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AN ASYMPTOTIC DECOMPOSITION OF HEDGING ERRORS

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ABSTRACT

This paper studies the problem of option hedging when the underlying asset price process is a compound Poisson process. By adopting an asymptotic approach to let the security price converge to a continuous process, we find a closed-form hedging strategy that improves the classical Black-Scholes hedging strategy in a quadratic sense. We first show that the scaled Black-Scholes hedging error has a limit in law, and that limit is decomposed into a part that can be traded away and a part that is purely unreplicable. The Black-Scholes hedging strategy is then modified by adding the replicable part of its hedging error and by adding the mean-variance hedging strategy to the nonreplicable part. Some results of simulation experiments are also provided.

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1. INTRODUCTION

Perfect hedging of an option is impossible in the real world. In a complete financial market, every contingent claim is exactly attainable by investing in the market. But in most real instances, the market is not complete. Under the classical Black-Scholes setting, in which the stock price process is a geometric Brownian motion, we can construct a perfect hedging strategy because their setup assures that the market is complete. However, the stock price process is not a geometric Brownian motion and even not continuous in reality. Stocks move in fixed increments that are multiples of the tick size and sometimes there are also

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big jumps, such as market crashes. When we look at the fluctuations of stock prices on an intraday trading scale, we can see that the more realistic model is a purely discontinuous process rather than a continuous one (Engle and Russell, 2002). In many cases, financial markets are incomplete when the underlying asset price process has jumps. The goal of this paper is to find an appropriate hedging strategy of a contingent claim under the market incompleteness due to jumps in the price process.

Models with jumps have been extensively investigated in the finance literature for several decades as an alternative to the Black-Scholes model. One example is jump-diffusion models, in which jumps are frequently modeled by Poisson processes. Although Poisson processes have been used more often with diffusion, they are also used to model the underlying price process on their own, as they could have such real market features that prices change at discrete random points in time. Also, even a simple geometric Poisson process can reproduce the volatility smile in foreign exchange markets as well as the volatility skew in equity markets (Kirch and Runggaldier, 2004). To name a few articles using Poisson processes, Frey (2000) used a doubly stochastic compound Poisson process to calculate the risk minimization strategy assuming that the asset price process is a martingale under the real probability measure. León *et al.* (2002) approximated a compound Poisson process by several independent Poisson processes and found the price and the hedging strategy. Kirch and Runggaldier (2004) considered the underlying price process as a geometric Poisson process with constant intensities.

In this paper, we will focus on the case where the log of the stock price process is modeled by a compound Poisson process alone. Jumps drive the whole process under this model, while in a typical jump-diffusion model, the diffusion part is the main source of randomness and the jump part models only abnormal components of the stock return distribution. In that sense, the model is similar to pure jump processes in Eberlein and Keller (1995), Madan and Seneta (1990) and Carr *et al.* (2002). A compound Poisson process has a very simple structure and yet, it can capture fat tails or the asymmetry of the return distribution. It is also intuitive in the sense that it jumps at random times and there are finitely many jumps in any finite interval. Moreover, since the component of arbitrarily small jumps of the Lévy process as well as the Brownian component can be obtained as a limit of compound Poisson processes, the theory developed in this paper can possibly be generalized to Lévy processes.

Here, we consider a sequence of compound Poisson processes whose limit is the Black-Scholes stock price model. The Black-Scholes model is still used as a reasonable approximation in practice, and it is often considered just as robust in theory. Therefore, it would be reasonable to consider an asymptotically geometric Brownian motion as the underlying price process so that the Black-Scholes model is an approximation to the true model.

Convergence of the sequence of discrete time price processes to a continuous time process has been widely studied in finance literature. Such literature goes back at least to the famous paper by Cox *et al.* (1979). Unlike most of the previous literature dealing with a set of fixed time points, we deal with a set of random time points with the time interval going to 0 as the jump intensity goes to infinity. One may want to see this compound Poisson process as a generalization of a binomial tree model, as an extension of the model by Rachev and Rushendorf (1994). The model defined in Section 2 can be viewed as randomizations of the jump time and the jump distribution from a binomial tree model in a particular way that the limit is a Brownian motion. Our convergence setup is more similar to that of Hong and Wee (2003). They considered a sequence of jump-diffusion models that converges weakly to the Black-Scholes model. The underlying price processes are driven by Lévy processes and they studied the convergence of option prices jointly with the costs from the local risk minimization strategies.

The purpose of this paper is to show that we can find an explicit form of the hedging strategy that improves the classical Black-Scholes strategy when we have a compound Poisson model. We first want to pull out the part of a contingent claim that can be replicated by looking at the hedging error of the classical Black-Scholes strategy. After a suitable normalization, we find the limit in law of the Black-Scholes hedging error. We then decompose it into replicable and non-replicable parts and find the pre-limiting processes that converge weakly to each part. By adding the pre-limiting process that converges weakly to the replicable part of the limit of the Black-Scholes hedging error, we update the Black-Scholes hedging strategy. To handle the non-replicable part of the hedging error, we employ the mean-variance hedging method.[3]

The remainder of the paper is organized as follows. Section 2 describes the detailed model and the convergence of the underlying asset price process. Section 3 shows the main results to deal with the convergence and the decomposition of the Black-Scholes hedging error process. Proofs are in the Appendix. Then, we use the mean-variance hedging method in Section 4 to handle the non-replicable part of the limiting Black-Scholes hedging error. Section 5 provides some simulation

[3]Readers may wonder what happens if we apply the mean-variance hedging directly to the limit of the Black-Scholes hedging error. See Remark 4.1.

results and Section 6 contains concluding comments.

2. THE MODEL

Consider a sequence of discontinuous processes that converges to a geometric Brownian motion. Each element of the sequence is a jump process, indexed by n. A larger n means that the degree of discontinuity is smaller, *i.e.*, the process is closer to a geometric Brownian motion.

We suppose that for each n, the log stock price process is defined on a probability space $(\Omega, \mathcal{F}^{(n)}, P^{(n)})$ and follows a compound Poisson process such as

$$\log S_t^{(n)} = \log S_0^{(n)} + \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)}, \tag{2.1}$$

where $N^{(n)}$ is a Poisson process with rate λ_n and $Z_i^{(n)}$'s are *iid* random variables, distributed as $Z^{(n)}$, that are independent of $N^{(n)}$. The filtration, $\{\mathcal{F}_t^{(n)}\}$, is generated by the stock price process $S^{(n)}$ defined above. We also assume the initial stock price $S_0^{(n)}$ is the same as S_0 for all n. As n goes to ∞, we assume that λ_n goes to ∞ and $Z^{(n)}$ converges to 0 in distribution. $N_t^{(n)}$ is the number of jumps in the log stock price process up to time t and $Z_i^{(n)}$ represents the size of the i^{th} jump of $\log S^{(n)}$.

The underlying stock price process follows the compound Poisson process with specific values of parameters including λ_n. The jump intensity, λ_n, is related to the level of the trading activity of an individual stock. A heavily traded stock is modeled with a large λ_n, and a less heavily traded stock is modeled with a smaller λ_n. According to the level of trading activity of a stock, we determine the value of λ_n so that the model fits with the data.[4] Each jump occurs when there is a trading that changes the underlying stock price.

We define the jump size distribution $Z^{(n)}$ more precisely as follows.

$$Z^{(n)} \stackrel{\mathcal{D}}{=} \frac{1}{\sqrt{\lambda_n}} Q + \frac{1}{\lambda_n}\Big(\mu - \frac{1}{2}\sigma^2\Big), \tag{2.2}$$

where Q is a random variable with $EQ = 0$, $EQ^2 = \sigma^2$, $EQ^3 = k_3$ and $EQ^4 = k_4$, under $P^{(n)}$, for all n. Q has a distribution that does not depend on n and it has finite moments of all orders. μ is a constant. $\stackrel{\mathcal{D}}{=}$ means that both sides of

[4] λ_n can be estimated by the number of tradings that change the price, when we deal with real datasets.

the equality have the same distribution. It is clear that $Z^{(n)}$ converges to 0 in probability as well as in distribution as n goes to ∞, $E(Z^{(n)}) = (1/\lambda_n)(\mu - \sigma^2/2)$, $E|Z^{(n)}|^p = O(\lambda_n^{-p/2})$ for $p = 2, 3$ and 4, and $E|Z^{(n)}|^p = o(\lambda_n^{-2})$ for $p > 4$. We can add a $o_p(\lambda_n^{-1})$ term to $Z^{(n)}$ if we want and it would not change the rest of the paper.

Now, consider the asymptotics as n goes to ∞. The conditions above assure that $\log S^{(n)}$ converges to a Brownian motion with drift.

PROPOSITION 2.1. *Assume all the above conditions. Then as n goes to ∞, the process $\log S^{(n)}$ converges in distribution to $\log S$ that is*

$$\log S_t = \log S_0 + \Big(\mu - \frac{1}{2}\sigma^2\Big)t + \sigma B_t, \tag{2.3}$$

where B is a Brownian motion under the limiting measure P.

Proof is standard so is omitted. One way to prove this proposition is to use the martingale central limit theorem.

By defining the jump size distribution as in (2.2), we can interpret the parameters as follows. μ and σ are the leading terms of the expected rate of return and the volatility, respectively, $k_3/\sqrt{\lambda_n}$ is the leading term of the skewness and k_4/λ_n is the leading term of the kurtosis of the log stock price process. For instance, $k_3/\sqrt{\lambda_n}$ is the leading term of the skewness as

$$E\{\log S_t^{(n)} - E(\log S_t^{(n)})\}^3 = \frac{k_3 t}{\sqrt{\lambda_n}} + \frac{3\sigma^4 t}{\lambda_n}\Big(\mu - \frac{1}{2}\sigma^2\Big) + \frac{(\mu - \frac{1}{2}\sigma^2)^3 t}{\lambda_n^2}.$$

The current model permits incorporating the skewness and the kurtosis of the return distribution, which gives an advantage over models considering only symmetric return distributions. In the following sections, we will see that the skewness of the log stock price process is involved in the proposed hedging strategy through k_3. If we consider the second order asymptotics, k_4 would also be included in the hedging strategy.

