퍼지 및 지능시스템학회 논문지 2006, Vol. 16, No. 3, pp. 321-325

A common fixed point theorem in the intuitionistic fuzzy metric space

Jong Seo Park[1], Seon Yu Kim and Hong Jae Kang

Department of Mathematics Education, Chinju National University of Education, Jinju, 660-756, Korea

Abstract

The purpose of this paper is to establish the common fixed point theorem in the intuitionistic fuzzy metric space in which it is a little revised in Park [11]. Our research are an extension of Jungck's common fixed point theorem [8] in the intuitionistic fuzzy metric space.

Key words : Common Fixed Point, Intuitionistic Fuzzy Metric Space

1. Introduction

Zadeh [17] was introduced to the concept of fuzzy sets, Lowen [10] is defined convergence in a fuzzy topological space which enables us to characterize fuzzy compactness. Grabiec [6], Park and Kim [12] are studied a fixed point in a fuzzy metric space introduced by Kramosil and Michalek [9], and Subrahmanyam [16] is proved a common fixed point theorem in fuzzy metric spaces.

On the other hand, Attanassov [1] generalized this idea to intuitionistic fuzzy sets, and later there has been much progress in the study of intuitionistic fuzzy sets by many authors [1-4, 11]. Also, Park [11] is defined an intuitionistic fuzzy metric space, and Park, Kwun and Park [13] are studied a fixed point theorem in an intuitionistic fuzzy metric space.

In this note, Jungck's common fixed point theorem in metric space is generalized in this intuitionistic fuzzy metric space in which it is a little revised in Park [11].

2. Preliminaries

Now, we will give some definitions, properties and notation of the intuitionistic fuzzy metric space.

Definition 2.1([15]). A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is continuous $t-$norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative,
(b) $*$ is continuous,
(c) $a*1=a$ for all $a\in[0,1]$,
(d) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ $(a,b,c,d\in[0,1])$.

Definition 2.2([15]). A binary operation $\diamond:[0,1]\times[0,1]\to[0,1]$ is continuous $t-$conorm if $\diamond$ is satisfying the following conditions:

(a) $\diamond$ is commutative and associative,
(b) $\diamond$ is continuous,
(c) $a\diamond 1=a$ for all $a\in[0,1]$,
(d) $a\diamond b\ge c\diamond d$ whenever $a\le c$ and $b\le d$ $(a,b,c,d\in[0,1])$.

Definition 2.3. The 5$-$tuple $(X,M,N,*,\diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and M,N are fuzzy sets on $X^2\times(0,\infty)$ satisfying the following conditions; for all $x,y,z\in X$, such that

(a) $M(x,y,t)>0$,
(b) $M(x,y,t)=1\Longleftrightarrow x=y$,
(c) $M(x,y,t)=M(y,x,t)$,
(d) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,
(e) $M(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous,
(f) $N(x,y,t)>0$,
(g) $N(x,y,t)=0\Longleftrightarrow x=y$,

접수일자 : 2006년 4월 13일
완료일자 : 2006년 5월 24일
[1]This paper is supported by the Chinju National University of Education Research Fund in 2006.

(h) $N(x,y,t) = N(y,x,t)$,

(i) $N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$,

(j) $N(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

In this note, we shall denote the intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ by X.

Lemma 2.1([6], [12]). In an intuitionistic fuzzy metric space X, $M(x,y,\cdot)$ is nondecreasing and $N(x,y,\cdot)$ is nonincreasing for all $x, y \in X$.

In all that follows **N** stands for the set of natural numbers and X stands for an intuitionistic fuzzy metric space X with the following properties:

$$\lim_{t\to\infty} M(x,y,t) = 1, \quad \lim_{t\to\infty} N(x,y,t) = 0 \tag{2.1}$$

for all $x, y \in X$

Lemma 2.2([13]). Let X be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology on X induced by the intuitionistic fuzzy metric. Then for a sequence $\{x_n\} \subset X$, $x_n \to x$ if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$.

Definition 2.4. Let X be an intuitionistic fuzzy metric space.

(a) A sequence $\{x_n\}$ in a intuitionistic fuzzy metric space X is called Cauchy if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$ for every $t > 0$ and each $p > 0$.

(b) X is complete if every Cauchy sequence in X converges in X.

(c) A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$, $\lim_{n\to\infty} N(x_n, x, t) = 0$ for each $t > 0$.

(d) A map $f : X \to X$ is called continuous at x_0 if $\{f(x_n)\}$ converges to $f(x_0)$ for each $\{x_n\}$ converging to x_0.

