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L-FUZZY GRADATION OF OPENNESS AND L-FUZZY GRADATION OF PROXIMITY

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ABSTRACT. In this paper we study lattice valued fuzzy gradation of openness so that fuzzy gradation of openness [13] could be obtained as a particular case. Some of its properties are studied. We also give definitions of lattice valued graded fuzzy filters, graded fuzzy grills, graded fuzzy preproximities and proximities.

1. INTRODUCTION

In the definition of Chang's fuzzy topology [1], fuzziness in the concept of openness of a fuzzy subset is absent. The fundamental idea of a fuzzy topology with fuzziness in the topology i.e., a topology being a fuzzy subset of a powerset was first appeared in a paper of Höhle [3]. Subsequently, different authors such as Kubiak [4], Šostak [11], Samanta, Chattopadhyay and Hazra [8,9], Ying [15], etc. developed this idea independently. In [9], Samanta, Chattopadhyay and Hazra gave a concept of gradation of openness as a mapping $\tau : I^X \to I$ (I = the closed unit interval $[0,1]$), satisfying the following axioms :

(1) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$,
(2) $\tau(\lambda_1 \cap \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$,
(3) $\tau(\cup_{i\in\Delta}\lambda_i) \geq \wedge_{i\in\Delta}\tau(\lambda_i)$.

Dually gradation of closedness $\mathcal{F}$ is also defined and using this concept properties of fuzzy closure operator was studied by Samanta and Chattopadhyay [10].

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On the otherhand, in [13], we introduced an idea of fuzzy gradation of openness by using a concept of fuzzy family as defined by Šostak [11]. In fact we have defined fuzzy gradation of openness as a mapping $\tau : I^X \to I$, where the arbitrary union condition and finite intersection condition are taken over fuzzy families of fuzzy subsets of X instead of crisp families of fuzzy subsets (which are taken for gradation of openness), thereby enhancing the involvement of fuzziness in the definition of fuzzy topology. We also extended this idea to generalized intuitionistic fuzzy setting [12].

In this paper, firstly we extend the concept of fuzzy gradation of openness to lattice valued setting. In fact, we define lattice valued fuzzy gradation of openness as a mapping $\tau : L^X \to L$, where the arbitrary union condition and finite intersection condition are takèn over L-valued fuzzy families of L-fuzzy subsets of X instead of crisp families of fuzzy subsets. We study the decomposition theorem of an L-fuzzy topological space to a family of Chang-fuzzy topological spaces. Characteristic properties of closure operator is also studied. We give the definitions of graded fuzzy filters, graded fuzzy grills and graded fuzzy proximities and obtain relation between a graded fuzzy proximity and a collection of graded fuzzy grills.

The organization of the paper is as follows:
Sections 1 and 2 consists of introduction and preliminaries respectively. In section 3, we study lattice valued fuzzy gradation of openness and obtain its basic properties. Definitions of graded fuzzy filters and graded fuzzy grills are given in section 4 and also their properties are studied. Lastly in section 5, we introduce definitions of graded fuzzy preproximities and graded fuzzy proximities and study their properties.

2. PRELIMINARIES

Let $(L, \leq)$ be a complete and completely distributive lattice with 0 and 1 as the least and the greatest elements respectively and $'$ be an order reversing involution. $\forall a \in L,\ a'$ is called the complement of a.

Following the definition of fuzzy family in [11], we define L-fuzzy family, union of L-fuzzy family and intersection of L-fuzzy family as given below:

Definition 2.1. An L-fuzzy set $\mathcal{G}$ on the set L^X i.e., a function $\mathcal{G} : L^X \to L$ is called an L-fuzzy family of L-fuzzy sets (briefly L-FF) of X.