3. DECOMPOSITION OF THE HEDGING ERROR

We consider a market with two securities; a stock as a risky asset and a cash bond as a riskless asset. The stock price follows the compound Poisson model as in (2.1) and the interest rate r is assumed to be 0 without loss of generality so that the value of a unit of the cash bond is always 1. Then consider a European style option whose payoff is $\eta(S_T^{(n)})$ with the expiration time T. We assume that $\eta(S_T^{(n)})$ is in $L^2(P^{(n)})$. Throughout the paper, we denote $C(x,t)$ the solution of the Black-Scholes PDE at time $t < T$, with the terminal condition, $C(x,T) = \eta(x)$. C_S, C_{SS} and C_{SSS} denote the first, second, and third derivatives of $C(x,t)$ with respect to x, respectively. $C_S^{(p)}$ is used for the p^{th} derivative of $C(x,t)$ with respect to x, for $p > 3$. For each n, we compute $C(S_t^{(n)},t)$ by plugging in the corresponding stock price $S_t^{(n)}$. In other words, $C(S_t^{(n)},t)$ is computed by the Black-Scholes PDE, but it may be different from what we observe from the market. On the other hand, $C(S_t,t)$ is also computed by the Black-Scholes PDE, but it is the true market price in the limit because the limiting stock price S follows a geometric Brownian motion.

Let $X^{(n)}$ be the value process of the Black-Scholes hedging portfolio, *i.e.*,

$$X_t^{(n)} = C(S_0^{(n)},0) + \int_0^t C_S(S_{u-}^{(n)},u)dS_u^{(n)}.$$

In a complete market, $X^{(n)}$ would be a perfect hedging for $\eta(S_T^{(n)})$, but it is not perfect under our compound Poisson model. Let us first look at the hedging error of the Black-Scholes hedging strategy.

THEOREM 3.1. *$\sqrt{\lambda_n}(C(S_\cdot^{(n)},\cdot) - X_\cdot^{(n)})$ converges jointly with $S^{(n)}$ in distribution to (R,S) where*

$$R_t = \frac{1}{2}\int_0^t S_u^2 C_{SS}(S_u,u)d\xi_u + \frac{k_3}{6}\int_0^t \{3S_u^2 C_{SS}(S_u,u) + S_u^3 C_{SSS}(S_u,u)\}\,du, \tag{3.1}$$

$$\xi_t = c_1\tilde{W}_t + \frac{k_3}{\sigma}B_t, \quad c_1 = \sqrt{k_4 - \left(\frac{k_3}{\sigma}\right)^2},$$

and $\{\tilde{W}_t\}$ and $\{B_t\}$ are independent Brownian motions under P.

PROOF. See Appendix. □

Theorem 3.1 says that when the Black-Scholes hedging error is suitably normalized, it converges weakly to a continuous process jointly with the underlying asset price process. The limiting asset price process S follows a geometric Brownian motion, as

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{3.2}$$

Let us look at R_T more closely.

$$\begin{aligned} R_T &= \frac{c_1}{2}\int_0^T S_u^2 C_{SS}(S_u,u)d\tilde{W}_u + \frac{1}{2}\int_0^T \frac{k_3}{\sigma} S_u^2 C_{SS}(S_u,u)dB_u + \int_0^T f(S_u,u)du \\ &= \frac{c_1}{2}\int_0^T S_u^2 C_{SS}(S_u,u)d\tilde{W}_u + \frac{1}{2}\int_0^T \frac{k_3}{\sigma^2} S_u C_{SS}(S_u,u)dS_u + \int_0^T g(S_u,u)du, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} f(S_u,u) &= \frac{1}{6}k_3\left\{3S_u^2 C_{SS}(S_u,u) + S_u^3 C_{SSS}(S_u,u)\right\}, \\ g(S_u,u) &= f(S_u,u) - \frac{k_3\mu}{2\sigma^2} S_u^2 C_{SS}(S_u,u). \end{aligned} \tag{3.4}$$

Define a measure P^* by

$$\frac{dP^*}{dP}|_{\mathcal{F}_t} = \exp\left(-\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right), \tag{3.5}$$

where $\{\mathcal{F}_t\}$ is a filtration generated by $(\tilde{W}, B)$. It is easy to show that P^* is an equivalent martingale measure for the stock price process in the limit. This measure is, in fact, the same as the minimal martingale measure introduced by Föllmer and Schweizer (1991).[5] We will use this P^* later in order to prove Theorem 3.2. Since $\tilde{W}$ is independent of dP^*/dP, $\tilde{W}$ remains a standard Brownian motion under P^*. It can also be shown that $\tilde{W}$ is independent of B under P^*.

THEOREM 3.2. *Assume the conditions in Section 2 and 3. The limiting stock price process is governed by (3.2) and R is defined as in (3.1). Also define Y to be $(c_1/2)S^2C_{SS}(S_\cdot,\cdot)$. Then*

$$R_T = \int_0^T Y_u d\tilde{W}_u + \int_0^T \frac{k_3}{2\sigma^2} S_u C_{SS}(S_u,u)dS_u + \int_0^T (T-u)g_S(S_u,u)dS_u. \tag{3.6}$$

In particular, when the second derivative of the Black-Scholes price with respect to S exists at the expiration time,[6] *then*

$$R_T = Y_T\tilde{W}_T + \int_0^T h(\tilde{W}_u, S_u)dS_u + \int_0^T (T-u)g_S(S_u,u)dS_u,$$

[5]Note that, of course, for the filtration generated only by B, (3.5) gives the usual unique equivalent martingale measure in the Black-Scholes model. Here, however, we are dealing with two dimensional processes, $(S^{(n)}, R^{(n)})$, so the limiting filtration is generated by $(\tilde{W}, S)$. Thus, the limiting market is incomplete in the sense that we have two independent Brownian motions but only one traded asset.

[6]For example, the second derivative of a call option is not defined right at the expiration time. Clearly, a call option has the second derivative, $C_{SS}(S_t,t)$, for all $t < T$.

where $h(\tilde{W}_u, S_u) = -c_1\tilde{W}_u\{S_u C_{SS}(S_u,u) + (1/2)S_u^2 C_{SSS}(S_u,u)\} + \{k_3/(2\sigma^2)\}S_u \times C_{SS}(S_u,u)$. $g_S(S_u,u)$ *denotes the first derivative of* $g(S_u,u)$ *in (3.4) with respect to* S.

PROOF. See Appendix. □

From Theorem 3.2, we can see that the limit of the Black-Scholes hedging error at the expiration time is divided into two parts: a replicable part, $\int_0^T [\{k_3/(2\sigma^2)\}S_u C_{SS}(S_u,u) + (T-u)g_S(S_u,u)]dS_u$ and a non-replicable part, $\int_0^T Y_u d\tilde{W}_u$. The replicable part is the stochastic integral with respect to the traded asset S, so we can replicate this object exactly by holding $\{k_3/(2\sigma^2)\}S_t \times C_{SS}(S_t,t) + (T-t)g_S(S_t,t)$ shares of the stock and put everything else in the cash bond at each time t. On the other hand, $\int_0^T Y_u d\tilde{W}_u$ is the stochastic integral with respect to a Brownian motion that is independent of S, so we cannot replicate this by trading the underlying asset S. Now, since we have a replicable part in the Black-Scholes hedging error, we try to update the Black-Scholes hedging strategy by including the replicable part. Define $H^{(n)}$ to be the value of the new hedging portfolio as follows.

$$
\begin{aligned}
H_t^{(n)} &= X_t^{(n)} + \frac{1}{\sqrt{\lambda_n}} \int_0^t \frac{k_3}{2\sigma^2} S_{u-}^{(n)} C_{SS}(S_{u-}^{(n)}, u) dS_u^{(n)} \\
&\quad + \frac{1}{\sqrt{\lambda_n}} \int_0^t (T-u) g_S(S_{u-}^{(n)}, u) dS_u^{(n)}. \qquad (3.7)
\end{aligned}
$$

When C_{SS} is bounded above and bounded away from 0 for all $t \le T$, we define $H^{(n)}$ by including more terms as

$$
H_t^{(n)} = X_t^{(n)} + \frac{1}{\sqrt{\lambda_n}} \int_0^t h(\tilde{W}_{u-}^{(n)}, S_{u-}^{(n)}) dS_u^{(n)} + \frac{1}{\sqrt{\lambda_n}} \int_0^t (T-u) g_S(S_{u-}^{(n)}, u) dS_u^{(n)}, \qquad (3.8)
$$

where $R_t^{(n)} = \sqrt{\lambda_n}\{C(S_t^{(n)}, t) - X_t^{(n)}\}$ and

$$
\begin{aligned}
\tilde{W}_t^{(n)} &= \int_0^t \frac{2}{c_1 {S_{u-}^{(n)}}^2 C_{SS}(S_{u-}^{(n)}, u)} dR_u^{(n)} - \int_0^t \frac{k_3}{c_1 \sigma^2 S_{u-}^{(n)}} dS_u^{(n)} \\
&\quad - \int_0^t \frac{k_3 S_{u-}^{(n)} C_{SSS}(S_{u-}^{(n)}, u)}{3 c_1 C_{SS}(S_{u-}^{(n)}, u)} du + \frac{k_3(\mu - \sigma^2)t}{c_1 \sigma^2}. \qquad (3.9)
\end{aligned}
$$

The function h is the same as before.

After updating Black-Scholes hedging portfolio by $H^{(n)}$, $\sqrt{\lambda_n}\{\eta(S_T^{(n)}) - H_T^{(n)}\}$ is the only uncontrollable part of the payoff. In fact, $\sqrt{\lambda_n}\{\eta(S_T^{(n)}) - H_T^{(n)}\}$ is purely non-replicable in the sense that it does not have any replicable component, because it converges to a stochastic integral with respect to untradable $\tilde{W}$. We can show the following convergence result for the new hedging error $\sqrt{\lambda_n}\{\eta(S_T^{(n)}) - H_T^{(n)}\}$.

