퍼지 및 지능시스템학회 논문지 2006, Vol. 16, No. 3, pp. 357-366

INTUITIONISTIC FUZZY (t, s)-CONGRUENCES

Tae Chon Ahn[1], Kul Hur[2] and Seok Beom Roh[3]

[1] **School of Electrical Electronic and Information Engineering Wonkwang University, Iksan, Chonbuk, Korea 570-749**
[2] **Division of Mathematics and Informational Statistic Wonkwang University Iksan, Chonbuk, Korea 570-749**
[3] **School of Electrical Electronic and Information Engineering Wonkwang University, Iksan, Chonbuk, Korea 570-749**

Abstract

We introduce the notion of intuitionistic fuzzy (t, s)-congruences on a lattice and study some of its properties. Moreover, we obtain some properties of intuitionistic fuzzy congruences on the direct product of two lattices. Finally, we prove that the set of all intuitionistic fuzzy congruences on a lattice forms a distributive lattice.

Key words : intuitionistic fuzzy set, intuitionistic fuzzy (t, s)-equivalence relation, intuitionistic fuzzy (t, s)-congruence.

0. Introduction

The subject of fuzzy sets as an approach to a mathematical representation of vagueness in everyday language was introduced by L.A.Zadeh [22] in 1965. He generalized the idea of the characteristic function of a subset of a set X by defining a fuzzy subset of X as a map from X into $[0, 1]$. After that time, Sidky and Atallah [21] introduced the concept of T-congruences on a lattice and investigated some of its properties.

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [1]. Since then, Çoker and his colleagues [5,6,9], and Lee and Lee [19], and Hur and his colleagues [14] introduced the concept of intuitionistic fuzzy topological spaces and studied various properties. Moreover, Hur and his colleagues [13] applied the notion of intuitionistic fuzzy set to topological group. Moreover, Banerjee and Basnet [2], Biswas [3], Hur and his colleagues [10,11,12,15] applied to group theory using intuitionistic fuzzy sets. In 1996, Bustince and Burillo [4] introduced the concept of intuitionistic fuzzy relations and investigated some of it's properties. In 2003, Deschrijver and Kerre [7] investigated some properties of the composition of intuitionistic fuzzy relations. In particular, Hur and his colleagues [16,18] introduced the concept of intuitionistic fuzzy congruences on a lattice (a semigroup) and studied some of it's properties. Also, Hur and his colleagues [17] investigated various properties of intuitionistic fuzzy equivalence relations.

In this paper, we introduce the notion of intuitionistic fuzzy (t, s)-congruences on a lattice and study some of its properties. Moreover, we obtain some properties of intuitionistic fuzzy congruences on the direct product of two lattices. Finally, we prove that the set of all intuitionistic fuzzy congruences on a lattice forms a distributive lattice.

1. Preliminaries

In this section, we list some basic concepts and one result which are needed in the later sections.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I. And for a general background of lattice theory, refer to [8]. Moreover, we will use t and s to denote a t-norm and a t-conorm, respectively. For a t-norm and a t-conorm, we refer to [20].

Definition 1.1[1,5]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$

Corresponding Author :Kul Hur,
접수일자 : 2005년 4월 18일
완료일자 : 2006년 5월 30일

for each $x \in X$, where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively.In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2 [1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then
(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

Definition 1.3 [5]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then
(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4 [5]. Let X be a set. Then a complex mapping $R = (\mu_R, \ \nu_R) : X \times X \to I \times I$ is called an *intuitionistic fuzzy relation* (in short, *IFR*) on X if $\mu_R(x,y) + \nu_R(x,y) \leq 1$ for each $(x,y) \in X \times X$, i.e., $R \in$ IFS $(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 1.5 [5,7] Let X be a set and let $P, Q \in$ IFR(X). Then the *composition* $Q \circ P$ of P and Q, is defined as follows : for any $x, y \in X$,

$$\mu_{Q \circ P}(x,y) = \bigvee_{z \in X}[\mu_P(x,z) \wedge \mu_Q(z,y)]$$

and

$$\nu_{Q \circ P}(x,y) = \bigwedge_{z \in X}[\nu_P(x,z) \vee \nu_Q(z,y)].$$

Definition 1.6 [5,7]. An Intutionistic fuzzy Relation R on a set X is called an *intutionsitic fuzzy equivalence relation* (in short, $IFER$) on X if it satisfies the following conditions :

(i) it is *intutionsitic fuzzy reflexive*, i.e., $R(x,x) = (1,0)$ for each $x \in X$.
(ii) it is *intutionsitic fuzzy symmetric*, i.e., $R(x,y) = R(y,x)$ for any $x, y \in X$.
(iii) it is *intutionsitic fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as IFE(X).

Result 1.A [17, Proposition 2.10]. Let $\{R_\alpha\}_{\alpha \in \Gamma}$ be a nonempty family of IFERs on a set X. Then $\bigcap_{\alpha \in \Gamma} R_\alpha \in$ IFE(X). However, in general, $\bigcup_{\alpha \in \Gamma} R_\alpha$ need not be an IFER on X.

2. Intuitionistic fuzzy (t,s)-equivalence relations

Throughout this section, let X, Y and Z be nonempty sets.

Definition 2.1. Let $R \in$ IFS$(X \times Y)$ and let $S \in$ IFS$(Y \times Z)$. Then the *(t,s)-composition* of R and S, $S \circ_s^t R$, is defined as follows : for each $(x,z) \in X \times Z$,

$$\mu_{S \circ_s^t R}(x,z) = \bigvee_{y \in Y}[\mu_R(x,y) \ t \ \mu_S(y,z)]$$

and

$$\nu_{S \circ_s^t R}(x,z) = \bigwedge_{y \in Y}[\nu_R(x,y) \ s \ \nu_S(y,z)].$$

Definition 2.2. Let $A \in$ IFS(X) and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$.

(1) [5] $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu\}$ is called the *(λ,μ)-level subset* of A.

(2) $A^{(\dot{\lambda},\mu)} = \{x \in X : \mu_A(x) > \lambda$ and $\nu_A(x) < \mu\}$ is called the *strong (λ,μ)-level subset* of A.

It is clear that $R \in$ IFR(X) if and only if $R^{(\lambda,\mu)}$ and $R^{(\dot{\lambda},\mu)}$ are relations on X.

Result 2.A [17, Theorem 2.17]. Let $R \in$ IFR(X). Then $R \in$ IFE(X) if and only if $R^{(\lambda,\mu)}$ is an equivalence relation for each $(\lambda,\mu) \in I \times I$ with $\lambda + \mu \leq 1$.

Proposition 2.3. Let $A, B \in$ IFS(X) and let $(\lambda,\mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then

(1) $(A \cap B)^{(\lambda,\mu)} = A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}$.

(2) $(A \cup B)^{(\lambda,\mu)} = A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}$.

Proof. (1) Let $x \in X$. Then

$$\begin{aligned}
& x \in (A \cap B)^{(\lambda,\mu)} \\
\Leftrightarrow \ & \mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x) \geq \lambda \\
& \text{and } \ \nu_{A \cap B}(x) = \nu_A(x) \vee \nu_B(x) \leq \mu \\
\Leftrightarrow \ & \mu_A(x) \geq \lambda, \mu_B(x) \geq \lambda \\
& \text{and } \ \nu_A(x) \leq \mu, \nu_B(x) \leq \mu \\
\Leftrightarrow \ & \mu_A(x) \geq \lambda, \nu_A(x) \leq \mu \\
& \text{and } \ \mu_B(x) \geq \lambda, \nu_B(x) \leq \mu \\
\Leftrightarrow \ & x \in A^{(\lambda,\mu)} \ \text{ and } \ x \in B^{(\lambda,\mu)} \\
\Leftrightarrow \ & x \in A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}.
\end{aligned}$$

(2) Let $x \in X$. Then

$$\begin{aligned} & x \in (A \cup B)^{(\lambda,\mu)} \\ \Leftrightarrow\ & \mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x) \geq \lambda \\ & \text{and } \ \nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x) \leq \mu \\ \Leftrightarrow\ & \mu_A(x) \geq \lambda \ \text{ or } \ \mu_B(x) \geq \lambda \\ & \text{and } \ \nu_A(x) \leq \mu \ \text{ or } \ \nu_B(x) \leq \mu \\ \Leftrightarrow\ & \mu_A(x) \geq \lambda, \nu_A(x) \leq \mu \\ & \text{or } \ \mu_B(x) \geq \lambda, \nu_B(x) \leq \mu \\ \Leftrightarrow\ & x \in A^{(\lambda,\mu)} \ \text{ or } \ x \in B^{(\lambda,\mu)} \\ \Leftrightarrow\ & x \in A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}. \end{aligned}$$

Definition 2.4. Let $R \in$ IFR(X). Then R is said to be :

(1) *intuitionistic fuzzy* (λ, μ)*-reflexive* if $\mu_R(x,x) \geq \lambda$ and $\nu_R(x,x) \leq \mu$ for each $x \in X$, where $(\lambda,\mu) \in I \times I$ with $\lambda + \mu \leq 1$.

