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WEAKLY B-REGULAR NEAR-RINGS

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ABSTRACT. The notion of regularity in near-ring was generalized by the concept of b-regular and some characterizations of the same was obtained through the substructures viz bi-ideals in near-rings. In this paper, we generalize further and introduce the notion of weakly b-regular near-rings and obtain a characterization of the same.

1. INTRODUCTION

Throughout this paper by a near-ring we mean a right near-ring. For basic definitions one may refer to Pilz [4]. Tamizh Chelvam and Ganesan [7] introduced the notion of bi-ideals in near-rings. Further Tamizh Chelvam [5] introduced the concept of b-regular near-rings and obtained equivalent conditions for regularity in terms of bi-ideals.

2. PRELIMINARIES

In fact, the following result in that context generalizes the result of Kovacs [2] for rings.

Theorem 1 (Tamizh Chelvam [5]). *Let N be a near-ring Then the following are equivalent.*

(i) *N is b-regular.*

(ii) *$RL = R \cap L$ for every left N-subgroup L of N and for every right N-subgroup R of N.*

(iii) *For every pair of elements a and b in N, $(a)_r \cap (b)_l = (a)_r (b)_l$.*

(iv) *For any element $a \in N$, $(a)_r \cap (a)_l = (a)_r (a)_l$.*

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Corollary 1 (Kovacs [2]). *A ring R is regular if and only if $A \cap B = AB$ for every right ideal A and every left ideal B of R.*

In this paper we further generalize and introduce the notion of weakly b-regular near-rings. In order to have a characterization for this, we also introduce the notion of strong bi-ideals, using which we obtain properties of weakly b-regular near-rings.

For any two subsets A and B of N, $AB = \{ab \mid a \in A, b \in B\}$ and $A * B = \{a_1(a_2 + b) - a_1a_2 \mid a_1, a_2 \in A$ and $b \in B\}$. A subgroup B of $(N, +)$ is said to be a *bi-ideal* of N if $BNB \cap (BN) * B \subseteq B$ [7]. In the case of a zero-symmetric near-ring, a subgroup B of $(N, +)$ is a bi-ideal if and only if $BNB \subseteq B$ [5]. A subgroup Q of $(N, +)$ is called a *quasi-ideal* of N if $QN \cap NQ \cap N * Q \subseteq Q$ [8]. If N is zero-symmetric, a subgroup Q of $(N, +)$ is a quasi-ideal of N if and only if $QN \cap NQ \subseteq Q$.

A near-ring N is said to have *additive property* if xN is a subgroup of $(N, +)$ for every $x \in N$. A near-ring N is said to be *sub commutative* if $xN = Nx$ for every $x \in N$. Note that every sub commutative near-ring N is a near-ring with additive property.

A near-ring N is said to be *left (right) unital* if $a \in Na(a \in aN)$ for all $a \in N$. A near-ring N is said to be *unital* if it is both left and right unital. An element $a \in N$ is said to be *regular* if $a = aba$ for some $b \in N$. A near-ring N is said to be *regular* if every element in N is regular. It may be noted that a regular near-ring is a unital near-ring, but not the converse.

An element $a \in N$ is said to be *strongly regular* if $a = ba^2$, for some $b \in N$. A near-ring N is called *strongly regular* if every element in N is strongly regular. N is said to satisfy *Insertion of Factors Property (: IFP)* if $ab = 0$ implies $axb = 0$ for all $x \in N$. A near-ring is called *left bi-potent* if $Na = Na^2$ for $a \in N$.

A subgroup M of $(N, +)$ is said to be a *left (right) N-subgroup* if $NM \subseteq M(MN \subseteq M)$. A near-ring N is said to be *two sided* if every left N-subgroup is a right N-subgroup and vice versa. A near-ring N is called *b-regular near-ring* if $a \in (a)_rN(a)_l$ for every $a \in N$ where $(a)_r$ $((a)_l)$ is the right (left) N-subgroup generated by $a \in N$[6]. Note that every regular near-ring is b-regular.

A near-ring N is said to be *left permutable* if $abc = bac$ for all $a, b, c \in N$. Let E be the set of all idempotents of N and L the set of all nilpotent elements of N. A near-ring N is called a *generalized near-field (: GNF)* if for each $a \in N$, there exists a unique $b \in N$ such that $a = aba$ and $b = bab$ [3].