THEOREM 3.3. *Assume the conditions of the Theorem 3.2. Then,*

$$\sqrt{\lambda_n}\left(\eta(S_T^{(n)}) - H_T^{(n)}\right) \xrightarrow{\mathcal{D}} \int_0^T Y_u d\tilde{W}_u,$$

where $H^{(n)}$ is defined as in (3.7). Moreover, when $\tilde{W}^{(n)}$ in (3.9) is well-defined, subject to the conditions (A.7) and (A.8) in Appendix,

$$\sqrt{\lambda_n}\left(\eta(S_T^{(n)}) - \dot{H}_T^{(n)}\right) \xrightarrow{\mathcal{D}} Y_T \tilde{W}_T,$$

where $H^{(n)}$ is defined as in (3.8). $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

PROOF. See Appendix. □

When we use $H^{(n)}$, we improve the Black-Scholes hedging strategy in the sense that the mean square of the limiting hedging error is reduced. Note that $H^{(n)}$ is the same as $X^{(n)}$ when the distribution of the log stock price is symmetric up to the order of $1/\sqrt{\lambda_n}$, *i.e.* $k_3 = 0$.

PROPOSITION 3.1. *Assume that $C(S_\cdot, \cdot)$ satisfies $E\{\int_0^T S_t^4 C'^2_{SS}(S_t, t)dt\} < \infty$, additional to the conditions of Theorem 3.2. Note that this additional assumption holds in case of a call option. Under these assumptions,*

$$E\left(\int_0^T Y_u d\tilde{W}_u\right)^2 \leq ER_T^2.$$

In other words,

$$E(\textit{limiting hedging error for } H^{(n)})^2 \leq E(\textit{limiting Black-Scholes hedging error})^2.$$

PROOF. It can be easily shown using $E(\int_0^T Y_u d\tilde{W}_u) = 0$ and the independence between $\tilde{W}$ and S. □

Since the new hedging error $\sqrt{\lambda_n}\{\eta(S_T^{(n)}) - H_T^{(n)}\}$ is purely non-replicable, we may want to use $H^{(n)}$ as the final choice for the hedging portfolio. But we can hope to do something with $\sqrt{\lambda_n}\{\eta(S_T^{(n)}) - H_T^{(n)}\}$ because the limit has the integrand Y that is a function of the underlying asset price process. If we specify a certain optimality criterion, we would be able to find the best possible hedging for the new hedging error. Consider a process

$$K_t^{(n)} = K_0 + \int_0^t \theta_u^{(n)} dS_u^{(n)},$$

where $\theta^{(n)}$ is a predictable process with respect to $\{\mathcal{F}_t^{(n)}\}$ satisfying $E\{\int_0^T (\theta_u^{(n)} \times S_{u-}^{(n)})^2 du\} < \infty$. We want to find an appropriate $\theta^{(n)}$ in order for us to use $K^{(n)}$ to hedge the new hedging error $\sqrt{\lambda_n}\{\eta(S_T^{(n)}) - H_T^{(n)}\}$. Define a process

$$K_t = K_0 + \int_0^t \theta_u dS_u$$

in the limiting market where θ is a predictable process with respect to $\{\mathcal{F}_t\}$ satisfying $E\{\int_0^T (\theta_u S_u^{(n)})^2 du\} < \infty$. Assuming that $\theta^{(n)}$ converges weakly to θ, we can show some weak convergence results for $(H_T^{(n)}, K^{(n)})$ in the next theorem. Note that the second term in the pair can be either a process, $K^{(n)}$, or a random variable $K_T^{(n)}$.

THEOREM 3.4. *Assume the conditions of the Theorem 3.2. Define $H^{(n)}$ and $K^{(n)}$ as before. Suppose that there exists a process θ that is predictable with respect to the limiting filtration $\{\mathcal{F}_t\}$ such that $E(\int_0^T \theta_u^2 S_u^2 du) < \infty$ and $\theta^{(n)} \xrightarrow{\mathcal{D}} \theta$ jointly with $S^{(n)}$ and $R^{(n)}$. Then*

$$\left(\sqrt{\lambda_n}\{\eta(S_T^{(n)}) - H_T^{(n)}\}, K^{(n)}\right) \xrightarrow{\mathcal{D}} \left(\int_0^T Y_u d\tilde{W}_u, K\right),$$

where $H^{(n)}$ is defined as in (3.7). If $\tilde{W}^{(n)}$ in (3.9) is well-defined, subject to the conditions (A.7) and (A.8) in Appendix,

$$\left(\sqrt{\lambda_n}\{\eta(S_T^{(n)}) - H_T^{(n)}\}, K^{(n)}\right) \xrightarrow{\mathcal{D}} \left(Y_T \tilde{W}_T, K\right),$$

where $H^{(n)}$ is defined as in (3.8).

PROOF. See Appendix. □

Now, the value of our new hedging strategy is

$$L_t^{(n)} = H_t^{(n)} + \frac{K_t^{(n)}}{\sqrt{\lambda_n}}. \tag{3.10}$$

$L^{(n)}$ converges to the value of the Black-Scholes hedging portfolio as n goes to ∞, but it includes correction terms for the Black-Scholes hedging error. We will call this as a *compound Poisson hedging strategy* in the rest of the paper.

4. MEAN-VARIANCE HEDGING FOR $\int_0^T Y_u d\tilde{W}_u$

In an incomplete market where contingent claims cannot be hedged perfectly, we need a certain criterion which our hedging strategy is based on. Although there is no uniformly superior hedging strategy found so far, the mean-variance hedging method is one of the most commonly used approaches. It minimizes the global risk over the whole life of contingent claims, in the sense that it minimizes the expected value of the square of the hedging error among all self-financing strategies. To name a few papers on this method, Duffie and Richardson (1991) and Schweizer (1992) studied it in a continuous time setting and Schäl (1994) and Schweizer (1995) examined the discrete time setup.

We will find the mean-variance hedging strategy to the non-replicable part of the limiting Black-Scholes hedging error. Note that everything in the current section is in the limit, in other words, we are going to find a mean-variance hedging strategy that hedges either $\int_0^T Y_u d\tilde{W}_u$ or $Y_T \tilde{W}_T$. The stock price process $\{S_t\}$ follows a geometric Brownian motion as in (3.2).

REMARK 4.1. We may want to find the mean-variance hedging strategy directly from the limit of the Black-Scholes hedging error, R_T in (3.3). In fact, it is not hard to show that we end up with the same strategy as (4.2) when we apply the mean-variance hedging strategy to R_T. The advantage of our method of decomposition is that we can clearly see which part of the Black-Scholes hedging error is completely hedgeable and which part is purely non-hedgeable.

Define M_T to be what we want to hedge; either $\int_0^T Y_u d\tilde{W}_u$ or $Y_T \tilde{W}_T$. First, we consider the case were $M_T = \int_0^T Y_u d\tilde{W}_u$. We want to find K_T minimizing $E(M_T - K_T)^2 = E\{M_T - K_0 - G_T(\theta)\}^2$, where $G_t(\theta) = \int_0^t \theta_u dS_u$ with $\theta \in \Theta = \{\theta : \text{predictable with respect to } \mathcal{F}, E(\int_0^T \theta_u^2 S_u^2 du) < \infty\}$. Define θ_{K_0} to be the $\text{argmin}_{\theta \in \Theta}\, E\{M_T - K_0 - G_T(\theta)\}^2$ for any given value K_0. By Schweizer (1992), $G(\theta_{K_0})$ is a solution, G^*, of the SDE, $dG_t^* = \{\mu/(\sigma^2 S_t)\}(\int_0^t Y_u d\tilde{W}_u - K_0 - G_t^*)dS_t$ with $G_0^* = 0$. Schweizer (1992) assumes that K_0 is negative, but the same argument works for nonnegative K_0's. The explicit form of G^* can be obtained easily and in particular, the explicit solution for our problem is as follows.

PROPOSITION 4.1. *The optimal hedging portfolio, $\{K_t\}$, that makes the expected squared loss, $E(M_T - K_T)^2$, minimized for a given initial value K_0 is*

$$K_t = K_0+\int_0^t \frac{\mu}{\sigma^2 S_u}\left(\frac{S_u}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)u}\left\{\int_0^u \left(\frac{S_v}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)v} Y_v d\tilde{W}_v - K_0\right\} dS_u.$$

Proof is omitted since one can easily solve for K by a simple modification of Schweizer (1992)'s derivation.

In the pre-limiting stage, we adopt the hedging portfolio $K^{(n)}$ for a given K_0 such as

$$\begin{aligned} K_t^{(n)} = K_0 + \int_0^t \frac{\mu}{\sigma^2 S_{u-}^{(n)}} \left(\frac{S_{u-}^{(n)}}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)u} \\ \times \left\{\int_0^u \left(\frac{S_{v-}^{(n)}}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)v} dV_v^{(n)} - K_0\right\} dS_u^{(n)}, \end{aligned} \quad (4.1)$$

where $dV_v^{(n)}$ denotes $dR_v^{(n)} - \{k_3/(2\sigma^2)\}S_{v-}^{(n)} C_{SS}(S_{v-}^{(n)}, v)dS_v^{(n)} - (T-v)g_S(S_{v-}^{(n)}, v)$ $\times dS_v^{(n)}$ and $R_t^{(n)} = \sqrt{\lambda_n}\{C(S_t^{(n)}, t) - X_t^{(n)}\}$.