(2) *intuitionistic fuzzy* (t,s)*-transitive* if $R \circ_s^t R \subset R$.

(3) an *intuitionistic fuzzy* (t,s)*-equivalence relation* on X if it is intuitionistic fuzzy reflexive, symmetric and (t,s)-transitive.

(4) an *intuitionistic fuzzy* (λ,μ)-(t,s)*-equivalence relation* on X if it is intuitionistic fuzzy (λ,μ)-reflexive, symmetric and (t,s)-transitive.

We will denote the set of intuitionistic fuzzy (t,s)-[resp.(λ,μ)-(t,s)-]equivalence relations on X as IFE$_{(t,s)}(X)$ [resp. IFE$_{(\lambda,\mu)-(t,s)}(X)$].

Proposition 2.5. The intersection of arbitrary subfamily of IFE$_{(t,s)}(X)$ [resp. IFE$_{(\lambda,\mu)-(t,s)}(X)$] is an intuitionistic fuzzy (t,s)-[resp. (λ,μ)-(t,s)-]equivalence relation on X.

Proof. Let $\{R_\alpha\}_{\alpha\in\Gamma}$ be a family of intuitionistic fuzzy (t,s)-[resp. (λ,μ)-(t,s)-]equivalence relations on X and let $R = \bigcap_{\alpha\in\Gamma} R_\alpha$. By Result 1.A, it is clear that R is intuitionistic fuzzy reflexive and symmetric. Thus it is sufficient to show that R is intuitionistic fuzzy (t,s)-transitive. Let $x,y,z \in X$. Then

$$\begin{aligned} \mu_R(x,y) &= \bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha}(x,y) \\ &\geq \bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha \circ_s^t R_\alpha}(x,y) \\ & \quad \text{(Since } R_\alpha \text{ is } (t,s)\text{-transitive)} \\ &= \bigwedge_{\alpha\in\Gamma} (\bigvee_{a\in X} [\mu_{R_\alpha}(x,a) t \mu_{R_\alpha}(a,y)]) \\ &\geq \bigwedge_{\alpha\in\Gamma} [\mu_{R_\alpha}(x,z) t \mu_{R_\alpha}(z,y)] \\ &= \{\bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha}(x,z)\} t \{\bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha}(z,y)\} \\ &= \mu_R(x,z) t \mu_R(z,y) \end{aligned}$$

and

$$\begin{aligned} \nu_R(x,y) &= \bigvee_{\alpha\in\Gamma} \nu_{R_\alpha}(x,y) \leq \bigvee_{\alpha\in\Gamma} \nu_{R_\alpha \circ_s^t R_\alpha}(x,y) \\ &= \bigvee_{\alpha\in\Gamma} (\bigwedge_{a\in X} [\nu_{R_\alpha}(x,a) s \nu_{R_\alpha}(a,y)]) \\ &\leq \bigvee_{\alpha\in\Gamma} [\nu_{R_\alpha}(x,z) s \nu_{R_\alpha}(z,y)] \\ &= \{\bigvee_{\alpha\in\Gamma} \nu_{R_\alpha}(x,z)\} s \{\bigvee_{\alpha\in\Gamma} \nu_{R_\alpha}(z,y)\} \\ &= \nu_R(x,z) s \mu_R(z,y). \end{aligned}$$

So R is intuitionistic fuzzy (t,s)-transitive. On the other hand, we can easily show that R is intuitionistic fuzzy (λ,μ)-reflexive. Hence $R = \bigcap_{\alpha\in\Gamma} R_\alpha \in$ IFE$_{(t,s)}(X)$ [resp. IFE$_{(\lambda,\mu)-(t,s)}(X)$. This completes the proof.

3. Intuitionistic fuzzy congruences on a lattice

Let L be a lattice with least element 0 and greatest element 1.

Definition 3.1. Let $R \in$ IFR(L). Then R is said to be :

(1) [16] *intuitionistic fuzzy compatible* if it satisfies the following conditions : for any $x_1, x_2, y_1, y_2 \in L$,

(i) $\mu_R(x_1 \wedge x_2, y_1 \wedge y_2) \geq \mu_R(x_1,y_1) \wedge \mu_R(x_2,y_2)$ and $\nu_R(x_1 \wedge x_2, y_1 \wedge y_2) \leq \nu_R(x_1,y_1) \vee \nu_R(x_2,y_2)$.

(ii) $\mu_R(x_1 \vee x_2, y_1 \vee y_2) \geq \mu_R(x_1,y_1) \wedge \mu_R(x_2,y_2)$ and $\nu_R(x_1 \vee x_2, y_1 \vee y_2) \leq \nu_R(x_1,y_1) \vee \nu_R(x_2,y_2)$.

(2) [16] an *intuitionistic fuzzy congruence* on L if it is an IFE on L and is intuitionistic fuzzy compatible.

(3) *intuitionistic fuzzy* (t,s)*-compatible* if it satisfies the following conditions : for any $x_1, x_2, y_1, y_2 \in L$,

(i) $\mu_R(x_1 \wedge x_2, y_1 \wedge y_2) \geq \mu_R(x_1,y_1)\ t\ \mu_R(x_2,y_2)$ and
$\nu_R(x_1 \wedge x_2, y_1 \wedge y_2) \leq \nu_R(x_1,y_1)\ s\ \nu_R(x_2,y_2)$,

(ii) $\mu_R(x_1 \vee x_2, y_1 \vee y_2) \geq \mu_R(x_1,y_1)\ t\ \mu_R(x_2,y_2)$ and
$\nu_R(x_1 \vee x_2, y_1 \vee y_2) \leq \nu_R(x_1,y_1)\ s\ \nu_R(x_2,y_2)$.

(4) *an intuitionistic fuzzy* (t,s)*-congruence* on L if $R \in$ IFE$_{(t,s)}(L)$ and R is intuitionistic fuzzy (t,s)-compatible.

(5) *an intuitionistic fuzzy* (λ,μ)-(t,s)*-congruence* on L if $R \in$ IFE$_{(\lambda,\mu)-(t,s)}(L)$ and R is intuitionistic fuzzy (t,s)-compatible.

We will denote the set of all intuitionistic fuzzy congruences [resp. (t,s)-congruences and (λ,μ)-(t,s)-congruences] on L as IFC(L) [resp. IFC$_{(t,s)}(L)$ and IFC$_{(\lambda,\mu)-(t,s)}(L)$].

Proposition 3.2. Let $R \in \mathrm{IFR}(L)$, let $\alpha = \bigvee_{x,y\in L}\mu_R(x,y), \beta = \bigwedge_{x,y\in L}\nu_R(x,y)$ and let $(\lambda,\mu) \in \mathrm{Im}R\backslash(\alpha,\beta)$. Then

(1) R is intuitionistic fuzzy (α,β)-reflexive if and only if $R^{(\lambda,\mu)}$ is reflexive.

(2) R is intuitionistic fuzzy symmetric if and only if $R^{(\lambda,\mu)}$ is symmetric.

(3) If R is intuitionistic fuzzy transitive, then $R^{(\lambda,\mu)}$ is transitive.

(4) If $R^{(\lambda,\mu)}$ is transitive, then R is intuitionistic fuzzy (t,s)-transitive.

(5) If R is intuitionistic fuzzy compatible, then $R^{(\lambda,\mu)}$ is compatible.