3. Strong Bi-Ideals

In this section, we introduce strong bi-ideals and obtain some of the properties of the same.

Definition 1. A bi-ideal B of N is said to be a *strong bi-ideal* if $NB^2 \subseteq B$.

Example 1. Let N be the near-ring constructed on the Klein's 4-group according to the scheme (0, 13, 0, 13) (p. 408, Pilz [4]). Note that $\{0, a\}$ is a strong bi-ideal of N.

Example 2. Let $N = \{0, a, b, c\}$ be a near-ring constructed on the Klein's 4-group according to the scheme (0, 1, 1, 1) (p. 408, Pilz [4]). Note that $\{0, a\}$ is a bi-ideal which is not a strong bi-ideal of N.

Proposition 1. *The set of all strong bi-ideals of a near-ring N forms a Moore system on N.*

Proof. Let $\{B_i | i \in I\}$ be a set of strong bi-ideals in N. Let $B = \cap B_i, i \in I$. Then $NB^2 \subseteq NB_iB_i = NB_i^2 \subseteq B_i$ for every $i \in I$. Hence $NB^2 \subseteq B$. Therefore B is a strong bi-ideal of N. □

Proposition 2. *Let N be a near-ring and B a strong bi-ideal of N. If elements of B are strongly regular, then B is a quasi-ideal of N.*

Proof. Let $x \in BN \cap NB$. Then $x = bn = n'b'$ for some $b, b' \in B$ and $n, n' \in N$. Since B is strongly regular, $b = cb^2$ and $b' = db'^2$ for some $c, d \in B$. Hence $x = bn = cb^2n$ $= cbbn = cbn'b' = cbn'db'^2 \subseteq NB^2 \subseteq B$, i.e., $BN \cap NB \subseteq B$. Hence B is a quasi-ideal of N. □

Proposition 3. *Let N be a left permutable near-ring and B a bi-ideal of N. If the elements of B are strongly regular, then B is strong bi-ideal of N if and only if B is a quasi-ideal of N.*

Proof. Only if part follows from Proposition 3.5. Conversely, if $x \in NB^2$, then $x = nb_1b_2 \in NB^2$. Since N is left permutable, $x = b_1nb_2 \subseteq BN \cap NB \subseteq B$. Therefore $NB^2 \subseteq B$ and hence B is a strong bi-ideal. □

Proposition 4. *Let N be a left permutable near-ring. If B is a strong bi-ideal of N, then nB and Bn' are strong bi-ideals of N, where n is a distributive element of N and $n' \in N$.*

Proof. Since $n, n' \in N$ and n is distributive, nB and Bn' are bi-ideals of N. If $x \in N(Bn')^2 = NBn'Bn'$, then $x = n_1bn'b'n' = n_1n'bb'n' \in NBBn' = NB^2n' \subseteq Bn'$. Similarly if $x \in N(nB)^2 = NnBnB$, then $x = n_1nbnb_1 = nn_1bnb_1 = nn_1nbb_1 \subseteq nNBB = nNB^2 \subseteq nB$. Therefore nB and Bn' are strong bi-ideals of N. □

Proposition 5. *If B is a strong bi-ideal of a near-ring N and S is a sub near-ring of N, then $B \cap S$ is a strong bi-ideal of S.*

Proof. Let $C = B \cap S$. Now $SC^2 = SCC = S(B \cap S)(B \cap S) = S((BB \cap SB) \cap (BS \cap SS)) \subseteq S(BB \cap SS) \subseteq SB^2 \cap SS \subseteq B \cap S = C$, i.e., $SC^2 \subseteq C$ and so $B \cap S$ is a strong bi-ideal of S. □

Proposition 6. *If N is a left permutable near-ring, then B is a bi-ideal if and only if B is a strong bi-ideal.*

Proof. If part is trivial. Conversely suppose B is a bi-ideal of N. For $x \in NB^2$, since N is left permutable, $x = nb_1b_2 = b_1nb_2 \in BNB \subseteq B$. i.e., B is a strong bi-ideal of N. □

Proposition 7. *Let N be a near-ring. If N is strongly regular, then $B = NB^2$ for every strong bi-ideal B of N.*

Proof. Let B be a strong bi-ideal of N. Trivially $NB^2 \subseteq B$. Let $b \in B$. Since N is strongly regular, there exists $x \in N$ such that $b = xb^2 \subseteq NB^2$, i.e., $B \subseteq NB^2$. □