Lemma 2.3. If $\{x_n\}$ is a sequence in an intuitionistic fuzzy metric space X and $M(x_n, x_{n+1}, t) \geq M(x_0, x_1, \frac{t}{\alpha^n})$, $N(x_n, x_{n+1}, t) \leq N(x_0, x_1, \frac{t}{\alpha^n})$ where α is a positive number with $0 < \alpha < 1$ and $n = 1, 2, \cdots$, $s * s \geq s$, $r \diamond r \leq r$ for $s, r \in [0,1]$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof. For each p, by Definition 2.3 (c),

$$\begin{aligned}
&M(x_{n+p}, x_n, t)\\
&\geq M(x_n, x_{n+1}, \frac{t}{p}) * M(x_{n+1}, x_{n+2}, \frac{t}{p})\\
&\quad * \cdots * M(x_{n+p-1}, x_{n+p}, \frac{t}{p})\\
&\geq M(x_0, x_1, \frac{t}{p\alpha^n}) * M(x_0, x_1, \frac{t}{p\alpha^{n+1}})\\
&\quad * \cdots * M(x_0, x_1, \frac{t}{p\alpha^{n+p-1}}),
\end{aligned}$$

$$\begin{aligned}
&N(x_{n+p}, x_n, t)\\
&\leq N(x_n, x_{n+1}, \frac{t}{p}) \diamond N(x_{n+1}, x_{n+2}, \frac{t}{p})\\
&\quad \diamond \cdots \diamond N(x_{n+p-1}, x_{n+p}, \frac{t}{p})\\
&\leq N(x_0, x_1, \frac{t}{p\alpha^n}) \diamond M(x_0, x_1, \frac{t}{p\alpha^{n+1}})\\
&\quad \diamond \cdots \diamond N(x_0, x_1, \frac{t}{p\alpha^{n+p-1}}).
\end{aligned}$$

Since

$$\lim_{n\to\infty} M(x_0, x_1, \frac{t}{p\alpha^n}) = 1, \ \lim_{n\to\infty} N(x_0, x_1, \frac{t}{p\alpha^n}) = 0,$$

by (2.1)

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) \geq 1 * 1 * \cdots * 1 \geq 1,$$

$$\lim_{n\to\infty} N(x_{n+p}, x_n, t) \leq 0 \diamond 0 \diamond \cdots \diamond 0 \leq 0.$$

Hence

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1,$$
$$\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$$

Therefore, by Definition 2.4 (a), $\{x_n\}$ is a Cauchy sequence in X.

3. Result

The following theorem has a intuitionistic fuzzy analogue for Jungck's [8].

Theorem 3.1. Let X be a complete intuitionistic fuzzy metric space and let $f, g : X \to X$ be maps that satisfy the following conditions:

(a) $g(X) \subseteq f(X)$.

(b) f is continuous.

(c) M and N are satisfied the following conditions:

$$\begin{aligned}
&M(g(x), g(y), \alpha t) \geq M(f(x), f(y), t),\\
&N(g(x), g(y), \alpha t) \leq N(f(x), f(y), t)
\end{aligned} \tag{3.1}$$

for all $x, y \in X$, $t > 0$ and $0 < \alpha < 1$.

Then f and g have a unique common fixed point provided f and g commute.

Proof. Let $x_0 \in X$. By condition (a), we can find x_1 such that $f(x_1) = g(x_0)$. Therefore we can define a sequence $\{x_n\} \subset X$ such that $f(x_n) = g(x_{n-1})$ by induction. Also,

$$\begin{aligned} & M(f(x_n), f(x_{n+1}), t) \\ & = M(g(x_{n-1}), g(x_n), t) \\ & \geq M(f(x_{n-1}), f(x_n), \frac{t}{\alpha}) \\ & = M(g(x_{n-2}), g(x_{n-1}), \frac{t}{\alpha}) \\ & \geq M(f(x_{n-2}), f(x_{n-1}), \frac{t}{\alpha^2}) \\ & \cdots\cdots\cdots\cdots \\ & \geq M(f(x_0), f(x_1), \frac{t}{\alpha^n}) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} & N(f(x_n), f(x_{n+1}), t) \\ & = N(g(x_{n-1}), g(x_n), t) \\ & \leq N(f(x_{n-1}), f(x_n), \frac{t}{\alpha}) \\ & = N(g(x_{n-2}), g(x_{n-1}), \frac{t}{\alpha}) \\ & \leq N(f(x_{n-2}), f(x_{n-1}), \frac{t}{\alpha^2}) \\ & \cdots\cdots\cdots\cdots \\ & \leq N(f(x_0), f(x_1), \frac{t}{\alpha^n}). \end{aligned} \tag{3.3}$$