(6) If $R^{(\lambda,\mu)}$ is compatible, then R is intuitionistic fuzzy (t,s)-compatible.

Proof. (1) $(\Rightarrow)$: Suppose R is intuitionistic fuzzy (α,β)-reflexive and let $x \in L$. Then clearly $\mu_R(x,x) = \alpha$ and $\nu_R(x,x) = \beta$. Since $(\lambda,\mu) \in \mathrm{Im}R\backslash(\alpha,\beta)$, $\mu_R(x,x) > \lambda$ and $\nu_R(x,x) < \mu$. Thus $(x,x) \in R^{(\lambda,\mu)}$. So $R^{(\lambda,\mu)}$ is reflexive.

$(\Leftarrow)$: Suppose $R^{(\lambda,\mu)}$ is reflexive and let $x \in L$. Then $\mu_R(x,x) > \lambda$ and $\nu_R(x,x) < \mu$ for each $(\lambda,\mu) \in \mathrm{Im}R\backslash(\alpha,\beta)$. Thus $\mu_R(x,x) \geq \alpha$ and $\nu_R(x,x) \leq \beta$. By the definition of (α,β), $\mu_R(x,x) = \alpha$ and $\nu_R(x,x) = \beta$. So R is intuitionistic fuzzy (α,β)-reflexive.

(2) $(\Rightarrow)$: Suppose R is intuitionistic fuzzy symmetric. For any $x,y \in L$, let $(x,y) \in R^{(\lambda,\mu)}$. Then $\mu_R(x,y) = \mu_R(y,x) > \lambda$ and $\nu_R(x,y) = \nu_R(y,x) < \mu$. Thus $(y,x) \in R^{(\lambda,\mu)}$. So $R^{(\lambda,\mu)}$ is symmetric.

$(\Leftarrow)$: $R^{(\lambda,\mu)}$ is symmetric. Assume that $R(x,y) = (\lambda_1,\mu_1)$ and $R(y,x) = (\lambda_2,\mu_2)$ such that $\lambda_1 > \lambda_2$ and $\mu_1 < \mu_2$. Then $(x,y) \in R^{(\lambda_2,\mu_2)}$. By the hypothesis, $(y,x) \in R^{(\lambda_2,\mu_2)}$. Thus $\lambda_2 = \mu_R(y,x) > \lambda_2$ and $\mu_2 = \nu_R(y,x) < \mu_2$. This is a contradiction. So $R(x,y) = R(y,x)$. Hence R is intuitionistic fuzzy symmetric.

(3) Suppose R is intuitionistic fuzzy transitive. For any $x,y,z \in L$, let $(x,y) \in R^{(\lambda,\mu)}$ and $(y,z) \in R^{(\lambda,\mu)}$. Then $\mu_R(x,y) > \lambda, \nu_R(x,y) < \mu$ and $\mu_R(y,z) > \lambda, \nu_R(y,z) < \mu$. By the hypothesis,

$$\begin{aligned}\mu_R(x,z) &\geq \textstyle\bigvee_{a\in X}[\mu_R(x,a)\wedge\mu_R(a,z)]\\ &\geq \mu_R(x,y)\wedge\mu_R(y,z)\\ &> \lambda\end{aligned}$$

and

$$\begin{aligned}\nu_R(x,z) &\leq \textstyle\bigwedge_{a\in X}[\nu_R(x,a)\vee\nu_R(a,z)]\\ &\leq \nu_R(x,y)\vee\nu_R(y,z)\\ &< \mu\,.\end{aligned}$$

Thus $(x,z) \in R^{(\lambda,\mu)}$. Hence $R^{(\lambda,\mu)}$ is transitive.

(4) Suppose $R^{(\lambda,\mu)}$ is transitive. For any $x,y,z \in L$, let $R(x,y) = (\lambda_1,\mu_1)$ and $R(y,z) = (\lambda_2,\mu_2)$ such that $\lambda_1 \leq \lambda_2$ and $\mu_1 \geq \mu_2$. Then

$$\mu_R(x,y)\, t\, \mu_R(y,z) \leq \lambda_1 \quad\text{and}\quad \nu_R(x,y)\, s\, \nu_R(y,z) \geq \mu_1.$$

Assume that $R(x,z) = (\lambda_3,\mu_3)$ such that $\lambda_3 < \lambda_1$ and $\mu_3 > \mu_1$. Then $(x,y) \in R^{(\lambda_3,\mu_3)}$ and $(y,z) \in R^{(\lambda_3,\mu_3)}$. By the hypothesis, $(x,z) \in R^{(\lambda_3,\mu_3)}$. Then $\lambda_3 = \mu_R(x,z) > \lambda_3$ and $\mu_3 = \nu_R(x,z) < \mu_3$. This contradicts the fact that $R(x,z) = (\lambda_3,\mu_3)$. Hence R is intuitionistic fuzzy (t,s)-transitive.

(5) Suppose R is intuitionistic fuzzy compatible and let $x_1,x_2,y_1,y_2 \in L$. Then

$$\mu_R(x_1\wedge x_2, y_1\wedge y_2) \geq \mu_R(x_1,y_1)\wedge\mu_R(x_2,y_2)$$

and

$$\nu_R(x_1\wedge x_2, y_1\wedge y_2) \leq \nu_R(x_1,y_1)\vee\nu_R(x_2,y_2).$$

Also,

$$\mu_R(x_1\vee x_2, y_1\vee y_2) \geq \mu_R(x_1,y_1)\wedge\mu_R(x_2,y_2)$$

and

$$\nu_R(x_1\vee x_2, y_1\vee y_2) \leq \nu_R(x_1,y_1)\vee\nu_R(x_2,y_2).$$

Let $(x_1,y_1) \in R^{(\lambda,\mu)}$ and $(x_2,y_2) \in R^{(\lambda,\mu)}$. Then $\mu_R(x_1,y_1) > \lambda, \nu_R(x_1,y_1) < \mu$ and $\mu_R(x_2,y_2) > \lambda, \nu_R(x_2,y_2) < \mu$. Thus

$$\mu_R(x_1\wedge x_2, y_1\wedge y_2) > \lambda, \nu_R(x_1\wedge x_2, y_1\wedge y_2) < \mu$$

and

$$\mu_R(x_1\vee x_2, y_1\vee y_2) > \lambda, \nu_R(x_1\vee x_2, y_1\vee y_2) < \mu.$$

So $(x_1\wedge x_2, y_1\wedge y_2) \in R^{(\lambda,\mu)}$ and $(x_1\vee x_2, y_1\vee y_2) \in R^{(\lambda,\mu)}$. Hence $R^{(\lambda,\mu)}$ is compatible.

(6) Suppose $R^{(\lambda,\mu)}$ is compatible. For any $x_1,x_2,y_1,y_2 \in L$, let $R(x_1,y_1) = (\lambda_1,\mu_1)$ and $R(x_2,y_2) = (\lambda_2,\mu_2)$ such that $\lambda_1 \leq \lambda_2$ and $\mu_1 \geq \mu_2$. Assume that $\mu_R(x_1\wedge x_2, y_1\wedge y_2) = \lambda_3 < \lambda_1$ and $\nu_R(x_1\wedge x_2, y_1\wedge y_2) = \mu_3 > \mu_1$. Then $(x_1,y_1) \in R^{(\lambda_3,\mu_3)}$ and $(x_2,y_2) \in R^{(\lambda_3,\mu_3)}$. By the hypothesis, $(x_1\wedge x_2, y_1\wedge y_2) \in R^{(\lambda_3,\mu_3)}$. Thus $\lambda_3 = \mu_R(x_1\wedge x_2, y_1\wedge y_2) > \lambda_2$ and $\mu_3 = \nu_R(x_1\wedge x_2, y_1\wedge y_2) < \mu_3$. This is a contradiction. Then $\lambda_3 \geq \lambda_1$ and $\mu_3 \leq \mu_1$. So

$$\begin{aligned}&\mu_R(x_1\wedge x_2, y_1\wedge y_2) = \lambda_3 \geq \lambda_1\\ &= \mu_R(x_1,y_1)\wedge\mu_R(x_2,y_2) \geq \mu_R(x_1,y_1)\, t\, \mu_R(x_2,y_2)\end{aligned}$$

and

$$\begin{aligned}&\nu_R(x_1\wedge x_2, y_1\wedge y_2) = \mu_3 \leq \mu_1\\ &= \nu_R(x_1,y_1)\vee\nu_R(x_2,y_2) \leq \nu_R(x_1,y_1)\, s\, \nu_R(x_2,y_2).\end{aligned}$$

By the similar arguments, we have

$$\mu_R(x_1\vee x_2, y_1\vee y_2) \geq \mu_R(x_1,y_1)\, t\, \mu_R(x_2,y_2)$$

and

$$\nu_R(x_1\vee x_2, y_1\vee y_2) \leq \nu_R(x_1,y_1)\, s\, \nu_R(x_2,y_2).$$

Hence R is intuitionistic fuzzy (t, s)-compatible. This completes the proof.