Proposition 8. *Let N be a left permutable and left unital near-ring. Then N is strongly regular if and only if $B = NB^2$ for every strong bi-ideal B of N.*

Proof. Only if part follows from Proposition 3.10. Conversely, since Na is a strong bi-ideal of N, $a \in Na = N(Na)^2 \subseteq NaNa$, i.e., $a = n_1an_2a = n_1n_2aa \in Na^2$. This implies that N is strongly regular. □

Since strongly regular and left bi-potent are equivalent in a left unital near ring (Theorem 1.12 [1]), we have the following proposition.

Proposition 9. *Let N be a left permutable and left unital near ring. Then N is left bi-potent if and only if $B = NB^2$ for every strong bi-ideal B of N.*

Proposition 10. *Let N be a left permutable near-ring and B a bi-ideal of N. Then $B = BNB$ if and only if $B = NB^2$.*

Proof. Assume that $B = BNB$ for a bi-ideal B of N. By the Proposition 3.9, B is a strong bi-ideal of N. If $x \in B = BNB$, then $x = b_1nb_2$ for some $b_1, b_2 \in B$ and $n \in N$. Since N is left permutable, $x = nb_1b_2 \in NB^2$. i.e., $B \subseteq NB^2$. Conversely if $B = NB^2$ for every bi-ideal B of N, then for $x \in B = NB^2$, $x = nb_1b_2 = b_1nb_2 \in BNB$, i.e., $B \subseteq BNB$. □

Theorem 2. *Let N be a left permutable and left unital near-ring. Then the following conditions are equivalent.*

(i) $NB^2 = B$ *for every bi-ideal B of N*
(ii) *N is regular*
(iii) *N is left bi-potent*
(iv) $B = BNB$ *for every bi-ideal B of N*
(v) $Q = QNQ$ *for every quasi-ideal Q of N*
(vi) *N is strongly regular.*

Proof. (i) ⇒ (ii) Let $a \in N$. Since Na is a strong bi-ideal of N and N is a left unital near ring, we have $a \in Na = N(Na)^2 \subseteq NaNa$. i.e., $a = xaya$ for some $x, y \in N$. Since N is left permutable, $a = axya \in aNa$. Hence N is regular.

(ii) ⇒ (iii) Let $x \in Na$. Since N is regular, $x \in Na \subseteq NaNa$ and so $x = n_1an_2a = n_1n_2a^2 \subseteq Na^2$. This implies that $Na \subseteq Na^2$ and so $Na = Na^2$.

(iii) ⇒ (iv) By Theorem 1.10 [1], N is regular and so $B = BNB$ for every bi-ideal B of N.

(iv) ⇒ (v) Assume that $B = BNB$ for every bi-ideal B of N. Let $a \in N$. Since N is a left unital near ring and N is left permutable Na is a bi-ideal of N containing $a \in N$. Thus by the assumption, we have $a \in Na = NaNNa \subseteq NaNa \subseteq aNNa$, i.e., N is regular. Let Q be a quasi-ideal of N. Since every quasi-ideal is a bi-ideal we get $QNQ \subseteq Q$. Also if $q \in Q$, then $q = qq_1q \in QNQ$. Thus $Q = QNQ$.

(v) ⇒ (vi) Since Na is a quasi-ideal, $Na = NaNNa$. Since N is a left permutable left unital near-ring $a \in Na \subseteq NaNa \subseteq Na^2$. i.e., N is strongly regular.

(vi) ⇒ (i) Follows from Proposition 3.11. □

4. Weakly b-Regular Near-Rings

In this section, we introduce the notion of weakly b-regular near-ring and obtain some characterization of the same.

Definition 2. A near-ring N is called *left (right) weakly b-regular,* if $a \in N(a)_l(a \in (a)_rN)$ for every $a \in N$ where $(a)_l((a)_r)$ is the left(right) N-subgroup generated by $a \in N$. A near-ring N is called *weakly b-regular* if N is both left and right weakly b-regular.

Every regular near-ring is b-regular and so weakly b-regular. However there exist near-rings which are weakly b-regular but not b-regular.

Example 3. Let N be the near-ring defined on the cyclic group $(Z_4, +)$ with multiplication as per the scheme 11: (0 1 3 2) (p. 407, Pilz [4]). This near-ring is weakly b-regular, but not b-regular since $2 \in (2)_rN(2)_l$.