We considered $M_T = \int_0^T Y_u d\tilde{W}_u$ in Proposition 4.1, but it is easy to see that we end up with the same hedging portfolio when $M_T = Y_T\tilde{W}_T$. The optimal hedging strategy θ_{K_0} will be different, but when we go back to the pre-limiting stage and calculate $L^{(n)}$, we get exactly the same strategy. Thus, in either case of $M_T = \int_0^T Y_u d\tilde{W}_u$ or $Y_T\tilde{W}_T$, we obtain the value of the resulting compound Poisson hedging portfolio as

$$\begin{aligned} L_t^{(n)} &= C(S_0, 0) + \int_0^t C_S(S_{u-}^{(n)}, u)dS_u^{(n)} + \frac{K_0}{\sqrt{\lambda_n}} \\ &+ \frac{1}{\sqrt{\lambda_n}}\int_0^t \frac{k_3}{2\sigma^2} S_{u-}^{(n)} C_{SS}(S_{u-}^{(n)}, u)dS_u^{(n)} + \frac{1}{\sqrt{\lambda_n}}\int_0^t (T-u)g_S(S_{u-}^{(n)}, u)dS_u^{(n)} \\ &+ \frac{1}{\sqrt{\lambda_n}}\int_0^t \frac{\mu}{\sigma^2 S_{u-}^{(n)}}\left(\frac{S_{u-}^{(n)}}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)u} \\ &\times \left\{\int_0^u \left(\frac{S_{v-}^{(n)}}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)v} dV_v^{(n)} - K_0\right\} dS_u^{(n)}. \end{aligned} \quad (4.2)$$

Notice that the compound Poisson hedging strategy, $L^{(n)}$, obtained above is determined uniquely for any given value of K_0. This naturally leads to the question of choice of K_0. Recall that the task in this section is to find $\{K_t\}$ that minimizes

$E(M_T - K_T)^2$. Since we found $\{\theta_{K_0,t}\}$ that minimizes $E(M_T - K_0 - \int_0^T \theta_u dS_u)^2$ for any given value K_0, we now want to find the value of K_0 that minimizes $E(M_T - K_0 - \int_0^T \theta_{K_0,u} dS_u)^2$.

PROPOSITION 4.2. *Assume that* $C(S_\cdot, \cdot)$ *satisfies* $E^*\{\int_0^T S_t^4 C_{SS}^2(S_t,t)dt\} < \infty$. *The value of* K_0 *that minimizes* $E(\int_0^T Y_u d\tilde{W}_u - K_0 - \int_0^T \theta_{K_0,u} dS_u)^2$ *is* $E^*(\int_0^T Y_u d\tilde{W}_u)$, *which is 0, where* E^* *is the expectation under the minimal martingale measure* P^*.

PROOF. To find K_0 that minimizes $E(\int_0^T Y_u d\tilde{W}_u - K_0 - \int_0^T \theta_{K_0,u} dS_u)^2$, we differentiate it with respect to K_0. Note that $G_T^* = \int_0^T \theta_{K_0,u} dS_u$ is a function of K_0.

$$\frac{\partial}{\partial K_0} E\Big(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T^*\Big)^2 = E\Big\{2\Big(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T^*\Big)\Big(-1 - \frac{\partial}{\partial K_0} G_T^*\Big)\Big\}.$$

The differentiation under the expectation can be easily shown to be legitimate. One can also get

$$\begin{aligned} & E\Big\{2\Big(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T^*\Big)\Big(-1 - \frac{\partial}{\partial K_0} G_T^*\Big)\Big\} \\ &= E\Big\{2\Big(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T^*\Big)\Big(-e^{A_T}\Big)\Big\} \\ &= E\Big\{2\Big(K_0 + G_T^* - \int_0^T Y_u d\tilde{W}_u\Big) e^{-\frac{\mu^2}{\sigma^2}T} \frac{dP^*}{dP}|_{\mathcal{F}_T}\Big\} \\ &= E^*\Big\{2e^{-\frac{\mu^2}{\sigma^2}T}\Big(K_0 + G_T^* - \int_0^T Y_u d\tilde{W}_u\Big)\Big\} \\ &= 2e^{-\frac{\mu^2}{\sigma^2}T}\Big\{K_0 - E^*\Big(\int_0^T Y_u d\tilde{W}_u\Big)\Big\}. \end{aligned}$$

G^* is a square-integrable martingale under the minimal martingale measure, so $E^*(G_T^*)$ is 0. Since $E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T^*)^2$ is convex in K_0, $E^*(\int_0^T Y_u d\tilde{W}_u)$ is the value of K_0 that minimizes $\min_{\theta \in \Theta} E\{\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta)\}^2$. $\tilde{W}$ is a Brownian motion under P^*, so $E^*(\int_0^T Y_u d\tilde{W}_u) = 0$. □

Similarly, K_0 that minimizes $E(Y_T \tilde{W}_T - K_0 - \int_0^T \theta_{K_0,u} S_u)^2$ is $E^*(Y_T \tilde{W}_T) = 0$. We can also easily modify the above proposition for more general setting that is given in Schweizer (1992).

The value of the compound Poisson hedging portfolio that we finally propose is $\{L_t^{(n)}\}$ in (4.2) with $K_0 = 0$. With $K_0 = 0$, we achieve $E(M_T - K_T)^2 \leq E(M_T)^2$, *i.e.*, the limiting hedging error for $L^{(n)}$ in (4.2) is smaller than the limiting hedging error for $H^{(n)}$ in (3.7) in terms of their mean squares. Combining this with Proposition 3.1, we get

$$E(\text{limiting Black-Scholes hedging error})^2 \geq E(\text{limiting hedging error for } H^{(n)})^2$$
$$\geq E(\text{limiting hedging error for } L^{(n)})^2.$$

5. SIMULATION

This section presents numerical results on the compound Poisson hedging strategy. Consider a European call option that expires in 3 months. The interest rate is assumed to be 0, μ is set to be 0.15 per annum and σ is set to be 0.2 per annum. We try the strike price $K = \$65$ and the initial stock price $S_0 = \$60$. We use three different jump intensities, $\lambda_n = 1,000$, $10,000$ and $100,000$. $\lambda_n = 10,000$ means that we expect 10,000 jumps per year in the stock price on average. Larger λ_n implies that the stock is more frequently traded. The hedging interval is 0.0001 years which means that we rebalance the hedging portfolio once in approximately one hour.

Any distribution with the moment conditions given in Section 2 can be used as the jump size distribution for the compound Poisson model. For example, $N(0, \sigma^2)$ can be used for the distribution of Q in (2.2) as a symmetric jump size case and $\sigma - \text{Exp}(1/\sigma)$ can be used as a left skewed jump size case. We use $\sigma - \text{Exp}(1/\sigma)$ as the distribution of Q in the simulation experiment. With this distribution, $k_3 = -2\sigma^3$ and $k_4 = 9\sigma^4$. The simulation size is 5,000, that is, the number of generated sample paths is 5,000. The Black-Scholes initial price is \$0.75, in this case.

We compare the performance of the Black-Scholes and the compound Poisson hedging strategies by calculating the mean squares of hedging errors (MSHE). By hedging error, we mean the option payoff subtracted by the value of the hedging portfolio at the expiration. For example, the MSHE of the Black-Scholes strategy is

$$E\left\{\eta(S_T^{(n)}) - C(S_0, 0) - \int_0^T C_S(S_{t-}^{(n)}, t)dS_t^{(n)}\right\}^2.$$

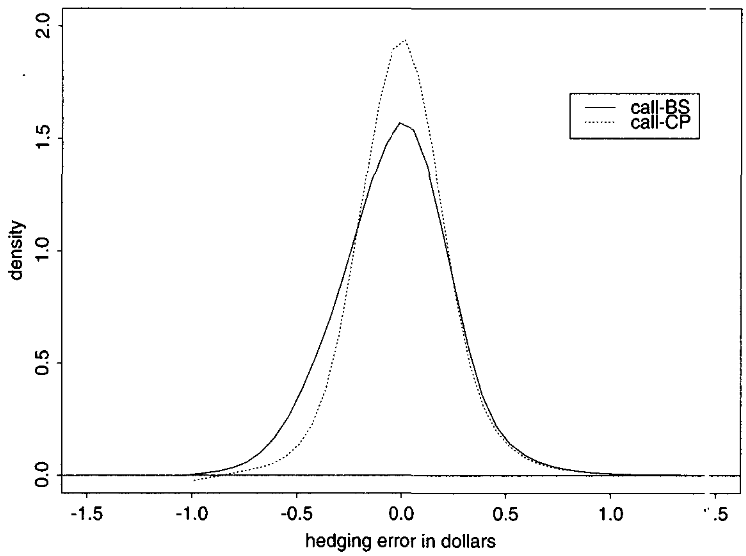

FIGURE 5.1 *Comparison of densities of hedging errors. Solid line is for Black-Scholes hedging error and dotted line is for compound Poisson hedging error. The jump size distribution is left-skewed and the jump intensity is 1000.*

In general, both of the hedging strategies perform better as λ_n gets larger in terms of the magnitude of MSHE, because the stock price process is getting closer to a geometric Brownian motion. Nevertheless, the compound Poisson hedging strategy provides smaller MSHE than the Black-Scholes hedging strategy overall. We also compare densities of hedging errors in Figure 5.1. The compound Poisson hedging error (dotted line) has less spread and is more symmetric than the Black-Scholes hedging error (solid line). In other words, the compound Poisson hedging strategy makes the distribution of hedging errors less biased as well as it makes the distribution less variable.[7]

The dollar terms of the mean square of hedging errors in Table 5.1 are small,

[7]We have also run a simulation when the underlying process is in fact a geometric Brownian Motion. In this case, k_3 will be 0, so $H_t^{(n)}$ is the same as the Black-Scholes hedging strategy. Since the Black-Scholes hedging strategy is a perfect one, the MSHE of Black-Scholes must be 0 and the MSHE of the compound Poisson strategy is $E(K_T^{(n)}/\sqrt{\lambda_n})^2$. With 5,000 sample paths and the same parameter values given in this section, the MSHE of the compound Poisson hedging strategy when $\lambda_n = 1,000$ is 0.0000102. We can see that the compound Poisson strategy can be used safely when the underlying process is a geometric Brownian Motion.

but the difference in MSHE between strategies is not negligible in terms of percentage. For example, the percentage gain in MSHE by using the compound Poisson hedging strategy over the Black-Scholes when λ_n is 1,000 is about 35%. Moreover, if we have different values of parameters, then we may also obtain more reduction in dollar terms.