The following is the immediate result of Definition 3.1 and Proposition 3.2.

Corollary 3.2. Let $R \in \text{IFR}(L)$, let $\alpha = \bigvee_{x,y\in L} \mu_R(x, y)$, $\beta = \bigwedge_{x,y\in L} \nu_R(x, y)$ and let $(\lambda, \mu) \in \text{Im}R\backslash(\alpha, \beta)$.

(1) $R \in \text{IFE}_{(\alpha,\beta)-(t,s)}(L)$ if and only if $R^{(\lambda,\mu)}$ is an equivalence relation on L.

(2) $R \in \text{IFC}_{(\alpha,\beta)-(t,s)}(L)$ if and only if $R^{(\lambda,\mu)}$ is a congruence.

Remark 3.3. If $R \in \text{IFC}_{(t,s)}(L)$, then $\mu_R(x \wedge z, y \wedge z) \geq \mu_R(x, y), \nu_R(x\wedge z, y\wedge z) \leq \nu_R(x, y)$ and $\mu_R(x\vee z, y\vee z) \geq \mu_R(x, y), \nu_R(x \vee z, y \vee z) \leq \nu_R(x, y)$ for any $x, y, z \in L$.

Proposition 3.4. The intersection of family of intuitionistic fuzzy (t, s)-congruences [resp. (λ, μ)-(t, s)-congruences)] on L is also so.

Proof. Let $\{R_\alpha\}_{\alpha\in\Gamma}$ be a family of intuitionistic fuzzy (t, s)-congruences [resp. (λ, μ)-(t, s)-congruences] on L and let $R = \cap_{\alpha\in\Gamma} R_\alpha$. Then, by Proposition 2.5, $R \in \text{IFE}_{(t,s)}(L)$ [resp. $\text{IFE}_{(\lambda,\mu)-(t,s)}(L)$]. Let $x_1, x_2, y_1, y_2 \in L$. Then

$$\begin{aligned}
&\mu_R(x_1 \wedge x_2, y_1 \wedge y_2)\\
=\;& \bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha}(x_1 \wedge x_2, y_1 \wedge y_2)\\
\geq\;& \bigwedge_{\alpha\in\Gamma} [\mu_{R_\alpha}(x_1, y_1)\; t\; \mu_{R_\alpha}(x_2, y_2)]\\
&\text{(Since } R_\alpha \text{ is } (t, s)\text{-compatible)}\\
=\;& (\bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha}(x_1, y_1))\; t\; (\bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha}(x_2, y_2))\\
=\;& \mu_{\cap_{\alpha\in\Gamma} R_\alpha}(x_1, y_1)\; t\; \mu_{\cap_{\alpha\in\Gamma} R_\alpha}(x_2, y_2)\\
=\;& \mu_R(x_1, y_1)\; t\; \mu_R(x_2, y_2)
\end{aligned}$$

and

$$\begin{aligned}
&\nu_R(x_1 \wedge x_2, y_1 \wedge y_2)\\
=\;& \bigvee_{\alpha\in\Gamma} \nu_{R_\alpha}(x_1 \wedge x_2, y_1 \wedge y_2)\\
\leq\;& \bigvee_{\alpha\in\Gamma} [\nu_{R_\alpha}(x_1, y_1)\; s\; \nu_{R_\alpha}(x_2, y_2)]\\
=\;& (\bigvee_{\alpha\in\Gamma} \nu_{R_\alpha}(x_1, y_1))\; s\; (\bigvee_{\alpha\in\Gamma} \nu_{R_\alpha}(x_2, y_2))\\
=\;& \nu_{\cap_{\alpha\in\Gamma} R_\alpha}(x_1, y_1)\; s\; \nu_{\cap_{\alpha\in\Gamma} R_\alpha}(x_2, y_2)\\
=\;& \nu_R(x_1, y_1)\; s\; \nu_R(x_2, y_2).
\end{aligned}$$

By the similar arguments, we have

$$\mu_R(x_1 \vee x_2, y_1 \vee y_2) \geq \mu_R(x_1, y_1)\; t\; \mu_R(x_2, y_2)$$

and

$$\nu_R(x_1 \vee x_2, y_1 \vee y_2) \leq \nu_R(x_1, y_1)\; s\; \nu_R(x_2, y_2).$$

So R is intuitionistic fuzzy (t, s)-compatible. Hence $R \in \text{IFC}_{(t,s)}(L)$ [resp. $\text{IFC}_{(\lambda,\mu)-(t,s)}(L)$].

Proposition 3.5. Let $R \in \text{IFR}(L)$. Then $R \in \text{IFC}(L)$ if and only if $R^{(\lambda,\mu)}$ is a congruence on L for each $(\lambda, \mu) \in \text{Im}R$.

Proof. By Result 1.A, $R \in \text{IFE}(L)$ if and only if $R^{(\lambda,\mu)}$ is an equivalence relation on L for each $(\lambda, \mu) \in \text{Im}R$. Then it is sufficient to show that R is intuitionistic fuzzy compatible if and only if $R^{(\lambda,\mu)}$ is compatible.

$(\Rightarrow)$: Suppose R is intuitionistic fuzzy compatible. For any $x_1, x_2, y_1, y_2 \in L$, let $(x_1, y_1) \in R^{(\lambda,\mu)}$ and $(x_2, y_2) \in R^{(\lambda,\mu)}$. Then

$$\mu_R(x_1, y_1) \geq \lambda, \quad \nu_R(x_1, y_1) \leq \mu$$

and

$$\mu_R(x_2, y_2) \geq \lambda, \quad \nu_R(x_2, y_2) \leq \mu.$$

Since R is intuitionistic fuzzy compatible,

$$\mu_R(x_1 \wedge x_2, y_1 \wedge y_2) \geq \mu_R(x_1, y_1) \wedge \mu_R(x_2, y_2) \geq \lambda$$

and

$$\nu_R(x_1 \wedge x_2, y_1 \wedge y_2) \leq \nu_R(x_1, y_1) \vee \nu_R(x_2, y_2) \leq \mu.$$

Thus $(x_1 \wedge x_2, y_1 \wedge y_2) \in R^{(\lambda,\mu)}$. By the similar arguments, we have $(x_1 \vee x_2, y_1 \vee y_2) \in R^{(\lambda,\mu)}$. So $R^{(\lambda,\mu)}$ is compatible.

$(\Leftarrow)$: Suppose $R^{(\lambda,\mu)}$ is compatible for each $(\lambda, \mu) \in \text{Im}R$. For any $x_1, x_2, y_1, y_2 \in L$, let $R(x_1, y_1) = (\lambda_1, \mu_1)$ and $R(x_2, y_2) = (\lambda_2, \mu_2)$ such that $\lambda_1 \geq \lambda_2$ and $\mu_1 \leq \mu_2$. Then $(x_1, y_1) \in R^{(\lambda_2,\mu_2)}$ and $(x_2, y_2) \in R^{(\lambda_2,\mu_2)}$. By the hypothesis, $(x_1 \wedge x_2, y_1 \wedge y_2) \in R^{(\lambda_2,\mu_2)}$ and $(x_1 \vee x_2, y_1 \vee y_2) \in R^{(\lambda_2,\mu_2)}$. Thus

$$\begin{aligned}
&\mu_R(x_1 \wedge x_2, y_1 \wedge y_2) \geq \lambda_2 \geq \lambda_1 \wedge \lambda_2\\
&\quad = \mu_R(x_1, y_1) \wedge \mu_R(x_2, y_2),\\
&\nu_R(x_1 \wedge x_2, y_1 \wedge y_2) \leq \mu_2 \leq \mu_1 \vee \mu_2\\
&\quad = \nu_R(x_1, y_1) \vee \nu_R(x_2, y_2),
\end{aligned}$$

and

$$\begin{aligned}
&\mu_R(x_1 \vee x_2, y_1 \vee y_2) \geq \lambda_2 \geq \lambda_1 \wedge \lambda_2\\
&\quad = \mu_R(x_1, y_1) \wedge \mu_R(x_2, y_2),\\
&\nu_R(x_1 \vee x_2, y_1 \vee y_2) \leq \mu_2 \leq \mu_1 \vee \mu_2\\
&\quad = \nu_R(x_1, y_1) \vee \nu_R(x_2, y_2).
\end{aligned}$$

So R is intuitionistic fuzzy compatible. This completes the proof.

