and where each corporation is at the risk of multiple defaults. So the sovereign bond is subject to multiple defaults, and the defaults can come from a variety of sources. In other words, we are considering multitype multiple defaultable bonds. In this paper we will directly derive a price process of multitype multiple defaultable bonds using conditional expectations.

The pricing of multiple defaultable bonds was first considered by Schonbucher (1998). Using the Heath, Jarrow, Morton framework, he represented the term structure of defaultable bond prices in terms of forward rates. In generalizing Schonbucher's model, Duffie, Pedersen and Singleton (DPS) (2003) constructed a model of the term structure of credit spreads on sovereign bonds. They showed that the cash flows promised by a sovereign bond can be discounted using a default-adjusted short-term discount rate. Wong's (2004) model is similar to the

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DPS model in that the model is constructed in the short-rate framework. Using the framework of Lando (1998), he assumes that the default generating process can be described by a Cox process. His model was built on this assumption by assuming that there exists an independent identically distributed process which marks each default. By taking k Cox processes (see Appendix) as a default generating process, we extend Wong's model to multitype multiple defaultable bonds.

Throughout this article we consider a k-variate point process (τ_n, z_n) as a basic process, where τ_n is a point process and z_n is a sequence of random variables which have values (marks) on $E = \{1, 2, \ldots, k\}$ (Bremaud, 1891). If the n^{th} default occurs at time τ_n, then the n^{th} loss rate L_n, which has values in [0,1), is generated by a given distribution depending on the mark at that time. The loss rate process L_t is defined by $L_t = L_n$ for $t \in [\tau_n, \tau_{n+1})$, $n \geq 0$, where L_0=0 and $\mathcal{L}_t$ is a filtration generated by the loss rate process. $\mathcal{F}_t$ is a filtration generated by the random vector-valued processes S_t of all relevant economic variables such as the short-term risk-free interest rate r, equity prices, inflation rate, *etc.*, in other words $\mathcal{F}_t = \sigma(S_u\,;\, 0 \leq u \leq t)$. For the next section we define a filtration $G = (\mathcal{G}_t)_{0 \leq t \leq T^*} = \big(\mathcal{L}_t \vee \mathcal{F}_t \vee_{i=1}^{k} \mathcal{N}_t(i)\big)_{0 \leq t \leq T^*}$, where T^* is a fixed finite planning horizon and $\mathcal{N}_t(i)$ is the smallest σ-algebra generated by all $N_s(i) = \sum_{n \geq 1} 1(\tau_n \leq s) 1(z_n = i)$, $s \leq t$. We also assume the existence of risk neutral probability measure P^* under which the discounted defaultable bond prices become a martingale. So all variables and processes are properly defined on the fixed probability space (Ω, G, P^*). We begin to derive our main result.

2. Derivation of a Price Process

In this section we will derive a price process for zero coupon bonds with defaults of multiple types under k Cox processes (see Appendix). Let us consider a multitype defaultable zero-coupon bond with maturity date $T \leq T*$, which promises to pay X at time T if no default occurs during the life of the bond. If the bond defaults before time T then the owner of the bond is given another bond of lower face value at each default event. In particular, if there are N_T defaults during the life of the bond, then the final pay-off would be

$$Z_T \equiv X \prod_{n=0}^{N_T} (1 - L_n),$$

where L_n denotes the loss rate at the time of n^{th} default. These are generated independently and identically by the distributions $F_i(l)$, where $i = 1, 2, \ldots, k$ corresponds to the type or "source" of the default. By the martingale property, the time-t price $\nu(t,T)$ of a multitype defautable zero-coupon bond which settles at time T is given as

$$\nu(t,T) = E\left[\exp\left\{-\int_t^T r(u)du\right\} \prod_{n=N_t+1}^{N_T} (1-L_n)Z_t \;\Big|\; \mathcal{G}_t\right],$$

where $r(u)$ is the short time interest rate and $Z_t \equiv \prod_{i=0}^{N_t}(1-L_i)X$. If L_n is the j^{th} loss of type-i default after time t, let us rewrite L_n as Y_{ij}. Doing this for all L_i, $i = N_t+1, N_t+2, \ldots, N_T$, the above expression becomes

$$E\left[\exp\left\{-\int_t^T r(u)du\right\} \prod_{j=1}^{N_t^T(1)} (1-Y_{1j}) \prod_{j=1}^{N_t^T(2)} (1-Y_{2j}) \cdots \prod_{j=1}^{N_t^T(k)} (1-Y_{kj})Z_t \;\Big|\; \mathcal{G}_t\right], \tag{2.1}$$

where $N_t^T(i)$ is the number of type-i defaults up through time T after time t.