TABLE 5.1 *Mean squares of hedging errors, unit=^{2}*

	$\lambda_n = 1,000$	$\lambda_n = 10,000$	$\lambda_n = 100,000$
$call - BS$	0.059172	0.007119	0.002272
$call - CP$	0.038612	0.005037	0.001976
$reduction(BS\ vs.\ CP)$	34.7%	29.3%	13.0%

6. CONCLUDING REMARK

In this paper, we have studied the problem of hedging derivative securities under a pure jump model. We use asymptotic theory to find a correction term to the Black-Scholes delta, which is a different viewpoint from many previous papers on the hedging problem in an incomplete market. When the asset price follows a compound Poisson process that converges to a geometric Brownian motion as the jump intensity increases, we obtained a new hedging strategy by dealing with the first order hedging error of the classical Black-Scholes hedging strategy.

The new hedging strategy performs better than Black-Scholes hedging strategy in terms of the mean square of the hedging error. Asymptotically, the mean square of the hedging error for $H^{(n)}$ defined in (3.7) is at least as small as the mean square of the hedging error for the Black-Scholes hedging strategy. Moreover, the mean square of the hedging error of the compound Poisson hedging strategy $L^{(n)}$ in (4.2) with $K_0 = 0$ is at least as small as the mean square of the hedging error for $H^{(n)}$. $K_0 = 0$ means that the initial investment of the compound Poisson hedging strategy is equal to the Black-Scholes option price. This is also consistent with the industry practice of using the Black-Scholes option price, but using a different hedging strategy rather than the Black-Scholes delta.

The simulation study shows that the smaller hedging error of the compound Poisson strategy is obtained even when n is not too big. When the jump intensity λ_n is 1,000, we obtained more than 30% reduction in the mean square of the hedging error.

What we have done here can be generalized to Lévy processes with proper conditions and we will leave it for future work.

APPENDIX

Proof of Theorem 3.1

LEMMA A.1. *Define* $\xi_t^{(n)} = \sqrt{\lambda_n}\{[\log S^{(n)}, \log S^{(n)}]_t - \sigma^2 t\}$ *and* $\xi_t = c_1 \tilde{W}_t + (k_3/\sigma) B_t$. c_1 *is* $\sqrt{k_4 - (k_3/\sigma)^2}$, *and* $\tilde{W}$ *and* B *are independent Brownian motions under the limiting measure* P. *Then,*

$$(\log S^{(n)}, \xi^{(n)}) \xrightarrow{\mathcal{D}} (\log S, \xi).$$

PROOF. Define $M_t^{(n)}$ to be $\sqrt{\lambda_n}\{[\log S^{(n)}, \log S^{(n)}]_t - \langle \log S^{(n)}, \log S^{(n)} \rangle_t\}$. Then $\xi_t^{(n)} = M_t^{(n)} + (1/\sqrt{\lambda_n})(\mu - \sigma^2/2)^2 t$. Since $(1/\sqrt{\lambda_n})(\mu - \sigma^2/2)^2 t$ is $o(1)$, it suffices to show that

$$(\log S^{(n)}, M^{(n)}) \xrightarrow{\mathcal{D}} (\log S, \xi).$$

Denote $\hat{M}_t^{(n)}$ to be $\log S_t^{(n)} - \log S_0^{(n)} - (\mu - \sigma^2/2)t$. Then $M^{(n)}$ and $\hat{M}^{(n)}$ are square-integrable martingales satisfying

$$\begin{aligned}
\langle M^{(n)}, M^{(n)} \rangle_t &= \lambda_n \langle \log S^{(n)}, \log S^{(n)}, \log S^{(n)}, \log S^{(n)} \rangle_t \\
&= \left\{ \frac{k_4}{\lambda_n^2} + o(\lambda_n^{-2}) \right\} \lambda_n^2 t = k_4 t + o(1) \underset{n\to\infty}{\longrightarrow} k_4 t, \\
\langle \hat{M}^{(n)}, \hat{M}^{(n)} \rangle_t &= \sigma^2 t + \frac{1}{\lambda_n} \left(\mu - \frac{1}{2}\sigma^2 \right) t \underset{n\to\infty}{\longrightarrow} \sigma^2 t,
\end{aligned}$$

and

$$\langle M^{(n)}, \hat{M}^{(n)} \rangle_t = \sqrt{\lambda_n} \langle \log S^{(n)}, \log S^{(n)}, \log S^{(n)} \rangle_t \underset{n\to\infty}{\longrightarrow} k_3 t.$$

Define

$$M_{\epsilon,t}^{(n)} = \sqrt{\lambda_n} \left[\sum_{i=1}^{N_t^{(n)}} (Z_i^{(n)})^2 I\Big(|Z_i^{(n)}|^2 > \frac{\epsilon}{\sqrt{\lambda_n}}\Big) - \lambda_n t E\Big\{(Z^{(n)})^2 I\Big(|Z^{(n)}|^2 > \frac{\epsilon}{\sqrt{\lambda_n}}\Big)\Big\} \right]$$

and

$$\hat{M}_{\epsilon,t}^{(n)} = \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)} I(|Z_i^{(n)}| > \epsilon) - \lambda_n t E\left\{ Z^{(n)} I(|Z^{(n)}| > \epsilon) \right\}.$$

Here, $I(\cdot)$ denotes the indicator function. Then $M_\epsilon^{(n)}$ and $\hat{M}_\epsilon^{(n)}$ are square-integrable martingales and they include all the jumps in absolute value greater than ϵ, for a given ϵ, of $M^{(n)}$ and $\hat{M}^{(n)}$, respectively. Their predictable quadratic variations are

$$\langle M_\epsilon^{(n)}, M_\epsilon^{(n)}\rangle_t = t\cdot E\left\{\lambda_n^2 (Z^{(n)})^4 I\Big(|Z^{(n)}|^2 > \frac{\epsilon}{\sqrt{\lambda_n}}\Big)\right\},$$
$$\langle \hat{M}_\epsilon^{(n)}, \hat{M}_\epsilon^{(n)}\rangle_t = t\cdot E\big\{\lambda_n (Z^{(n)})^2 I\big(|Z^{(n)}| > \epsilon\big)\big\}.$$

$\langle M_\epsilon^{(n)}, M_\epsilon^{(n)}\rangle_t \longrightarrow 0$ and $\langle \hat{M}_\epsilon^{(n)}, \hat{M}_\epsilon^{(n)}\rangle_t \longrightarrow 0$ as $n\to\infty$ because $E\{\lambda_n^2(Z^{(n)})^4\} < \infty$, $P(|Z^{(n)}|^2 > \epsilon/\sqrt{\lambda_n}) \longrightarrow 0$ and $P(|Z^{(n)}| > \epsilon) \longrightarrow 0$ as $n\to\infty$. By Rebolledo's theorem (Andersen *et al.*, 1993, p.83), $(M^{(n)}, \hat{M}^{(n)}) \xrightarrow{\mathcal{D}} (\xi, \sigma B)$ and therefore, by Proposition VI. 3.17 in Jacod and Shiryaev (1987),

$$(\log S^{(n)}, \xi^{(n)}) \xrightarrow{\mathcal{D}} (\log S, \xi). \qquad \square$$

Let us now define $R^{(n)}$ as a process, $\sqrt{\lambda_n}(C(S_\cdot,\cdot) - X_\cdot^{(n)})$. Since we are assuming $E|Z^{(n)}|^p = o(\lambda_n^{-2})$ for $p = 5, 6, \ldots$, $(\Delta \log S_t^{(n)})^p = (Z^{(n)}_{N_t^{(n)}})^p I(\Delta N_t^{(n)} = 1) = o_p(\lambda_n^{-2})$ for $p = 5, 6, \ldots$. We also know that $(\Delta \log S_t^{(n)})^3 = O_p(\lambda_n^{-3/2})$ and $(\Delta \log S_t^{(n)})^4 = O_p(\lambda_n^{-2})$. By Itô's formula and Taylor expansion,

$$\begin{aligned}
R_t^{(n)} &= \sqrt{\lambda_n}\left\{C(S_t^{(n)}, t) - C(S_0, 0) - \int_0^t C_S(S_{u-}^{(n)}, u)\, dS_u^{(n)}\right\}\\
&= \sqrt{\lambda_n}\left[\int_0^t C_t(S_{u-}^{(n)}, u)\, du + \sum_{u\le t}\left\{\Delta C(S_u^{(n)}, u) - C_S(S_{u-}^{(n)}, u)\Delta S_u^{(n)}\right\}\right]\\
&= \sqrt{\lambda_n}\left\{\int_0^t C_t(S_{u-}^{(n)}, u)\, du + \sum_{u\le t}\frac{1}{2} C_{SS}(S_{u-}^{(n)}, u)(\Delta S_u^{(n)})^2\right.\\
&\qquad \left. + \sum_{u\le t}\frac{1}{6} C_{SSS}(\tilde{Z}_u^{(n)}, u)(\Delta S_u^{(n)})^3\right\},
\end{aligned}$$

where $\tilde{Z}^{(n)}$ is a process satisfying $\min(S_{t-}^{(n)}, S_t^{(n)}) \le \tilde{Z}_t^{(n)} \le \max(S_{t-}^{(n)}, S_t^{(n)})$, for all $0 < t < T$. By Black-Scholes PDE and using $\Delta S_t^{(n)} = S_{t-}^{(n)}(e^{\Delta \log S_t^{(n)}} - 1)$,

$$\begin{aligned}
R_t^{(n)} &= -\frac{\sqrt{\lambda_n}}{2}\int_0^t \sigma^2 C_{SS}(S_{u-}^{(n)}, u)(S_{u-}^{(n)})^2\, du\\
&\quad + \frac{\sqrt{\lambda_n}}{2}\sum_{u\le t} C_{SS}(S_u^{(n)}, u)(S_{u-}^{(n)})^2 (e^{\Delta \log S_u^{(n)}} - 1)^2\\
&\quad + \frac{\sqrt{\lambda_n}}{6}\sum_{u<t} C_{SSS}(\tilde{Z}_u^{(n)}, u)(S_{u-}^{(n)})^3 (e^{\Delta \log S_u^{(n)}} - 1)^3.
\end{aligned} \qquad \text{(A.1)}$$

We can rewrite $R_t^{(n)}$ as

$$\begin{aligned}R_t^{(n)} &= \frac{1}{2}\int_0^t C_{SS}(S_{u-}^{(n)},u)(S_u^{(n)})^2 d\xi_u^{(n)} \\ &+\sqrt{\lambda_n}\sum_{u\le t}\frac{1}{6}\left\{3C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2 + C_{SSS}(\tilde{Z}_u^{(n)},u)(S_{u-}^{(n)})^3\right\}(\Delta\log S_u^{(n)})^3 \\ &+\sqrt{\lambda_n}\sum_{u\le t}\frac{1}{4}\left\{\frac{7}{6}C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2 + C_{SSS}(\tilde{Z}_u^{(n)},u)(S_{u-}^{(n)})^3\right\}(\Delta\log S_u^{(n)})^4 \\ &+\sum_{u\le t}\Big\{C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2 \times o_p(\lambda_n^{-3/2}) \\ &\qquad +C_{SSS}(\tilde{Z}_u^{(n)},u)(S_{u-}^{(n)})^3 \times o_p(\lambda_n^{-3/2})\Big\}. \qquad (A.2)\end{aligned}$$

We can also show the followings.