Proposition 11. *Let N be a near-ring. Then the following are equivalent.*

(i) *N is weakly b-regular*

(ii) *$RN \cap NL = R \cap L$ for every right N-subgroup R and left N-subgroup L of N.*

(iii) *For any element a of N, $(a)_rN \cap N(a)_l = (a)_r \cap (a)_l$.*

Proof. (i) $\Rightarrow$ (ii) Let R and L be right and left N-subgroups of N respectively. Let $x \in R \cap L$. Since N is weakly b-regular, $x \in (x)_rN \cap N(x)_l \subseteq RN \cap NL$, i.e., $R \cap L \subseteq RN \cap NL$. But trivially $RN \cap NL \subseteq R \cap L$. Hence $RN \cap NL = R \cap L$.

(ii) $\Rightarrow$ (iii) Trivially true.

(iii) $\Rightarrow$ (i) Let $a \in N$. Then $a \in (a)_r \cap (a)_l = (a)_rN \cap N(a)_l$, i.e., N is weakly b-regular. □

Proposition 12. *Let N be a near-ring. Then the following conditions are equivalent.*

(i) *Every right N subgroup is idempotent and $N(x)_l = (x)_rN$ for every $x \in N$.*

(ii) *N is weakly b-regular, two sided and $(x)_rN = (x)_r^2N$ for every $x \in N$.*

(iii) *N is b-regular and two sided.*

Proof. (i) $\Rightarrow$ (ii) If $x \in N$, then $x \in (x)_r = (x)_r^2 \subseteq (x)_rN = N(x)_l$. This implies that N is weakly b-regular. By the assumption $(x)_rN = (x)_r^2N$. Let L be any left N-subgroup. For $x \in L$, $xN \subseteq (x)_rN = N(x)_l \subseteq (x)_l \subseteq L$. Hence L is a right N-subgroup of N. Similarly one can prove that every right N-subgroup is also a left N-subgroup Thus N is two sided.

(ii) $\Rightarrow$ (iii) Let $x \in N$. By the assumption, N is two sided, i.e., $(x)_l = (x)_r$. Since N is weakly b-regular, by the Proposition 4.3 we get that $x \in (x)_r \cap (x)_l = (x)_rN \cap N(x)_l = (x)_r^2N \cap N(x)_l \subseteq (x)_r(x)_r = (x)_r(x)_l$. i.e., $(x)_r \cap (x)_l \subseteq (x)_r(x)_l$.

Trivially $(x)_r(x)_l \subseteq (x)_r \cap (x)_l$ and so $(x)_r(x)_l = (x)_r \cap (x)_l$. By Proposition 3.2 [5], N is b-regular.

(iii) $\Rightarrow$ (i) Let M be a right N-subgroup of N. Then M is a sub near-ring of N and $MNM \cap MN \cap M$. Since N is b-regular, for $m \in M$, $m \in (m)_rN(m)_l \subseteq MNM$, i.e., $M \subseteq MNM$. Hence we observe that $M = MNM \subseteq MM = M^2$ and so M is idempotent. Since $(x)_l$ is a bi-ideal and N is b-regular and two sided, $(x)_lN(x)_l = (x)_l$. Now $(x)_rN \subseteq (x)_r = (x)_l = (x)_lN(x)_l \subseteq N(x)_l \subseteq (x)_l = (x)_r \subseteq (x)_rN(x)_r \subseteq (x)_rN$ and so $(x)_rN = N(x)_l$. □

Theorem 3. *Let N be a unital near-ring with additive property. Then the following are equivalent.*

(i) *N is two sided and every quasi-ideal of N is idempotent*
(ii) *N is b-regular and two sided*
(iii) *N is weakly b-regular and $RN \cap NL = LR$ for every left N-subgroup L and right N-subgroup R of N*
(iv) *N is regular and sub commutative*
(v) *N is a GNF*
(vi) *$B = NB^2$ for every strong bi-ideal B of N and N is sub commutative*
(vii) *$B = BNB$ for every bi-ideal B of N and N is sub commutative.*