Definition 2.2. Let $\mathcal{G}$ be an L-FF of X. Then the union of this L-FF is a function $\vee\mathcal{G} : X \to L$ defined by

$$(\vee\mathcal{G})(x) = \vee_{\mu\in L^X}(\mathcal{G}(\mu) \wedge \mu(x)).$$

Definition 2.3. Let $\mathcal{G}$ be an L-FF of X. Then the intersection of this L-FF is a function $\wedge\mathcal{G} : X \to L$ defined by the equality

$$(\wedge\mathcal{G})(x) = \wedge_{\mu\in L^X}((\mathcal{G}(\mu))' \vee \mu(x)).$$

Notation 2.4. For $\alpha \in L$, $\tilde{\alpha}$ denotes the constant L-fuzzy set of X with value α i.e., $\tilde{\alpha}(x) = \alpha$, $\forall x \in X$. For $A \in L^X$, the L-fuzzy set A' means the complement of A. Further, we denote $L \setminus \{0\}$ by L_0, $L \setminus \{1\}$ by L_1, $L \setminus \{0, 1\}$ by $L_{0,1}$, the set of all molecules of L by $M(L)$ and the set of all prime elements of L by $Pr(L)$. For an L-FF $\mathcal{G}$ of X we denote

$$S(\mathcal{G}) = \{A \in L^X; \mathcal{G}(A) > 0\}.$$

Definition 2.5. An L-FF $\mathcal{B}$ of X is said to be a finite L-FF if for some positive integer $n, \exists\ B_1, B_2, ..., B_n \in L^X$ such that $\mathcal{B}(B_i) > 0$, $1 \le i \le n$ and $\mathcal{B}(A) = 0$, if $A \in L^X \setminus \{B_1, B_2, ..., B_n\}$.
If $\mathcal{B}(B_i) = p_i$, $1 \le i \le n$, then $\mathcal{B}$ is expressed as

$$\mathcal{B} = \left\{\frac{B_1}{p_1}, \frac{B_2}{p_2}, ..., \frac{B_n}{p_n}\right\}.$$

Definition 2.6. A lattice L is said to be order dense if for any r, $s \in L$ with $r < s$, $\exists t \in L$ such that $r < t < s$.

Definition 2.7 ([14]). Let L be a complete lattice. Then L is said to possess sup property if for any $P \subset L$, $\vee P > s \Rightarrow \exists p \in P$ s.t. $p > s$.

According to [2,7], we take the following definitions in L-fuzzy setting.

Definition 2.8. A stack of fuzzy sets S on X is a subset of L^X such that $\lambda \ge \mu \in S \Rightarrow \lambda \in S$.

Definition 2.9. A filter of fuzzy sets F on X is a non-empty stack of fuzzy sets on X such that μ_1, $\mu_2 \in F \Rightarrow \mu_1 \wedge \mu_2 \in F$, $\forall \mu_1, \mu_2 \in L^X$.

If $\tilde{0} \notin F$, then F is called a proper filter of fuzzy sets on X.

Definition 2.10. A grill of fuzzy sets G on X is a stack of fuzzy sets on X such that

(i) $\tilde{0} \notin G$

(ii) $\lambda \vee \mu \in G \Rightarrow \lambda \in G$ or $\mu \in G$, $\forall \lambda, \mu \in L^X$.

If $\tilde{1} \in G$, then G is called a proper grill of fuzzy sets on X.

Definition 2.11. Let $\mathcal{F} : L^X \to L$ be a mapping satisfying

(i) $\mathcal{F}(\tilde{1}) = 1$

(ii) $\mathcal{F}(\lambda_1 \wedge \lambda_2) = \mathcal{F}(\lambda_1) \wedge \mathcal{F}(\lambda_2)$.

Then $\mathcal{F}$ is called a fuzzy filter on X. If $\mathcal{F}(\tilde{0}) = 0$ then $\mathcal{F}$ is called a proper fuzzy filter on X.

Definition 2.12. Let $\mathcal{G} : L^X \to L$ be a mapping satisfying

(i) $\mathcal{G}(\tilde{0}) = 0$

(ii) $\mathcal{G}(\lambda_1 \vee \lambda_2) = \mathcal{G}(\lambda_1) \vee \mathcal{G}(\lambda_2)$.

Then $\mathcal{G}$ is called a fuzzy grill on X. If $\mathcal{G}(\tilde{1}) = 1$ then $\mathcal{G}$ is called a proper fuzzy grill on X.