(i) By Lemma VI. 3.31 in Jacod and Shiryaev (1987) and Lemma A.1, $(\log\tilde{Z}^{(n)}, \log S^{(n)}, \xi^{(n)}) \xrightarrow{\mathcal{D}} (\log S, \log S, \xi)$, since $(\log S^{(n)}, \log S^{(n)}, \xi^{(n)}) \xrightarrow{\mathcal{D}} (\log S, \log S, \xi)$, and $\sup_{t\le T}|\log\tilde{Z}_t^{(n)} - \log S_t^{(n)}| < \sup_{t\le T}|\Delta\log S_t^{(n)}| = o_p(1)$.

(ii) By the weak law of large numbers and Doeblin-Anscombe's theorem (Chow and Teicher, 1997),

$$\sum_{u\le t}\sqrt{\lambda_n}(\Delta\log S_u^{(n)})^3 = \frac{1}{\lambda_n}\sum_{i=1}^{N_t^{(n)}}\left\{Q_i + \frac{1}{\sqrt{\lambda_n}}\left(\mu - \frac{1}{2}\sigma^2\right)\right\}^3 \xrightarrow{\mathcal{D}} k_3 t,$$

$$\sum_{u\le t}\sqrt{\lambda_n}(\Delta\log S_u^{(n)})^4 = \frac{1}{\lambda_n^{3/2}}\sum_{i=1}^{N_t^{(n)}}\left\{Q_i + \frac{1}{\sqrt{\lambda_n}}\left(\mu - \frac{1}{2}\sigma^2\right)\right\}^4 \xrightarrow{\mathcal{D}} 0.$$

(iii) By Theorem 2.7 in Kurtz and Protter (1991) and (i) and (ii) above,

$$\begin{aligned}&\Bigg(\log\tilde{Z}^{(n)}, \log S^{(n)}, \xi^{(n)}, \frac{1}{2}\int_0^{\cdot}(S_{u-}^{(n)})^2 C_{SS}(S_{u-}^{(n)},u)d\xi_u^{(n)}, \\ &\quad\sqrt{\lambda_n}\sum_{u\le\cdot}\frac{1}{6}\left\{3C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2 + C_{SSS}(\tilde{Z}_u^{(n)},u)(S_{u-}^{(n)})^3\right\}(\Delta\log S_u^{(n)})^3, \\ &\quad\sqrt{\lambda_n}\sum_{u\le\cdot}\frac{1}{4}\left\{\frac{7}{6}C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2 + C_{SSS}(\tilde{Z}_u^{(n)},u)(S_{u-}^{(n)})^3\right\}(\Delta\log S_u^{(n)})^4\Bigg) \\ &\xrightarrow{\mathcal{D}} \Bigg(\log S,\ \log S,\ \xi,\ \frac{1}{2}\int_0^{\cdot} S_u^2 C_{SS}(S_u,u)d\xi_u, \\ &\qquad \int_0^{\cdot}\frac{k_3}{6}\left\{3C_{SS}(S_u,u)S_u^2 + C_{SSS}(S_u,u)S_u^3\right\}du,\ 0\Bigg).\end{aligned}$$

For this, we check conditions including the followings.

– For $\xi^{(n)}$,

$$\begin{aligned}\xi_t^{(n)} &= \sqrt{\lambda_n}\Big([\log S^{(n)}, \log S^{(n)}]_t - \langle \log S^{(n)}, \log S^{(n)}\rangle_t\Big)\\ &\quad + \sqrt{\lambda_n}\Big(\langle \log S^{(n)}, \log S^{(n)}\rangle_t - \sigma^2 t\Big)\\ &=: M_{\xi,t}^{(n)} + A_{\xi,t}^{(n)},\end{aligned}$$

and $\sup_n E\{[M_\xi^{(n)}, M_\xi^{(n)}]_t + T_t(A_\xi^{(n)})\} < \infty$. $T_t(A)$ is the total variation of a process A on $[0,t]$.

– For $\sqrt{\lambda_n}\sum_{u\le t}(\Delta \log S_u^{(n)})^3$,

$$\begin{aligned}\sqrt{\lambda_n}\sum_{u\le t}(\Delta \log S_u^{(n)})^3 &= \sqrt{\lambda_n}\Big([\log S^{(n)}, \log S^{(n)}, \log S^{(n)}]_t\\ &\qquad - \langle \log S^{(n)}, \log S^{(n)}, \log S^{(n)}\rangle_t\Big)\\ &\quad + \sqrt{\lambda_n}\langle \log S^{(n)}, \log S^{(n)}, \log S^{(n)}\rangle_t\\ &=: M_{(\Delta \log S)^3,t}^{(n)} + A_{(\Delta \log S)^3,t}^{(n)},\end{aligned}$$

and $\sup_n E\{[M_{(\Delta \log S)^3}^{(n)}, M_{(\Delta \log S)^3}^{(n)}]_t + T_t(A_{(\Delta \log S)^3}^{(n)})\} < \infty$.

– For $\sqrt{\lambda_n}\sum_{u\le t}(\Delta \log S_u^{(n)})^4$,

$$\sqrt{\lambda_n}\sum_{u\le t}(\Delta \log S_u^{(n)})^4 =: M_{(\Delta \log S)^4,t}^{(n)} + A_{(\Delta \log S)^4,t}^{(n)},$$

and $\sup_n E\{[M_{(\Delta \log S)^4}^{(n)}, M_{(\Delta \log S)^4}^{(n)}]_t + T_t(A_{(\Delta \log S)^4}^{(n)})\} < \infty$.

(iv) The last term in (A.2) has a higher order than

$$\sqrt{\lambda_n}\sum_{u\le \cdot}\frac{1}{4}\Big\{\frac{7}{6}C_{SS}(S_{u-}^{(n)}, u)(S_{u-}^{(n)})^2 + C_{SSS}(\tilde{Z}_u^{(n)}, u)(S_{u-}^{(n)})^3\Big\}(\Delta \log S_u^{(n)})^4.$$

Therefore,

$$R_t^{(n)} \xrightarrow{\mathcal{D}} \int_0^t \frac{1}{2}S_u^2 C_{SS}(S_u, u)d\xi_u + \int_0^t \frac{k_3}{6}\Big\{3S_u^2 C_{SS}(S_u, u) + S_u^3 C_{SSS}(S_u, u)\Big\}du,$$

as a process jointly with $S^{(n)}$.

Proof of Theorem 3.2

From (3.3),

$$R_T = \int_0^T Y_u d\tilde{W}_u + \frac{1}{2}\int_0^T \frac{k_3}{\sigma^2} S_u C_{SS}(S_u, u) dS_u + \int_0^T g(S_u, u) du.$$

We want to show that

$$\int_0^T g(S_u, u) du = \int_0^T (T-u) g_S(S_u, u) dS_u.$$

Since the interest rate r is assumed to be 0, S is a martingale under the minimal martingale measure P^* as $dS_u = \sigma S_u dB_u^*$, where B^* is a Brownian motion under P^*. Let $\{\tilde{\mathcal{F}}_t\}$ be an augmentation of the filtration generated by B^*. Then $\int_0^T g(S_u, u) du$ is $\tilde{\mathcal{F}}_T$-measurable and finite almost surely. By Dudley's theorem (Karatzas and Shreve, 1991, p.188 or Duffie, 1996, p.287), there exists a progressively measurable process $\tilde{Y} = \{\tilde{Y}_t, \tilde{\mathcal{F}}_t; 0 \le t \le T\}$ satisfying $\int_0^T \tilde{Y}_t^2 dt < \infty$ almost surely under P^* such that

$$\int_0^T g(S_u, u) du = \int_0^T \tilde{Y}_u dB_u^* = \int_0^T \tilde{Y}_u \frac{1}{\sigma S_u} dS_u. \tag{A.3}$$

Note that $g(S_u, u)$ is a P^*-martingale because $S_t^p C_S^{(p)}(S_t, t)$ is a P^*-martingale for any positive integer p. Define ζ_t to be $E^*\{\int_0^T g(S_u, u) du \mid \mathcal{F}_t\}$. Then,

$$\begin{aligned}
\zeta_t &= \int_0^t g(S_u, u) du + E^*\left\{\int_t^T g(S_u, u) du \mid \mathcal{F}_t\right\} \\
&= \int_0^t g(S_u, u) du + (T-t) g(S_t, t).
\end{aligned} \tag{A.4}$$

On the other hand,

$$\zeta_t = E^*\left\{\int_0^T g(S_u, u) du \mid \mathcal{F}_t\right\} = E^*\left(\int_0^T \tilde{Y}_u \frac{1}{\sigma S_u} dS_u \mid \mathcal{F}_t\right) = \int_0^t \tilde{Y}_u \frac{1}{\sigma S_u} dS_u.$$