Throughout the paper, E denotes the expectation operator under the risk neutral measure. Now we begin to derive a formula for time-t discounted bond price $\nu(t,T)$, which is an extension of Wong's (2004) result. Since Z_t is adapted to $\mathcal{G}_t$ and the short time interest rate $r(u)$ is adapted to $\mathcal{F}_u$, the above expression (2.1) becomes

$$\begin{aligned} &Z_t E\left[\exp\left\{-\int_t^T r(u)du\right\} \prod_{j=1}^{N_t^T(1)} (1-Y_{1j}) \prod_{j=1}^{N_t^T(2)} (1-Y_{2j}) \cdots \prod_{j=1}^{N_t^T(k)} (1-Y_{kj}) \;\Big|\; \mathcal{G}_t\right] \\ &= Z_t E\left[\exp\left\{-\int_t^T r(u)du\right\} E\left[E\left\{\prod_{j=1}^{N_t^T(1)} (1-Y_{1j}) \prod_{j=1}^{N_t^T(2)} (1-Y_{2j}) \cdots \right.\right.\right. \\ &\qquad \left.\left.\left. \times \prod_{j=1}^{N_t^T(k)} (1-Y_{kj}) \;\Big|\; \mathcal{F}_T \vee \mathcal{G}_t, N_t^T(1), \ldots, N_t^T(k)\right\} \;\Big|\; \mathcal{F}_T \vee \mathcal{G}_t\right] \;\Big|\; \mathcal{G}_t\right]. \end{aligned}$$

Assume the loss rates Y_{ij}'s are independent of each other and of all other processes. Then the above expression becomes

$$Z_t E\left[\exp\left\{-\int_t^T r(u)du\right\} E\left\{\prod_{i=1}^{k} \prod_{j=1}^{N_t^T(i)} \Big(1-E(Y_{ij})\Big) \;\Big|\; \mathcal{F}_T \vee \mathcal{G}_t\right\} \;\Big|\; \mathcal{G}_t\right]$$

$$= Z_t E\left[E\left\{\exp\Big(-\int_t^T r(u)du\Big)\prod_{i=1}^{k}\Big(1-y(i)\Big)^{N_t^T(i)} \;\Big|\; \mathcal{F}_T \vee \mathcal{G}_t\right\} \;\Big|\; \mathcal{G}_t\right],$$

where $y(i) = E[Y_{ij}]$, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, N_t^T(i)$.

Now we calculate the value of the inside expectation of the above expression, that is,

$$E\left[\exp\left\{-\int_t^T r(u)du\right\}\prod_{i=1}^{k}\{1-y(i)\}^{N_t^T(i)} \;\Big|\; \mathcal{F}_T \vee \mathcal{G}_t\right].$$

Noting $N_t^T = \sum_{i=1}^k N_t^T(i)$ and denoting N_t^T by j for simplicity, the above expectation becomes

$$\exp\left\{-\int_t^T r(u)du\right\}\sum_{j=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=j}\prod_{i=1}^{k}(1-y(i))^{n_i}$$
$$\times P^*\left\{N_t^T(1)=n_1,\ldots,N_t^T(k)=n_k \mid \mathcal{F}_T \vee \mathcal{G}_t\right\}. \quad (2.2)$$

Given $\mathcal{F}_T$, default processes of each type $N_t(i)$, $i = 1, 2, \ldots, k$ are inhomogeneous Poisson processes with parameter $\Theta_i = \int_t^T \lambda_u(i)du$ (See Grandell, 1976, p.5). Assume, given $\mathcal{F}_T$, these default processes are conditionally independent. Then we have

$$P\left\{N_t^T(1)=n_1,\ldots,N_t^T(k)=n_k \mid \mathcal{F}_T \vee \mathcal{G}_t\right\}$$
$$=\prod_{h=1}^{k}\frac{\left\{\int_t^T \lambda_u(h)du\right\}^{n_h}}{n_h!}\exp\left\{-\int_t^T \lambda_u(h)du\right\}.$$

Using this the equation (2.3) becomes

$$\exp\left\{-\int_t^T r(u)du\right\}\sum_{j=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=j}\prod_{i=1}^{k}\{1-y(i)\}^{n_i}$$
$$\times\prod_{h=1}^{k}\frac{\left\{\int_t^T \lambda_u(h)du\right\}^{n_h}}{n_h!}\exp\left\{-\int_t^T \lambda_u(h)du\right\}$$
$$=\exp\left\{-\int_t^T r(u)du\right\}\sum_{j=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=j}\frac{j!}{n_1!n_2!\cdots n_k!}$$
$$\times\left[\prod_{i=1}^{k}\left\{\int_t^T\Big(1-y(i)\Big)\lambda_u(i)du\right\}^{n_i}\right]\frac{\exp\left\{-\int_t^T\sum_{h=1}^k \lambda_u(h)du\right\}}{j!}$$