Definition 2.13. A mapping $\Delta : L^X \times L^X \to L$ satisfying

(1) $\Delta = \Delta^{-1}$

(2) $\Delta(\lambda, \tilde{0}) = 0$

(3) $\Delta(\lambda, \mu_1 \vee \mu_2) = \Delta(\lambda, \mu_1) \vee \Delta(\lambda, \mu_2)$, is called a fuzzy preproximity on X.

Definition 2.14. A mapping $\delta : L^X \times L^X \to L$ having the following properties is called a fuzzy proximity on X:

(1) $\delta(\tilde{0}, \tilde{1}) = 0$,

(2) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$,

(3) $\delta(\lambda_1 \vee \lambda_2, \mu) = \delta(\lambda_1, \mu) \vee \delta(\lambda_2, \mu)$,

(4) $\delta(\lambda, \mu) \ngeq r' \Rightarrow \delta(cl(\lambda, r), \mu) \ngeq r'$, where

$$cl(\lambda, r) = \wedge\{\eta' \geq \lambda;\ \delta(\lambda, \eta) \ngeq r'\},\ r \in L_1.$$

3. L-Fuzzy Gradation of Openness

In this section, we give the definition of L-fuzzy gradation of openness and deduce some of its basic results.

Definition 3.1. A mapping $\tau : L^X \to L$ is said to be an L-fuzzy gradation of openness (shortly L-FGO) on X if it satisfies the following axioms :

(L-FGO1) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$,

(L-FGO2) for any L-FF $\mathcal{G}$ of X,

$$\tau(\vee\mathcal{G}) \geq \wedge_{A\in S(\mathcal{G})}(\tau(A) \wedge \mathcal{G}(A)),$$

(L-FGO3) for any finite L-FF $\mathcal{B} = \{\frac{B_1}{p_1}, \frac{B_2}{p_2}, ..., \frac{B_n}{p_n}\}$ of X,

$$\tau(\wedge\mathcal{B}) \geq \wedge_{i=1}^{n}(\tau(B_i) \wedge \mathcal{B}(B_i)).$$

Definition 3.2. A mapping $\mathcal{F} : L^X \to L$ is said to be an L-fuzzy gradation of closedness (shortly L-FGC) on X if it satisfies the following axioms :

(L-FGC1) $\mathcal{F}(\tilde{0}) = \mathcal{F}(\tilde{1}) = 1$,

(L-FGC2) for any L-FF $\mathcal{G}$ of X,

$$\mathcal{F}(\wedge\mathcal{G}) \geq \wedge_{A\in S(\mathcal{G})}(\mathcal{F}(A) \wedge \mathcal{G}(A)),$$

(L-FGC3) for any finite L-FF $\mathcal{B} = \{\frac{B_1}{p_1}, \frac{B_2}{p_2}, ..., \frac{B_n}{p_n}\}$ of X,

$$\mathcal{F}(\vee\mathcal{B}) \geq \wedge_{i=1}^{n}(\mathcal{F}(B_i) \wedge \mathcal{B}(B_i)).$$

The pair (X, τ) (or $(X, \mathcal{F})$) is called an L-fuzzy topological space.

Definition 3.3. Let $\mathcal{G}$ be an L-FF of X. Then the L-FF $\mathcal{G}^*$ of complemented L-fuzzy sets of X is defined by $\mathcal{G}^*(A) = \mathcal{G}(A')$, $\forall A \in L^X$.

Theorem 3.4 (Generalized De Morgan's Laws). *Let $\mathcal{G}$ be an L-FF of X. Then we have*

(a) $[\vee\mathcal{G}]' = \wedge\mathcal{G}^*$,

(b) $[\wedge\mathcal{G}]' = \vee\mathcal{G}^*$.

The proof is straightforward.

Remark 3.5. If τ is an L-FGO on X then $\mathcal{F}_\tau$ defined by $\mathcal{F}_\tau(A) = \tau(A')$ is an L-FGC on X associated to τ. Similarly, for an L-FGC $\mathcal{F}$ on X the mapping $\tau_{\mathcal{F}} : L^X \to L$ defined by $\tau_{\mathcal{F}}(A) = \mathcal{F}(A')$ is an L-FGO on X associated to $\mathcal{F}$. Further $\mathcal{F}_{\tau_{\mathcal{F}}} = \mathcal{F}$ and $\tau_{\mathcal{F}_\tau} = \tau$.