So for any positive integer p,

$$\begin{aligned} & M(f(x_n), f(x_{n+p}), t) \\ & \geq M(f(x_n), f(x_{n+1}), \frac{t}{p}) * M(f(x_{n+1}), f(x_{n+2}), \frac{t}{p}) \\ & \quad * \cdots * M(f(x_{n+p-1}), f(x_{n+p}), \frac{t}{p}) \\ & \geq M(f(x_0), f(x_1), \frac{t}{p\alpha^n}) * \cdots \\ & \quad \cdots * M(f(x_0), f(x_1), \frac{t}{p\alpha^{n+p-1}}) \end{aligned}$$

and

$$\begin{aligned} & N(f(x_n), f(x_{n+p}), t) \\ & \leq N(f(x_n), f(x_{n+1}), \frac{t}{p}) \diamond N(f(x_{n+1}), f(x_{n+2}), \frac{t}{p}) \\ & \quad \diamond \cdots\cdots \diamond N(f(x_{n+p-1}), f(x_{n+p}), \frac{t}{p}) \\ & \leq N(f(x_0), f(x_1), \frac{t}{p\alpha^n}) \diamond \cdots \\ & \quad \cdots \diamond N(f(x_0), f(x_1), \frac{t}{p\alpha^{n+p-1}}). \end{aligned}$$

By (2.1),

$$\lim_{n\to\infty} M(f(x_0), f(x_1), \frac{t}{p\alpha^n}) = 1,$$
$$\lim_{n\to\infty} N(f(x_0), f(x_1), \frac{t}{p\alpha^n}) = 0,$$

from (3.2) and (3.3),

$$\begin{aligned} \lim_{n\to\infty} M(f(x_n), f(x_{n+p}), t) & \geq 1 * 1 * \cdots * 1 \\ & \geq 1, \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} N(f(x_n), f(x_{n+p}), t) & \leq 0 \diamond 0 \diamond \cdots \diamond 0 \\ & \leq 0. \end{aligned}$$

Therefore by Definition 2.4 and Lemma 2.3, $\{f(x_n)\}$ is Cauchy sequence. By the completeness of X in assumption, there exist $w \in X$ such that $\{f(x_n)\}$ converges to w. So $g(x_{n-1}) = f(x_n)$ tends to w as $n \to \infty$. It can be seen from the condition (b) of theorem that the continuity of f implies that of g.

So, $\{g(f(x_n))\} \to g(w)$. However, $g(f(x_n)) = f(g(x_n))$ from commutativity of f and g. Hence $f(g(x_n))$ converges to $f(w)$. Since the limits are unique, $f(w) = g(w)$. Also, $f(f(w)) = f(g(w))$ by commutativity and

$$\begin{aligned} & M(g(w), g(g(w)), t) \\ & \geq M(f(w), f(g(w)), \frac{t}{\alpha}) \\ & \cdots\cdots\cdots \\ & \geq M(g(w), g(g(w)), \frac{t}{\alpha^n}) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & N(g(w), g(g(w)), t) \\ & \leq N(f(w), f(g(w)), \frac{t}{\alpha}) \\ & \cdots\cdots\cdots \\ & \leq N(g(w), g(g(w)), \frac{t}{\alpha^n}). \end{aligned} \tag{3.5}$$

By Definition 2.3, (2.1), (3.4) and (3.5),

$$M(g(w), g(g(w)), t) = 1, \qquad N(g(w), g(g(w)), t) = 0.$$

Hence $g(w) = g(g(w))$, and $g(w) = g(g(w)) = f(g(w))$. Therefore $g(w)$ is a common fixed point of f and g.

If x, z are two fixed points common to f and g, then

$$\begin{aligned} 1 &\geq M(x,z,t) \\ &= M(g(x), g(z), t) \\ &\geq M(f(x), f(z), \frac{t}{\alpha}) \\ &= M(x, z, \frac{t}{\alpha}) \cdots\cdots \\ &\geq M(x, z, \frac{t}{\alpha^n}) \to 1 \quad \text{as} \quad n \to \infty, \\ 0 &\leq N(x,z,t) \\ &= N(g(x), g(z), t) \\ &\leq N(f(x), f(z), \frac{t}{\alpha}) \\ &= N(x, z, \frac{t}{\alpha}) \cdots\cdots \\ &\leq N(x, z, \frac{t}{\alpha^n}) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore $x = z$ by Definition 2.3.

Example 3.1. Let (X, d) be a metric space. Denote $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and let M_d, N_d be fuzzy sets on $X^2 \times (0, \infty)$ defines as follows:

$$(3.6) \qquad \begin{aligned} M_d(x,y,t) &= \frac{t}{t + d(x,y)} \\ N_d(x,y,t) &= \frac{d(x,y)}{t + d(x,y)} \quad \text{if} \quad x, y \in X \end{aligned}$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X and $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space.