Definition 3.6[8]. Let L and M be lattices. We define two operations $\wedge$ and $\vee$ on $L \times M$, respectively as follows : for any $(a, b), (a_1, b_1) \in L \times M$,

$$(a, b) \wedge (a_1, b_1) = (a \wedge a_1, b \wedge b_1)$$

and

$$(a, b) \vee (a_1, b_1) = (a \vee a_1, b \vee b_1).$$

Then $L \times M$ is a lattice. In this case, $L \times M$ is called the *direct product* of L and M and denoted by $L \underline{\times} M$.

Result 3.A[8, Theorem 13 in P.28]. Let L and M be lattices, let P be a congruence on L and let Q a congruence on M. We define the relation $P\underline{\times}Q$ on $L\underline{\times}M$ by

$$(a, b) \equiv (c, d)(P\underline{\times}Q)$$

if and only if

$$a \equiv c(P) \quad \text{and} \quad b \equiv d(Q).$$

Then $P\underline{\times}Q$ is congruence on $L\underline{\times}M$. Conversely, every congruence on $L\underline{\times}M$ is of this form.

Result 3.B[8, Lemma 8 in P.24]. Let R be a reflexive and symmetric relation on a lattice L. Then R is a congruence on L if and only if the following conditions hold : for any $x, y, z, t \in L$,

(i) $x \equiv y(R)$ iff $x \wedge y \equiv x \vee y(R)$.

(ii) $x \leq y \leq z, x \equiv y(R)$ and $y \equiv z(R)$ imply that $x \equiv z(R)$.

(iii) $x \leq y$ and $x \equiv y(R)$ imply that $x \wedge t \equiv y \wedge t(R)$ and $x \vee t \equiv y \vee t(R)$.

Result 3.C[8, Theorem 9 in P.25]. Let $C(L)$ denote the set of all congruences on a poset L. For any $P, Q \in C(L)$, we define $P \wedge Q = P \cap Q$ and the join, $P \vee Q$, as follows : for any $x, y \in L$, $x \equiv y(P \vee Q)$ if and only if there is a sequence $z_0 = x \wedge y, z_1, \cdots, z_{n-1} = x \vee y$ in L such that $z_0 \leq z_1 \leq \cdots \leq z_{n-1}$ and for each i, $0 \leq i < n-1, z_i \equiv z_{i+1}(P)$ or $z_i \equiv z_{i+1}(Q)$. Then $C(L)$ is called the *congruence lattice* of L, where $C(L)$ denotes the set of all congruences on L partially ordered by set inclusion.

Proposition 3.7. Let L and M be lattices, and let $P \in$ IFR(L), $Q \in$ IFR(M). We define a complex mapping $P\underline{\times}Q = (\mu_{P\underline{\times}Q}, \nu_{P\underline{\times}Q}) : (L\underline{\times}M) \times (L\underline{\times}M) \longrightarrow I \times I$ as follows : for any $(x_1, y_1), (x_2, y_2) \in L\underline{\times}M$,

$$\mu_{P\underline{\times}Q}((x_1, y_1), (x_2, y_2)) = \mu_P(x_1, x_2) \wedge \mu_Q(y_1, y_2)$$

and

$$\nu_{P\underline{\times}Q}((x_1, y_1), (x_2, y_2)) = \nu_P(x_1, x_2) \vee \mu_Q(y_1, y_2).$$

Then $(P\underline{\times}Q)^{(\lambda,\mu)} = P^{(\lambda,\mu)}\underline{\times}Q^{(\lambda,\mu)}$ for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$.

Proof. It is clear that $P\underline{\times}Q$ is an IFR on $L\underline{\times}M$. Let $(x_1, y_1), (x_2, y_2) \in L\underline{\times}M$ and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then

$$\begin{aligned} & ((x_1, y_1), (x_2, y_2)) \in (P\underline{\times}Q)^{(\lambda,\mu)} \\ \Leftrightarrow\ & \mu_{P\underline{\times}Q}((x_1, y_1), (x_2, y_2)) \\ =\ & \mu_P(x_1, x_2) \wedge \mu_Q(y_1, y_2) \\ \geq\ & \lambda \end{aligned}$$

and

$$\begin{aligned} & \nu_{P\underline{\times}Q}((x_1, y_1), (x_2, y_2)) = \nu_P(x_1, x_2) \vee \nu_Q(y_1, y_2) \leq \mu \\ \Leftrightarrow\ & \mu_P(x_1, x_2) \geq \lambda, \mu_Q(y_1, y_2) \geq \lambda \\ & \text{and} \quad \nu_P(x_1, x_2) \leq \mu, \nu_Q(y_1, y_2) \leq \mu \\ \Leftrightarrow\ & \mu_P(x_1, x_2) \geq \lambda, \nu_P(x_1, x_2) \leq \mu \\ & \text{and} \quad \mu_Q(y_1, y_2) \geq \lambda, \nu_Q(y_1, y_2) \leq \mu \\ \Leftrightarrow\ & (x_1, x_2) \in P^{(\lambda,\mu)} \quad \text{and} \quad (y_1, y_2) \in Q^{(\lambda,\mu)} \\ \Leftrightarrow\ & ((x_1, x_2), (y_1, y_2)) \in P^{(\lambda,\mu)}\underline{\times}Q^{(\lambda,\mu)}. \\ & \text{(By Result 3.B)} \end{aligned}$$

Hence $(P\underline{\times}Q)^{(\lambda,\mu)} = P^{(\lambda,\mu)}\underline{\times}Q^{(\lambda,\mu)}$. This completed the proof.

Remark 3.8. From Proposition 3.7 and Result 3.A, it follows that if $P \in$ IFC(L) and $Q \in$ IFC(M), then $P\underline{\times}Q \in$ IFC$(L\underline{\times}M)$.

Lemma 3.9. Let $R \in$ IFC$(L\underline{\times}M)$. Then

(1) $R((x_1, y), (x_2, y)) = R((x_1, y'), (x_2, y'))$ for any $x_1, x_2 \in L$ and $y, y' \in M$.

(2) $R((x, y_1), (x, y_2)) = R((x', y_1), (x', y_2))$ for any $x, x' \in L$ and $y_1, y_2 \in M$.