Thus,

$$\langle \zeta\ , B^* \rangle_t = \int_0^t \tilde{Y}_u \frac{1}{\sigma S_u} d\langle S, B^* \rangle_u = \int_0^t \tilde{Y}_u \frac{1}{\sigma S_u} \sigma S_u du = \int_0^t \tilde{Y}_u du,$$

and $\tilde{Y}_t = \frac{d}{dt}\langle \zeta\ , B^* \rangle_t$. By the way, from (A.4),

$$\begin{aligned}
d\zeta_t &= g(S_t, t) dt - g(S_t, t) dt + (T-t) dg(S_t, t) \\
&= (T-t) dg(S_t, t) \\
&= (T-t)\left\{g_S(S_t, t) dS_t + g_t(S_t, t) dt + \frac{1}{2} g_{SS}(S_t, t) \sigma^2 S_t^2 dt\right\}.
\end{aligned}$$

Thus,

$$\tilde{Y}_t = \frac{d}{dt}\langle \zeta\ , B^*\rangle_t = (T-t)g_S(S_t,t)\sigma S_t. \tag{A.5}$$

This process is progressively measurable and satisfies $\int_0^T \tilde{Y}_t^2 dt < \infty$ almost surely. Combining (A.3) and (A.5),

$$R_T = \int_0^T Y_u d\tilde{W}_u + \int_0^T \frac{k_3}{2\sigma^2} S_u C_{SS}(S_u,u)dS_u + \int_0^T (T-u)g_S(S_u,u)dS_u.$$

In case where the second derivative of the Black-Scholes price exists at time T, by applying Itô's formula to $\int_0^T Y_u d\tilde{W}_u$, we get

$$R_T = Y_T\tilde{W}_T - \int_0^T \tilde{W}_u dY_u + \frac{1}{2}\int_0^T \frac{k_3}{\sigma^2} S_u C_{SS}(S_u,u)dS_u + \int_0^T g(S_u,u)du.$$

Again, apply the Itô's formula to Y_t to obtain

$$\begin{aligned} dY_t &= c_1\Big\{S_t C_{SS}(S_t,t) + \frac{1}{2}S_t^2 C_{SSS}(S_t,t)\Big\}dS_t + \frac{c_1}{2}S_t^2\Big[C_{SSt}(S_t,t) \\ &\quad + \frac{1}{2}\sigma^2\{2C_{SS}(S_t,t) + 4S_t C_{SSS}(S_t,t) + S_t^2 C_S^{(4)}(S_t,t)\}\Big]dt. \end{aligned}$$

On the other hand, we know from the Black-Scholes PDE, $C_t + \sigma^2 S^2 C_{SS}/2 = 0$ when $r = 0$. Thus,

$$\frac{\partial^2}{\partial S^2}\Big(C_t + \frac{1}{2}\sigma^2 S^2 C_{SS}\Big) = C_{SSt} + \frac{1}{2}\sigma^2(2C_{SS} + 4SC_{SSS} + S^2 C_S^{(4)}) = 0.$$

Therefore, $dY_t = c_1\{S_t C_{SS}(S_t,t) + (1/2)S_t^2 C_{SSS}(S_t,t)\}dSt$ and R_T becomes

$$\begin{aligned} R_T &= Y_T\tilde{W}_T - c_1\int_0^T \tilde{W}_u\Big\{S_u C_{SS}(S_u,u) + \frac{1}{2}S_u^2 C_{SSS}(S_u,u)\Big\}dS_u \\ &\quad + \frac{1}{2}\int_0^T \frac{k_3}{\sigma^2} S_u C_{SS}(S_u,u)dS_u + \int_0^T g(S_u,u)du \\ &= Y_T\tilde{W}_T + \int_0^T h(\tilde{W}_u,S_u)dS_u + \int_0^T g(S_u,u)du, \end{aligned} \tag{A.6}$$

where $h(\tilde{W}_u,S_u) = -c_1\tilde{W}_u\{S_u C_{SS}(S_u,u) + (1/2)S_u^2 C_{SSS}(S_u,u)\} + \{k_3/(2\sigma^2)\}$ $S_u C_{SS}(S_u,u)$. Combining (A.6) and previous arguments,

$$R_T = Y_T\tilde{W}_T + \int_0^T h(\tilde{W}_u,S_u)dS_u + \int_0^T (T-u)g_S(S_u,u)dS_u.$$

Proof of Theorems 3.3 and 3.4

Let $M_t^{(n)}$ be $\sqrt{\lambda_n}\{C(S_t^{(n)},t) - H_t^{(n)}\}$. First, for $H_t^{(n)}$ is defined as in (3.7), we want to show

$$M_T^{(n)} \xrightarrow{\mathcal{D}} \int_0^T Y_u d\tilde{W}_u.$$

(i) We know that $(R^{(n)}, S^{(n)})$ converges to (R, S) weakly. By the continuous mapping theorem, for any continuous function f,

$$\left(S^{(n)}, R^{(n)}, f(S^{(n)})\right) \xrightarrow{\mathcal{D}} \left(S, R, f(S)\right).$$

(ii) $S^{(n)}$, $R^{(n)}$, $f(S^{(n)})$ are adapted to the filtration generated by $S^{(n)}$, cádlág processes, and $S^{(n)}$ is a semimartingale. Suppose $\hat{M}^{(n)}$ is a martingale of $S^{(n)}$ and $\hat{A}^{(n)}$ is the finite variation process in the Doob-Meyer decomposition (Karatzas and Shreve, 1991, p.24–25). Then

$$\hat{M}_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} S_{\tau_{i-}^{(n)}}^{(n)} \left\{\exp(Z_i^{(n)}) - 1\right\} - E\left\{\exp(Z^{(n)}) - 1\right\} \int_0^t S_{u-}^{(n)} \lambda_n du,$$

and

$$\hat{A}_t^{(n)} = E\left\{\exp(Z^{(n)}) - 1\right\} \int_0^t S_{u-}^{(n)} \lambda_n du.$$

$\tau_i^{(n)}$ is the time of the i^{th} jump of $\log S^{(n)}$. We can show that $\sup_n E[\hat{M}^{(n)}, \hat{M}^{(n)}]_t < \infty$, since

$$\begin{aligned} E[\hat{M}^{(n)}, \hat{M}^{(n)}]_t &= E(e^{Z^{(n)}} - 1)^2 \int_0^t (S_{u-}^{(n)})^2 \lambda_n du \\ &= \{\mu + O(\lambda_n^{-1/2})\} \exp\left\{2(\mu + \frac{1}{2}\sigma^2)t + O(\lambda_n^{-1/2})\right\}. \end{aligned}$$

The total variation of $\hat{A}^{(n)}$ is

$$\begin{aligned} T_t(\hat{A}^{(n)}) &= \sup_{m\geq 1} \sum_{k=1}^{2^m} \left| \hat{A}^{(n)}\left(\frac{tk}{2^m}\right) - \hat{A}^{(n)}\left(\frac{t(k-1)}{2^m}\right) \right| \\ &= |E\{\exp(Z^{(n)}) - 1\}| \int_0^t S_{u-}^{(n)} \lambda_n du, \end{aligned}$$

because $\hat{A}^{(n)}$ is monotone. Since $E\{\exp(Z^{(n)}) - 1\} = \mu\lambda_n^{-1} + O(\lambda_n^{-3/2})$,

$$E\{T_t(\hat{A}^{(n)})\} = |\mu + O(\lambda_n^{-1/2})| S_0^{(n)} \int_0^t \exp\left\{\mu u + O(\lambda_n^{-1/2})\right\} du$$

and $\sup_n E\{T_t(\hat{A}^{(n)})\} < \infty$. Thus, $S^{(n)}$ satisfies the condition of the integrator in Theorem 2.7 in Kurtz and Protter (1991) with $\tau_n^\alpha = T \vee (\alpha+1)$ Therefore, $\left(S^{(n)}, R^{(n)}, \int f(S^{(n)})dS^{(n)}\right) \xrightarrow{\mathcal{D}} \left(S, R, \int f(S)dS\right)$, for any continuous function f.

(iii) By (i) and (ii),

$$\begin{aligned} M_T^{(n)} &= R_T^{(n)} - \int_0^T \left\{ \frac{k_3}{2\sigma^2} S_{u-}^{(n)} C_{SS}(S_{u-}^{(n)}, u) + (T-u) g_S(S_{u-}^{(n)}, u) \right\} dS_u^{(n)} \\ &\xrightarrow{\mathcal{D}} R_T - \int_0^T \left\{ \frac{k_3}{2\sigma^2} S_u C_{SS}(S_u, u) + (T-u) g_S(S_u, u) \right\} dS_u \\ &= \int_0^T Y_u d\tilde{W}_u. \end{aligned}$$

Secondly, when $\tilde{W}^{(n)}$ in (3.9) is well-defined, we want to show

$$M_T^{(n)} \xrightarrow{\mathcal{D}} Y_T \tilde{W}_T,$$

with $H_t^{(n)}$ defined as in (3.8). Assume the following conditions.

$$\begin{aligned} &\sup_n E \int_0^T |C_{SS}(S_{u-}^{(n)}, u)| (S_{u-}^{(n)})^2 du < \infty, \\ &\sup_n E \int_0^T \sup_{x\in D} |C_{SSS}(x, u)(S_{u-}^{(n)})^3| du < \infty, \end{aligned} \tag{A.7}$$

where D is the interval $\left(\min(S_{u-}^{(n)}, S_u^{(n)}), \max(S_{u-}^{(n)}, S_u^{(n)})\right)$ and

$$\begin{aligned} &\sup_n E \int_0^T C_{SS}^2(S_{u-}^{(n)}, u)(S_{u-}^{(n)})^4 du < \infty, \\ &E \int_0^T \sup_{x\in D} |C_{SS}(S_{u-}^{(n)}, u) C_{SSS}(x, u)(S_{u-}^{(n)})^5| du = O(\lambda_n^{1/2}), \\ &E \int_0^T \sup_{x\in D} C_{SSS}^2(x, u)(S_{u-}^{(n)})^6 du = O(\lambda_n). \end{aligned} \tag{A.8}$$