In this case, let X be the set $\{\frac{1}{n} : n \in \mathbf{N}\} \cup \{0\}$ with the metric d defined by $d(x,y) = |x - y|$, then from (3.6)

$$\begin{aligned} M_d(x,y,t) &= \frac{t}{t + |x-y|}, \\ N_d(x,y,t) &= \frac{|x-y|}{t + |x-y|} \quad \text{if} \quad x, y \in X. \end{aligned}$$

Clearly, $(X, M_d, N_d, *, \diamond)$ is a complete intuitionistic fuzzy metric space. Define $g(x) = \frac{x}{6}$, $f(x) = \frac{x}{3}$ on X. It is evident that $g(X) \subseteq f(X)$. Also, for $\alpha = \frac{1}{2}$,

$$\begin{aligned} M(g(x), g(y), \frac{t}{2}) &= \frac{\frac{t}{2}}{\frac{t}{2} + \frac{|x-y|}{6}} \\ &= \frac{3t}{3t + |x-y|} \\ &\geq M(f(x), f(y), t) \\ &= \frac{t}{t + \frac{|x-y|}{3}} \\ &= \frac{3t}{3t + |x-y|}, \end{aligned}$$

$$\begin{aligned} N(g(x), g(y), \frac{t}{2}) &= \frac{\frac{|x-y|}{6}}{\frac{t}{2} + \frac{|x-y|}{6}} \\ &= \frac{|x-y|}{3t + |x-y|} \\ &\leq N(f(x), f(y), t) \\ &= \frac{\frac{|x-y|}{3}}{t + \frac{|x-y|}{3}} \\ &= \frac{|x-y|}{3t + |x-y|}. \end{aligned}$$

Thus all the conditions of the above Theorem 3.1 are satisfied, and f and g have the common fixed point 0.

Appendix(Jungck's Theorem). Let f be a continuous mapping of a complete metric space (X, d) into itself and let $g : X \to X$ be a map. If

(a) $g(X) \subseteq f(X)$,

(b) g commutes with f,

(c) $d(g(x), g(y)) \leq \alpha d(f(x), f(y))$ for some $\alpha \in (0, 1)$ and all x and y in X,

then f and g have a unique common fixed point.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87–96.

[2] K. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **61** (1994), 137–142.

[3] K. Atanassov and S. Stoeva, *Intuitionistic fuzzy sets*, in:Polish Symp. on Interval and Fuzzy Mathematics, Poznan(August), (1986), 23–26.

[4] D. Coker, *On fuzzy inclusion in intuitionistic sense*, J. Fuzzy Math., **4** (1996), 701–714.

[5] M. Edelstein, *On fixed point and periodic points under contractive mappings*, J. London Math. Soc., **37** (1962), 74–79.

[6] M. Grabiec, *Fixed point in fuzzy metric spaces*, Fuzzy Sets and Systems, **27** (1988), 385–389.

[7] G. Jungck, *Commutating maps and fixed points*, Amer. Math. Monthly, **83** (1976), 261–263.

[8] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., **9** (1986), 771–774.

[9] J. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, **11** (1975), 326–334.

[10] R. Lowen, *Convergence in fuzzy topological spaces*, General Topology and its Applns., **10** (1979), 147–160.

[11] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals, **22(5)** (2004), 1039–1046.

[12] J. S. Park and S. Y. Kim, *A fixed point Theorem in a fuzzy metric space*, F. J. M. S., **1(6)** (1999), 927–934.

[13] J. S. Park, Y. C. Kwun and J. H. Park, *A fixed point theorem in the intuitionistic fuzzy metric spaces*, F. J. M. S. **16(2)** (2005), 137–149.

[14] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 314–334.

[15] B. Schweizer and A. Sklar, *Probabilistic metric spaces*, North-Holland, New York, Oxford, (1983).

[16] P. V. Subrahmanyam, *A common fixed point theorem in fuzzy metric spaces*, Inform. Sci., **83** (1995), 109–112.

[17] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8** (1965), 338–353.

저 자 소 개

Jong Seo Park

Associate Professor of Chinju National University of Education
Research Area: Fuzzy mathematics, System theory and control, Fuzzy differential equation
E-mail : parkjs@cue.ac.kr

Seon Yu Kim

Professor of Chinju National University of Education
Research Area: Fuzzy mathematics, Fuzzy differential equation
E-mail : sykim@cue.ac.kr

Hong Jae Kang

Full-time Lecturer of Chinju National University of Education
Research Area: Fuzzy mathematics, Fuzzy linear programming
E-mail : kanghj@cue.ac.kr