Proof. (1) Let $x_1, x_2 \in L$ and $y, y' \in M$. Then

$$\begin{aligned} & \mu_R((x_1, y), (x_2, y)) \\ =\ & \mu_R((x_1, y), (x_2, y)) \wedge \mu_R((x_1 \wedge x_2, y'), \\ & \quad (x_1 \wedge x_2, y')) \\ & \text{(Since } R \text{ is intuitionistic fuzzy reflexive)} \\ \leq\ & \mu_R((x_1, y) \vee (x_1 \wedge x_2, y'), \\ & \quad (x_2, y) \vee (x_1 \wedge x_2, y')) \\ & \text{(Since } R \text{ is intuitionistic fuzzy compatible)} \\ =\ & \mu_R((x_1 \vee (x_1 \wedge x_2), y \vee y'), \\ & \quad (x_2 \vee (x_1 \wedge x_2), y \vee y')) \\ =\ & \mu_R((x_1, y \vee y'), (x_2, y \vee y')) \\ =\ & \mu_R((x_1, y \vee y'), (x_2, y \vee y')) \\ & \quad \wedge \mu_R((x_1 \vee x_2, y'), (x_1 \vee x_2, y')) \\ & \text{(Since } R \text{ is intuitionistic fuzzy reflexive)} \\ \leq\ & \mu_R((x_1, y \vee y') \wedge (x_1 \vee x_2, y'), (x_2, y \vee y') \\ & \quad \wedge (x_1 \vee x_2, y')) \\ & \text{(Since } R \text{ is intuitionistic fuzzy compatible)} \\ =\ & \mu_R((x_1 \wedge (x_1 \vee x_2), (y \vee y') \wedge y'), \\ & \quad (x_2 \wedge (x_1 \vee x_2), (y \vee y') \wedge y')) \\ =\ & \mu_R((x_1, y'), (x_2, y')) \end{aligned}$$

and

$$\begin{aligned}
&\nu_R((x_1,y),(x_2,y))\\
=\ &\nu_R((x_1,y),(x_2,y)) \vee \nu_R((x_1 \wedge x_2, y'),\\
&(x_1 \wedge x_2, y'))\\
\geq\ &\nu_R((x_1,y) \vee (x_1 \wedge x_2, y'),\\
&(x_2,y) \vee (x_1 \wedge x_2, y'))\\
=\ &\nu_R((x_1 \vee (x_1 \wedge x_2), y \vee y'),\\
&(x_2 \vee (x_1 \wedge x_2), y \vee y'))\\
=\ &\nu_R((x_1, y \vee y'),(x_2, y \vee y'))\\
=\ &\nu_R((x_1, y \vee y'),(x_2, y \vee y'))\\
&\vee \nu_R((x_1 \vee x_2), y'),(x_1 \vee x_2, y'))\\
\geq\ &\nu_R((x_1, y \vee y') \wedge (x_1 \vee x_2, y'),(x_2, y \vee y')\\
&\wedge (x_1 \vee x_2, y'))\\
=\ &\nu_R((x_1 \wedge (x_1 \vee x_2),(y \vee y') \wedge y'),\\
&(x_2 \wedge (x_1 \vee x_2),(y \vee y') \wedge y'))\\
=\ &\nu_R((x_1,y'),(x_2,y')).
\end{aligned}$$

By the similar arguments, we have $\mu_R((x_1,y),(x_2,y)) \geq \mu_R((x_1,y'),(x_2,y'))$ and $\nu_R((x_1,y),(x_2,y)) \leq \nu_R((x_1,y'),(x_2,y'))$ for each $(y,y') \in M \times M$. Hence $R((x_1,y),(x_2,y)) = R((x_1,y'),(x_2,y'))$ for any $x_1, x_2 \in L$ and any $y, y' \in M$.

(2) By the similar arguments of the proof of (1), we can see that (2) holds.

Proposition 3.10. Let $R \in$ IFC($L \underline{\times} M$). Then there exist $P \in$ IFC(L) and $Q \in$ IFC(L) such that $R = P \underline{\times} Q$.

Proof. We define two complex mappings $P = (\mu_P, \nu_P) : L \times L \to I \times I$ and $Q = (\mu_Q, \nu_Q) : M \times M \to I \times I$ as follows :

(i) $P(x_1,x_2) = R((x_1,y),(x_2,y))$ for any $x_1, x_2 \in L$ and each $y \in M$,

(ii) $Q(y_1,y_2) = R((x,y_1),(x,y_2))$ for each $x \in L$ and any $y_1, y_2 \in M$.

Then clearly $P \in$ IFR(L) and $Q \in$ IFR(M). Moreover, by Lemma 3.9, P and Q are well-defined. Let $(\lambda,\mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then

$(x_1,x_2) \in P^{(\lambda,\mu)}$

$\Leftrightarrow ((x_1,y),(x_2,y)) \in R^{(\lambda,\mu)}$ for each $y \in M$

and

$(y_1,y_2) \in Q^{(\lambda,\mu)}$

$\Leftrightarrow ((x,y_1),(x,y_2)) \in R^{(\lambda,\mu)}$ for each $x \in L$.

By Proposition 3.5, $P^{(\lambda,\mu)}$ and $Q^{(\lambda,\mu)}$ are congruences on L and M, respectively. Moreover, by Result 3.B, $R^{(\lambda,\mu)} = P^{(\lambda,\mu)} \underline{\times} Q^{(\lambda,\mu)}$. By proposition 3.8, $P^{(\lambda,\mu)} \underline{\times} Q^{(\lambda,\mu)} = (P \underline{\times} Q)^{(\lambda,\mu)}$. So $R = P \underline{\times} Q$. Moreover, by Proposition 3.5, $P \in$ IFC(L) and $Q \in$ IFC(M). This completes the proof.

Proposition 3.11. Let $R \in$ IFR(L) be intuitionistic fuzzy (t,s)-transitive such that $\mu_R(x \wedge z, y \wedge z) \geq \mu_R(x,y), \nu_R(x \wedge z, y \wedge z) \leq \nu_R(x,y)$ and $\mu_R(x \vee z, y \vee z) \geq \mu_R(x,y), \nu_R(x \vee z, y \vee z) \leq \nu_R(x,y)$ for any $x,y,z \in L$. Then R is intuitionistic fuzzy (t,s)-compatible.

Proof. Let $x_1, x_2, y_1, y_2 \in L$. Then

$$\begin{aligned}
&\mu_R(x_1 \wedge x_2, y_1 \wedge y_2)\\
\geq\ &\bigvee_{z \in L} [\mu_R(x_1 \wedge x_2, z)\ t\ \mu_R(z, y_1 \wedge y_2)]\\
&\text{(Since } R \text{ is } (t,s)\text{-transitive)}\\
\geq\ &\mu_R(x_1 \wedge x_2, y_1 \wedge x_2)\ t\ \mu_R(y_1 \wedge x_2, y_1 \wedge y_2)\\
\geq\ &\mu_R(x_1,y_1)\ t\ \mu_R(x_2,y_2) \quad \text{(By the hypotheses)}
\end{aligned}$$

and

$$\begin{aligned}
&\nu_R(x_1 \wedge x_2, y_1 \wedge y_2)\\
\leq\ &\bigwedge_{z \in L} [\nu_R(x_1 \wedge x_2, z)\ s\ \nu_R(z, y_1 \wedge y_2)]\\
\leq\ &\nu_R(x_1 \wedge x_2, y_1 \wedge x_2)\ s\ \nu_R(y_1 \wedge x_2, y_1 \wedge y_2)\\
\leq\ &\nu_R(x_1,y_1)\ s\ \nu_R(x_2,y_2).
\end{aligned}$$

Similarly,we have

$$\mu_R(x_1 \vee x_2, y_1 \vee y_2) \geq \mu_R(x_1,y_1)\ t\ \mu_R(x_2,y_2)$$

and

$$\nu_R(x_1 \vee x_2, y_1 \vee y_2) \leq \nu_R(x_1,y_1)\ s\ \nu_R(x_2,y_2).$$

Hence R is intuitionistic fuzzy (t,s)-compatible.

The following is the immediate result of Proposition 3.11.

Corollary 3.11 Let $R \in$IFR(L) be intuitionistic fuzzy transitive such that $\mu_R(x \wedge z, y \wedge z) \geq \mu_R(x,y)$, $\nu_R(x \wedge z, y \wedge z) \leq \nu_R(x,y)$ and $\mu_R(x \vee z, y \vee z) \geq \mu_R(x,y), \nu_R(x \vee z, y \vee z) \leq \nu_R(x,y)$ for any $x,y,z \in L$. Then R is intuitionistic fuzzy compatible.

Proposition 3.12. If $R \in$ IFC(L), then $R(x,y) = R(x \wedge y, x \vee y)$ for any $x,y \in L$.