(i) From (A.1),

$$\begin{aligned} R_t^{(n)} = &-\frac{\sigma^2\sqrt{\lambda_n}}{2} \int_0^t C_{SS}(S_{u-}^{(n)}, u)(S_{u-}^{(n)})^2 du \\ &+ \frac{\sqrt{\lambda_n}}{2} \int_0^t C_{SS}(S_{u-}^{(n)}, u) d[S^{(n)}, S^{(n)}]_u \\ &+ \frac{\sqrt{\lambda_n}}{6} \int_0^t C_{SSS}(\tilde{Z}_u^{(n)}, u) d[S^{(n)}, S^{(n)}, S^{(n)}]_u, \end{aligned}$$

since $S^{(n)}$ is a pure jump process. Suppose $\tilde{M}^{(n)}$ is the local martingale part of $R^{(n)}$ and $\tilde{A}^{(n)}$ is the finite variation process in the Doob-Meyer decomposition. Then

$$\begin{aligned}\tilde{A}_t^{(n)} &= -\frac{\sigma^2\sqrt{\lambda_n}}{2}\int_0^t C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2du \\ &+ \frac{\sqrt{\lambda_n}}{2}\int_0^t C_{SS}(S_{u-}^{(n)},u)E(e^{Z^{(n)}}-1)^2\lambda_n(S_{u-}^{(n)})^2du \\ &+ \frac{\sqrt{\lambda_n}}{6}\int_0^t C_{SSS}(\tilde{Z}_u^{(n)},u)E(e^{Z^{(n)}}-1)^3\lambda_n(S_{u-}^{(n)})^3du,\end{aligned}$$

and

$$\begin{aligned}\tilde{M}_t^{(n)} &= \frac{\sqrt{\lambda_n}}{2}\int_0^t C_{SS}(S_{u-}^{(n)},u)d\big([S^{(n)},S^{(n)}]_u - \langle S^{(n)},S^{(n)}\rangle_u\big) \\ &+ \frac{\sqrt{\lambda_n}}{6}\int_0^t C_{SSS}(\tilde{Z}_u^{(n)},u)d\big([S^{(n)},S^{(n)},S^{(n)}]_u - \langle S^{(n)},S^{(n)},S^{(n)}\rangle_u\big).\end{aligned}$$

We can see that

$$\begin{aligned}E\big([\tilde{M}^{(n)},\tilde{M}^{(n)}]_t\big) &= \frac{1}{4}\{k_4+o(1)\}\int_0^t E[\{C_{SS}(S_{u-}^{(n)},u)\}^2(S_{u-}^{(n)})^4]du \\ &+ \frac{1}{6}O(\lambda_n^{-1/2})\int_0^t E\{C_{SS}(S_{u-}^{(n)},u)C_{SSS}(\tilde{Z}_u^{(n)},u)(S_{u-}^{(n)})^5\}du \\ &+ \frac{1}{36}O(\lambda_n^{-1})\int_0^t E[\{C_{SSS}(\tilde{Z}_u^{(n)},u)\}^2(S_{u-}^{(n)})^6]du,\end{aligned}$$

and by the assumption (A.8),

$$\sup_n E[\tilde{M}^{(n)},\tilde{M}^n]_t < \infty. \tag{A.9}$$

The total variation of $\tilde{A}^{(n)}$ on $[0,t]$ is

$$\begin{aligned}T_t(\tilde{A}^{(n)}) &= \sup_{m\geq 1}\sum_{k=1}^{2^m}\left|\tilde{A}^{(n)}\Big(\frac{tk}{2^m}\Big) - \tilde{A}^{(n)}\Big(\frac{t(k-1)}{2^m}\Big)\right| \\ &\leq \int_0^T \frac{1}{2}|C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2||k_3+o(1)|du \\ &+ \int_0^T \frac{1}{6}|C_{SSS}(\tilde{Z}_u^{(n)},u)(S_{u-}^{(n)})^3||k_3+o(1)|du.\end{aligned}$$

Thus.

$$E\{T_t(\tilde{A}^{(n)})\} \le \frac{1}{2}|k_3+o(1)|\int_0^T E|C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2|du$$
$$+\frac{1}{6}|k_3+o(1)|\int_0^T E|C_{SSS}(\tilde{Z}_u^{(n)},u)(S_{u-}^{(n)})^3|du.$$

By the assumption (A.7),

$$\sup_n E\{T_t(\tilde{A}^{(n)})\} < \infty. \tag{A.10}$$

Combining (A.9) and (A.10), we can see that $R^{(n)}$ satisfies the conditions of the integrator in Theorem 2.7 in Kurtz and Protter (1991) with $\tau_n^\alpha = T \vee (\alpha+1)$. Therefore,

$$\left(S^{(n)}, R^{(n)}, \int f(S^{(n)})dR^{(n)}\right) \xrightarrow{\mathcal{D}} \left(S, R, \int f(S)dR\right),$$

for any continuous function f.

(ii) $\tilde{W}^{(n)}$ can be written as

$$\tilde{W}^{(n)} = \int_0^t f_1(S_u^{(n)})dR^{(n)} + \int_0^t f_2(S_u^{(n)})dS_u^{(n)} + \int_0^t f_3(S_u^{(n)})du,$$

where f_1, f_2 and f_3 are continuous functions. We know the joint weak convergence of $S^{(n)}$, $R^{(n)}$, $f_1(S^{(n)})$, $f_2(S^{(n)})$, $\int f_1(S^{(n)})dR^{(n)}$ and $\int f_2(S^{(n)}) dS^{(n)}$ to $S, R, f_1(S), f_2(S), \int f_1(S)dR$, and $\int f_2(S)dS$. Since $f_3(S)$ is continuous, by Proposition VI. 1.17 in Jacod and Shiryaev (1987) and the continuous mapping theorem,

$$(\tilde{W}^{(n)}, R^{(n)}, S^{(n)}) \xrightarrow{\mathcal{D}} (\tilde{W}, R, S).$$

(iii) By Theorem 2.7 in Kurtz and Protter (1991),

$$\left(\tilde{W}^{(n)}, R^{(n)}, S^{(n)}, \int f(\tilde{W}^{(n)}, S^{(n)})dS^{(n)}\right) \xrightarrow{\mathcal{D}} \left(\tilde{W}, R, S, \int f(\tilde{W}, S)dS\right)$$

for any continuous f.

(iv) By (i), (ii) and (iii),

$$M_T^{(n)} = R_T^{(n)} - \int_0^T \left\{h(\tilde{W}_{u-}^{(n)}, S_{u-}^{(n)}) + (T-u)g_S(S_{u-}^{(n)},u)\right\} dS_u^{(n)}$$
$$\xrightarrow{\mathcal{D}} R_T - \int_0^T \left\{h(\tilde{W}_u, S_u) + (T-u)g_S(S_u,u)\right\} dS_u = Y_T\tilde{W}_T.$$

Now, for Theorem 3.4, we want to show the joint convergence of $(M_T^{(n)}, K^{(n)})$. Since we assume that $\theta^{(n)}$ converges in distribution to θ jointly with $S^{(n)}$ and $R^{(n)}$, the convergence is trivial from the proof of the Theorem 3.3. In case where $M_T = Y_T \tilde{W}_T$, we need the assumptions (A.7) and (A.8).

REFERENCES

ANDERSEN, P. K., BORGAN, Ø., GILL, R. D. AND KEIDING, N. (1993). *Statistical Models Based on Counting Processes*, Springer-Verlag, New York.

CARR, P., GEMAN, H., MADAN, D. B. AND YOR, M. (2002). "The fine structure of asset returns: An empirical investigation", *Journal of Business*, **75**, 305–332.

CHOW, Y. S. AND TEICHER, H. (1997). *Probability Theory: independence, interchangeability, martingales*, Springer-Verlag, New York.

COX, J. C., ROSS, S. A. AND RUBINSTEIN, M. (1979). "Option pricing: A simplified approach", *Journal of Financial Economics*, **7**, 229–263.

DUFFIE, D. (1996). *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton.

DUFFIE, D. AND RICHARDSON, H. R. (1991). "Mean-variance hedging in continuous time", *The Annals of Applied Probability*, **1**, 1–15.

EBERLEIN, E. AND KELLER, U. (1995). "Hyperbolic distributions in finance", *Bernoulli*, **1**, 281–299.

ENGLE, R. F. AND RUSSELL, J. R. (2002). "Analysis of high frequency financial data", In *Handbook of Financial Econometrics* (Ait-Sahalia, Y. and Hansen, L. P., eds).

FREY, R. (2000). "Risk minimization with incomplete information in a model for high-frequency data", *Mathematical Finance*, **10**, 215–225.

FÖLLMER, H. AND SCHWEIZER, M. (1991). "Hedging of contingent claims under incomplete information", In *Applied Stochastic Analysis* (Davis, M. H. A. and Elliott, R. J., eds.), 389–414, Gordon and Breach, New York.

HONG, D. AND WEE, I. (2003). "Convergence of jump-diffusion models to the Black-Scholes model", *Stochastic Analysis and Applications*, **21**, 141–160.

JACOD, J. AND SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin.

KARATZAS, I. AND SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.

KIRCH, M. AND RUNGGALDIER, W. J. (2004). "Efficient hedging when asset prices follow a geometric Poisson process with unknown intensities", *SIAM Journal on Control and Optimization*, **43**, 1174–1195.

KURTZ, T. G. AND PROTTER, P. (1991). "Weak limit theorems for stochastic integrals and stochastic differential equations", *The Annals of Probability*, **19**, 1035–1070.

LEÓN, J. A., SOLÉ, J. L., UTZET, F. AND VIVES, J. (2002). "On Lévy processes, Malliavin calculus and market models with jumps", *Finance and Stochastics*, **6**, 197–225.

MADAN, D. B. AND SENETA, E. (1990). "The variance gamma (V.G.) model for share market returns", *Journal of Business*, **63**, 511–524.

RACHEV, S. T. AND RUSHENDORF, L. (1994). "Models for option prices", *Theory of Probability and Its Applications*, **39**, 120–152.

SCHÄL, M. (1994). "On quadratic cost criteria for option hedging", *Mathematics of Operations Research*, **19**, 121–131.

SCHWEIZER, M. (1992). "Mean-variance hedging for general claims", *The Annals of Applied Probability*, **2**, 171–179.

SCHWEIZER, M. (1995). "Variance-optimal hedging in discrete time", *Mathematics of Operations Research*, **20**, 1–32.