Proof. (i) $\Rightarrow$ (ii) Suppose A and B are two quasi-ideals of N. Then $A \cap B$ is also a quasi-ideal of N. By the assumption we have, $A \cap B = (A \cap B)^2 \subseteq AB \cap BA$. Trivially we have $AB \cap BA \subseteq AN \cap NA \subseteq A$. Thus we get that $AB \cap BA \subseteq A \cap B$. From this, for $a \in N$, we have $(a)_r \cap (a)_l = (a)_r(a)_l$ since left and right N-subgroups are also quasi-ideals. Since $a \in (a)_r \cap (a)_l = (a)_r(a)_l, a = bc$ for some $b \in (a)_r$ and $c \in (a)_l$. Similarly $b = de$ for some $d \in (b)_r \subseteq (a)_r$ and $e \in (b)_l$. Thus $a = dec \in (a)_rN(a)_l$. i.e., N is b-regular.

(ii) $\Rightarrow$ (iii) Trivially N is weakly b-regular. Thus by Proposition 4.3, $RN \cap NL = R \cap L$. Since N is b-regular and two sided, by Lemma 3 [6], $R \cap L = LR$. i.e., $RN \cap NL = LR$.

(iii) $\Rightarrow$ (iv) Let R and L be right and left N-subgroups respectively. Since N is weakly b-regular, by Proposition 4.3, $RN \cap NL = R \cap L$. By our assumption $RN \cap NL = LR$ and so $R \cap L = LR$. Taking L as N, we get that $R = NR$ and so R is a left N-subgroup of N. Similarly L is a right N-subgroup of N. Hence N is two-sided. Since N is a unital near-ring with additive property, $a \in (a)_r \cap (a)_l =$

$aN \cap Na = aNNa$. i.e, N is regular. Also by assumption $Na = (a)_l = (a)_r = aN$ and hence N is sub commutative.

(iv) $\Rightarrow$ (v) Follows from Theorem 1 [3].

(v) $\Rightarrow$ (vi) Let B be a strong bi-ideal B of N and $b \in B$. By the assumption, we have $b = bcb$. Also by the Theorem 1[3], idempotents lie in the center. Since cb is an idempotent, $b = cb^2 = Nb^2 \subseteq NB^2$. Thus $B = NB^2$ for every strong bi-ideal B of N.

(vi) $\Rightarrow$ (vii) Since N is sub commutative, every bi-ideal is a strong bi-ideal and hence $B = BNB$ for every bi-ideal B of N.

(vii) $\Rightarrow$ (i) Let $a \in N$. Since N is a unital near-ring with additive property, $a \in aN \cap Na = (aN \cap Na)N(aN \cap Na) \subseteq aNNa$. i.e., N is regular. Let Q be any quasi-ideal in N. Trivially $Q^2 \subseteq QN \cap NQ \subseteq Q$. On the other hand let $a \in Q$. Then $a = aba = a^2c \in Q^2N$. Thus $a \in Q^2N \cap QNQ = Q(QN \cap NQ) \subseteq Q^2$. Thus $Q = Q^2$. i.e., every quasi-ideal is idempotent. Since N is a unital near-ring with additive property, N is two sided. □

REFERENCES

1. J. L. Jat & S. C. Choudhary: On left bipotent near-rings. *Proc. Edinburgh Math. Soc. (2)* **22** (1979), no. 2, 99-107. MR **80j**:16024
2. L. Kovacs: A note on regular rings. *Publ. Math. Debrecen* **4** (1956), 465-468. MR **18**,188c
3. C. V. L. N. Murty: Generalized near-fields. *Proc. Edinburgh Math. Soc.* **27** (1984), no. 2, 21-24. MR **85c**:16054
4. G. Pilz: *Near-rings, the theory and its applications.* North-HollandPublishing Co., Amsterdam (1983). MR **85h**:16046
5. T. Tamizh Chelvam: Bi-ideals and *b*-regular near-rings. *J. Ramanujan Math. Soc.* **7** 1982, no. 2, 155-164. MR **94a**:16080
6. ______: Bi-regular near-rings. *Math. Student* **62** (1993), no. 1-4, 89-92. MR **94f**:16065
7. T. Tamizh Chelvam & N. Ganesan: On bi-ideals of near-rings. *Indian J. Pure Appl. Math.* **18**, (1987), no. 11, 1002-1005. MR **88m**:16041
8. I. Yakabe: Quasi-ideals in near-rings. *Math. Rep. Kyushu Univ.* **14** (1983), 41-46. MR **85b**:16036

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