Proof. Let $x,y \in L$. Then

$$\begin{aligned}
&\mu_R(x \wedge y, x \vee y)\\
=\ &\mu_R(x \wedge y, x \vee y) \wedge \mu_R(x,x)\\
&\text{(Since } R \text{ is intuitionistic fuzzy reflexive)}\\
\leq\ &\mu_R((x \wedge y) \wedge x, (x \vee y) \wedge x)\\
&\text{(Since } R \text{ is intuitionistic fuzzy compatible)}\\
\leq\ &\mu_R(x \wedge y, x)\\
=\ &\mu_R(x, x \wedge y)\\
&\text{(Since } R \text{ is intuitionistic fuzzy symmetric)}
\end{aligned}$$

and

$$\begin{aligned} & \nu_R(x \wedge y, x \vee y) \\ = \ & \nu_R(xwedgey, x \vee y) \vee \nu_R(x,x) \\ \geq \ & \nu_R((x \wedge y) \wedge x, (x \vee y) \wedge x) \\ \geq \ & \nu_R(x \wedge y, x) = \nu_R(x, x \wedge y). \end{aligned}$$

Also

$$\begin{aligned} & \mu_R(x \wedge y, x \vee y) \\ = \ & \mu_R(x \wedge y, x \vee y) \wedge \mu_R(y,y) \\ & \text{(Since } R \text{ is intuitionistic fuzzy reflexive)} \\ \leq \ & \mu_R((x \wedge y) \vee y, (x \vee y) \vee y) \\ & \text{(Since } R \text{ is intuitionistic fuzzy compatible)} \\ \leq \ & \mu_R(y, x \vee y) \\ = \ & \mu_R(x \vee y, y) \\ & \text{(Since } R \text{ is intuitionistic fuzzy symmetric)} \end{aligned}$$

and

$$\begin{aligned} & \nu_R(x \wedge y, x \vee y) \\ = \ & \nu_R(x \wedge y, x \vee y) \vee \nu_R(y,y) \\ \geq \ & \nu_R((x \wedge y) \vee y, (x \vee y) \vee y) \\ \geq \ & \nu_R(y, x \vee y) = \nu_R(x \vee y, y). \end{aligned}$$

Thus

$$\begin{aligned} & \mu_R(x,y) \\ \geq \ & \bigvee_{z \in L} [(\mu_R(x,z) \wedge \mu_R(z,y)] \\ & \text{(Since } R \text{ is intuitionistic fuzzy transitive)} \\ \geq \ & \mu_R(x, x \wedge y) \wedge \mu_R(x \wedge y, x \vee y) \wedge \mu_R(x \vee y, y) \\ = \ & \mu_R(x \wedge y, x \vee y) \end{aligned}$$

and

$$\begin{aligned} & \nu_R(x,y) \\ \leq \ & \bigwedge_{z \in L} [(\nu_R(x,z) \vee \nu_R(z,y)] \\ \leq \ & \nu_R(x, x \wedge y) \vee \nu_R(x \wedge y, x \vee y) \vee \nu_R(x \vee y, y) \\ = \ & \nu_R(x \wedge y, x \vee y). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mu_R(x,y) \\ = \ & \mu_R(x,y) \wedge \mu_R(y,y) \\ & \text{(Since } R \text{ is intuitionistic fuzzy reflexive)} \\ \leq \ & \mu_R(x \wedge y, y \wedge y) \\ & \text{(Since } R \text{ is intuitionistic fuzzy compatible)} \\ = \ & \mu_R(x \wedge y, y) \end{aligned}$$

and

$$\begin{aligned} \nu_R(x,y) &= \nu_R(x,y) \vee \nu_R(y,y) \geq \nu_R(x \wedge y, y \wedge y) \\ &= \nu_R(x \wedge y, y). \end{aligned}$$

Similarly, we have

$$\mu_R(x,y) \leq \mu_R(x \vee y, y)$$

and

$$\nu_R(x,y) \geq \nu_R(x \vee y, y).$$

Thus

$$\begin{aligned} \mu_R(x,y) &\leq \mu_R(x \wedge y, y) \wedge \mu_R(x \vee y, y) \\ &= \mu_R(x \wedge y, y) \wedge \mu_R(y, x \vee y) \\ &\leq \mu_R(x \wedge y, x \vee y) \end{aligned}$$

and

$$\begin{aligned} \nu_R(x,y) &\geq \nu_R(x \wedge y, y) \vee \nu_R(x \vee y, y) \\ &= \nu_R(x \wedge y, y) \vee \nu_R(y, x \vee y) \\ &\geq \nu_R(x \wedge y, x \vee y). \end{aligned}$$

Hence $R(x,y) = R(x \wedge y, x \vee y)$ for any $x, y \in L$.

Proposition 3.13. Let $R \in \mathrm{IFR}(L)$ be intuitionistic fuzzy reflexive and compatible such that $R(b,c) = R(b \wedge c, b \vee c)$ for any $b, c \in [a,d] \subset L$. Then $\mu_R(b,c) \geq \mu_R(a,d)$ and $\nu_R(b,c) \leq \nu_R(a,d)$.

Proof. Let $b, c \in [a,d] \subset L$. Then

$$\begin{aligned} & \mu_R(b \wedge c, d) \\ = \ & \mu_R(a \vee (b \wedge c), d \vee (b \wedge c)) \\ & \text{(Since } b \wedge c \geq a \text{ and } b \wedge c \leq d) \\ \geq \ & \mu_R(a,d) \wedge \mu_R(b \wedge c, b \wedge c) \\ & \text{(Since } R \text{ is intuitionistic fuzzy compatible)} \\ = \ & \mu_R(a,d) \\ & \text{(Since } R \text{ is intuitionistic fuzzy reflexive)} \end{aligned}$$

and

$$\begin{aligned} \nu_R(b \wedge c, d) &= \nu_R(a \vee (b \wedge c), d \vee (b \wedge c)) \\ &\leq \nu_R(a,d) \vee \nu_R(b \wedge c, b \wedge c) \\ &= \nu_R(a,d). \end{aligned}$$

Thus

$$\begin{aligned} & \mu_R(b,c) \\ = \ & \mu_R(b \wedge c, b \vee c) \quad \text{(By the hypothesis)} \\ = \ & \mu_R((b \wedge c) \wedge (b \vee c), d \wedge (b \vee c)) \\ & \text{(Since } b \wedge c \leq b \vee c \text{ , and } b \vee c \leq d) \\ \geq \ & \mu_R(b \wedge c, d) \wedge \mu_R(b \vee c, b \vee c) \\ & \text{(Since } R \text{ is intuitionistic fuzzy compatible)} \\ = \ & \mu_R(b \wedge c, d) \\ & \text{(Since } R \text{ is intuitionistic fuzzy reflexive)} \\ \geq \ & \mu_R(a,d) \end{aligned}$$

and

$$\begin{aligned} & \nu_R(b,c) \\ = & \ \nu_R(b \wedge c, b \vee c) \\ = & \ \nu_R((b \wedge c) \wedge (b \vee c), d \wedge (b \vee c)) \\ \leq & \ \nu_R(b \wedge c, d) \vee \nu_R(b \vee c, b \vee c) \\ = & \ \nu_R(b \wedge c, d) \leq \nu_R(a,d). \end{aligned}$$

This completes the proof.

Question. Can we find an analogue of Result 3.B by intuitionistic fuzzy setting? That is, let R be an intiutionistic fuzzy reflexive and symmetric relation on a lattice L. Then, under certain conditions, $R \in \text{IFC}(L)$ [resp. $R \in \text{IFC}_{(t,s)}(L)$]?

It is clear that ITC(L) is closed with respect to $\cap$ from Proposition 3.4. So $P \wedge Q = P \cap Q$ for any $P, Q \in \text{ITC}(L)$.

Definition 3.14. Let $P, Q \in \text{ITC}(L)$. The supremum of P and Q is defined to be the IFS $P \vee Q = \cap\{R \in \text{ITC}(L) : P \cup Q \subset R\}$.

It is clear that $P \vee Q \in \text{ITC}(L)$. So $(\text{ITC}(L), \subset, \wedge, \vee)$ is a lattice.

Proposition 3.15. Let $P, Q \in \text{IFC}(L)$. Then $(P \vee Q)^{(\lambda,\mu)} = P^{(\lambda,\mu)} \vee Q^{(\lambda,\mu)}$ for each $(\lambda, \mu) \in \text{Im}(P \vee Q)$.