$$= \exp\left\{-\int_t^T r(u)du\right\}\sum_{j=0}^{\infty}\frac{\left[\sum_{i=1}^k\int_t^T\{1-y(i)\}\lambda_u(i)du\right]^j}{j!}$$

$$\times \exp(-\int_t^T(\sum_{h=1}^k \lambda_u(h))du)$$

$$= \exp\left\{-\int_t^T r(u)du\right\}\exp\left[\sum_{i=1}^k\int_t^T\{1-y(i)\}\lambda_u(i)du\right]\exp\left\{-\sum_{h=1}^k\int_t^T\lambda_u(h)du\right\}$$

$$= \exp\left\{-\int_t^T r(u)du\right\}\exp\left\{-\sum_{i=1}^k\int_t^T y(i)\lambda_u(i)du\right\}.$$

Now we insert this result to the equation (2.2) and we obtain

$$\nu(t,T) = Z_t E\left[\exp\left\{-\int_t^T r(u)du\right\}\exp\left\{-\sum_{i=1}^k\int_t^T y(i)\lambda_u(i)du\right\} \,\Big|\, \mathcal{G}_t\right].$$

Since $Z_t \in \mathcal{G}_t$, we finally obtain

$$\begin{aligned}\nu(t,T) &= E\left[\exp\left\{-\int_t^T r(u)du\right\}\exp\left\{-\sum_{i=1}^k\int_t^T y(i)\lambda_u(i)du\right\}Z_t \,\Big|\, \mathcal{G}_t\right]\\ &= E\left[\exp\left\{-\int_t^T r(u)du-\sum_{i=1}^k\int_t^T y(i)\lambda_u(i)du\right\}Z_t \,\Big|\, \mathcal{G}_t\right].\end{aligned}$$

We write the above derived result in the following theorem.

THEOREM 2.1. *Let the i-type defaults occur according to Cox processes $N_t(i)$ with the stochastic intensity measure $\Theta_i([s,t)) = \int_s^t \lambda_u(i)du$, $i = 1,2,\ldots,k$. Assume*

(*i*) *given $\mathcal{F}_T$, default processes of each type $N_t(1),\ldots,N_t(k)$ are conditionally independent ,*

(*ii*) *type-i loss rates Y_{ij}, j=1,2,...,$N_t^T(i)$, are identically and independently distributed with distribution $F_i(y)$, i=1,2,...,k and these rates are independent of each other and of all other processes.*

Then the time-t price of the zero coupon bond with maturity T is

$$\nu(t,T) = E\left[\exp\left\{-\int_t^T r(u)du-\sum_{i=1}^k\int_t^T y(i)\lambda_u(i)du\right\}Z_t \,\Big|\, \mathcal{G}_t\right],$$

where $y(i) = E[Y_i]$, $i = 1,2,\ldots,k$, and $Z_t = X\prod_{n=0}^{N_t}(1-L_n)$.

Wong (2004) found a zero-coupon bond price formula for the case of single-type multiple defaults using Cox process. We have extended it to the case of multi-type multiple defaults using k Cox processes.

3. Concluding Remarks

In this paper we derived a price process for multitype multiple defaultable zero coupon bonds. The result was similar to that of DPS (2003). DPS obtained this process by using a default adjusted short-term discount rate and Ito's formula. Instead, following the structure of Lando (1998), we used the properties of conditional expectations and got a similar result with reduced technical difficulties.

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Appendix

Referring to Grandell (1976), we redefine Cox process as follows. Let N_t be a counting process adapted to a right continuous filtration $H = (\mathcal{H}_t)_{0\le t\le T} = (\mathcal{F}_t \vee \mathcal{N}_t)_{0\le t\le T}$, where $\mathcal{N}_t = \sigma(N_s\,;\, s \le t)$ and $\mathcal{F}_t$ is defined as in introduction. Let N_t be defined on a probability space (Ω, H, P). It is called a Cox process with respect to H, with the associated stochastic intensity measure $\Theta(t)$, if for any $0 \le s < t$ and every k=0,1,. . . .

$$P\left\{N([s,t)) = k \mid \mathcal{F}_T \vee \mathcal{H}_s\right\} = \frac{\Theta([s,t))^k}{k!} \exp\left\{-\Theta([s,t))\right\}, \ a.s.\ [P],$$

where N([s,t))=$N_t - N_s$ and the stochastic intensity measure is given by $\Theta([s,t)) = \int_s^t \lambda(u)du$ for some F-progressively measurable process λ with locally integrable sample paths.

Note that by conditioning on the path of stochastic intensity we obtain the intensity measure of the inhomogeneous Poisson process.

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