Proof. Let $R \in \text{IFR}(L)$ such that $\text{Im}R = \text{Im}(P \vee Q)$ and let $R^{(\lambda,\mu)} = P^{(\lambda,\mu)} \vee Q^{(\lambda,\mu)}$ for each $(\lambda, \mu) \in \text{Im}R$. Then $R^{(\lambda,\mu)}$ is a congruence on L for each $(\lambda, \mu) \in \text{Im}R$. By Proposition 3.5, $R \in \text{IFC}(L)$. Moreover, by Proposition 2.3, $P \cup Q \subset R$. Let $D \in \text{IFC}(L)$ with $P \cup Q \subset D$. Then $(P \cup Q)^{(s,t)} = P^{(s,t)} \cup Q^{(s,t)} \subset D^{(s,t)}$ for each $(s,t) \in I \times I$ with $s + t \leq 1$. Thus $R^{(s,t)} \subset D^{(s,t)}$ for each $(s,t) \in I \times I$ with $s + t \leq 1$. So $R \subset D$, i.e, $R = P \vee Q$. Hence $(P \vee Q)^{(\lambda,\mu)} = P^{(\lambda,\mu)} \vee Q^{(\lambda,\mu)}$ for each $(\lambda, \mu) \in \text{Im}(P \vee Q)$.

From Proposition 2.3, Proposition 3.15 and Result 3.C, we obtain the following result

Proposition 3.16. Let $P, Q \in \text{IFC}(L)$ and let $(s,t) \in I \times I$ with $s + t \leq 1$. Then $\mu_{P \vee Q}(x,y) \geq s$ and $\nu_{P \vee Q}(x,y) \leq t$ for any $x, y \in L$ if and only if there is a sequence $x \wedge y = z_1 \leq \cdots \leq z_n = x \vee y$ in L such that $\mu_{A \cup B}(z_i, z_{i+1}) \geq s$ and $\nu_{A \cup B}(z_i, z_{i+1}) \leq t$ for each $i \in \{1, 2, \cdots, n-1\}$.

Proposition 3.17. The lattice IFC(L) is distributive.

Proof. Let $P, Q, R \in \text{IFC}(L)$. For any $x, y \in L$, let $P(x,y) = (s_1, t_1)$ and $(Q \vee R)(x,y) = (s_2, t_2)$. Then, by Proposition 3.16, there is a sequence $x \wedge y = z_1 \leq \cdots \leq z_n = x \vee y$ in L such that $\mu_{B \cup C}(z_i, z_{i+1}) \geq s_2$ and $\nu_{B \cup C}(z_i, z_{i+1}) \leq t_2$ for each $i \in \{1, 2, \cdots, n-1\}$. Since $(x,y) \in P^{(s_1,t_1)}$ and $P^{(s_1,t_1)}$ is a congruence on L, by Result 3.B, $(x \wedge y, x \vee y) \in P^{(s_1,t_1)}$. By Result 3.C, $(z_i, z_{i+1}) \in P^{(s_1,t_1)}$ for each $i \in \{1, 2, \cdots, n-1\}$. Let $i \in \{1, 2, \cdots, n-1\}$. Then

$$\begin{aligned} & \mu_{P \wedge (Q \vee R)}(x,y) \\ = & \ \mu_P(x,y) \wedge \mu_{Q \vee R}(x,y) \\ = & \ s_1 \wedge s_2 \\ \leq & \ \mu_P(z_i, z_{i+1}) \wedge (\mu_Q(z_i, z_{i+1}) \vee \mu_R(z_i, z_{i+1})) \\ = & \ \mu_{P \wedge Q}(z_i, z_{i+1}) \vee \mu_{P \wedge R}(z_i, z_{i+1}) \\ = & \ \mu_{(P \wedge Q) \cup (P \wedge R)}(z_i, z_{i+1}) \end{aligned}$$

and

$$\begin{aligned} & \nu_{P \wedge (Q \vee R)}(x,y) \\ = & \ \nu_P(x,y) \vee \nu_{Q \vee R}(x,y) = t_1 \vee t_2 \\ \geq & \ \nu_P(z_i, z_{i+1}) \vee (\nu_Q(z_i, z_{i+1}) \wedge \nu_R(z_i, z_{i+1})) \\ = & \ \nu_{P \wedge Q}(z_i, z_{i+1}) \wedge \nu_{P \wedge R}(z_i, z_{i+1}) \\ = & \ \nu_{(P \wedge Q) \cup (P \wedge R)}(z_i, z_{i+1}). \end{aligned}$$

Thus

$$\mu_{(P \wedge Q) \vee (P \wedge R)}(x,y) \geq \mu_{P \wedge (Q \vee R)}(x,y)$$

and

$$\nu_{(P \wedge Q) \vee (P \wedge R)}(x,y) \leq \nu_{P \wedge (Q \vee R)}(x,y).$$

So $P \wedge (Q \vee R) \subset (P \wedge Q) \vee (P \wedge R)$. Similarly, we have $(P \wedge Q) \vee (P \wedge R) \subset P \wedge (Q \vee R)$.
Hence $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$. This completes the proof.

4. Acknowledgements

The authors are highly grateful to referees for their valuable comments and suggestions for improving the paper.

References

1. K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1986), 87-96.

2. Baldev Banerjee and Dhiren Kr. Basnet, Intuitionistic fuzzy subrings and ideals, J.Fuzzy Math. 11(1)(2003), 139-155.

3. R.Biswas, Intuitionistic fuzzy subgroups, Mathematical Forum x(1989), 37-46.

4. H.Bustince and P.Burillo, Structures on intuitionistic fuzzy relations, Fuzzy Sets and Systems 78(1996), 293-303

5. D. Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88(1997), 81-89.

6. D. Çoker and A.Haydar Es, On fuzzy compactness in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 3(1995), 899-909.

7. G.Deschrijver and E.E.Kerre, On the composition of intuitionistic fuzzy relations, Fuzzy Sets and Systems 136(2003), 333-361.

8. G.Grätzer, Lattice Theory, First concepts and Distributive lattices, W.H.Freemann and Company, San Francisco (1971).

9. H.Gürçay, D. Çoker and A.Haydar Es, On fuzzy continuity in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 5(1997), 365-378.

10. K.Hur, S.Y.Jang and H.W.Kang,Intuitionistic fuzzy subgroupoids, International Journal of Fuzzy Logic and Intelligent Systems 3(1) (2003), 72-77.

11. K.Hur, H.W.Kang and H.K.Song, Intuitionistic fuzzy subgroups and subrings, Honam Mathematical J.25(1)(2003), 19-41.

12. K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroups and cosets, Honam Math.J,26(1)(2004), 17-41.

13. K.Hur, Y.B.Jun and J.H.Ryou, Intuitionistic fuzzy topological groups, Honam Mathematical J.26(2)(2004), 163-192.

14. K.Hur, J.H.Kim and J.H.Ryou, Intuitionistic fuzzy topological spaces, J.Korea Soc. Math.Educ.Ser.B : Pure Appl.Math.11(3)(2004), 243-265.

15. K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets, Honam Math.J.26(4)(2004), 559-587.

16. ____________________, Intuitionistic fuzzy congruences on a lattice, J.Appl.Math.and Computing 18(1-2)(2005), 465-486.

17. K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy equivalence relations, Honam Math. J. 27(2)(2005), 163-181.

18. K.Hur, S.Y.Jang and Y.B.Jun, Intuitionistic fuzzy congruences, Far East J.Math. Sci.(FJMS) 17(1)(2005), 1-29.

19. S.J.Lee and E.P.Lee, The category of intuitionistic fuzzy topological spaces, Bull. Korean Math. Soc. 37(1)(2000), 63-76.

20. B.Schweizer and A.Sklar, Statistical metric spaces, Pacific J.Math.10(1960), 313-334.

21. F.I.Sidky and M.M.Atallah, On T-congruences of lattices, The J.Fuzzy Math.6(4)(1998), 993-1000.

22. L.A.Zadeh, Fuzzy sets, Inform. and Control 8(1965), 338-353.

Tae Chon Ahn

Professor of Wonkwang University
Research Area:
E-mail : {tcahn, nado}@wonkwang.ac.kr

Kul Hur

Professor of Wonkwang University
Research Area: Fuzzy mathematics, Fuzzy topology, Fuzzy hyperspace, Fuzzy algebra, General topology
E-mail : kulhur@wonkwang.ac.kr

Seok Beom Roh

Professor of Wonkwang University
Research Area:
E-mail : {tcahn, nado}@wonkwang.ac.kr