Proof. Let τ be an L-FGO on X. Then

(1) $\mathcal{F}_\tau(\tilde{0}) = \tau(\tilde{1}) = 1,\ \mathcal{F}_\tau(\tilde{1}) = \tau(\tilde{0}) = 1.$

(2)

$$\begin{aligned}\mathcal{F}_\tau(\wedge\mathcal{G}) &= \tau[(\wedge\mathcal{G})'] \\ &= \tau[\vee\mathcal{G}^*] \geq \wedge_{A\in S(\mathcal{G}^*)}[\tau(A) \wedge \mathcal{G}^*(A)] \\ &= \wedge_{A'\in S(\mathcal{G})}[\mathcal{F}_\tau(A') \wedge \mathcal{G}(A')] \text{ (since } A \in S(\mathcal{G}^*) \Leftrightarrow A' \in S(\mathcal{G})) \\ &= \wedge_{B\in S(\mathcal{G})}[\mathcal{F}_\tau(B) \wedge \mathcal{G}(B)].\end{aligned}$$

(3) Let $\mathcal{B} = \{\frac{B_1}{p_1}, \frac{B_2}{p_2}, ..., \frac{B_n}{p_n}\}$ be a finite fuzzy family. Then

$$\begin{aligned}\mathcal{F}_\tau(\vee\mathcal{B}) &= \tau[(\vee\mathcal{B})'] \\ &= \tau[\wedge\mathcal{B}^*] \\ &\geq \wedge_{i=1}^n[\tau(B_i') \wedge \mathcal{B}^*(B_i')] \\ &= \wedge_{i=1}^n[\mathcal{F}_\tau(B_i) \wedge \mathcal{B}(B_i)].\end{aligned}$$

Hence $\mathcal{F}_\tau$ is an L-FGC on X.

Similarly, $\tau_\mathcal{F}$ can be shown to be an L-FGO on X associated to $\mathcal{F}$.
The last part of the Remark is obvious. □

Remark 3.6. Every L-fuzzy gradation of openness (closedness) is a gradation of openness (closedness). But the converse is not necessarily true (Remark 2.4 of [13]).

Theorem 3.7. *A gradation of openness τ on X is an L-fuzzy gradation of openness on X iff*

(c1) $\tau(A \wedge \tilde{\alpha}) \geq \tau(A) \wedge \alpha,\ \forall\alpha \in L_{0,1}$,

(c2) $\tau(A \vee \tilde{\alpha}) \geq \tau(A) \wedge \alpha',\ \forall\alpha \in L_{0,1}$.

Proof. Suppose τ is a gradation of openness satisfying (c1) and (c2). Let $\mathcal{G}$ be an L-fuzzy family. Then

$$\begin{aligned}\tau(\vee\mathcal{G}) &= \tau[\vee_{\mu\in S(\mathcal{G})}(\widetilde{\mathcal{G}(\mu)} \wedge \mu)] \\ &\geq \wedge_{\mu\in S(\mathcal{G})}\tau(\widetilde{\mathcal{G}(\mu)} \wedge \mu) \\ &\geq \wedge_{\mu\in S(\mathcal{G})}(\mathcal{G}(\mu) \wedge \tau(\mu)), \text{ by (c1)}.\end{aligned}$$

For a finite L-fuzzy family $\mathcal{B} = \{\frac{B_1}{p_1}, \frac{B_2}{p_2}, \dots, \frac{B_n}{p_n}\}$ of X,

$$\begin{aligned}\tau(\wedge\mathcal{B}) &= \tau[\wedge_{i=1}^{n}(B_i \vee \tilde{\mathcal{B}}'(B_i))] \\ &= \tau[\wedge_{i=1}^{n}(B_i \vee \tilde{p}_i')] \\ &\geq \wedge_{i=1}^{n}\tau(B_i \vee \tilde{p}_i') \\ &\geq \wedge_{i=1}^{n}(\tau(B_i) \wedge p_i), \text{ by (c2)} \\ &= \wedge_{i=1}^{n}(\tau(B_i) \wedge \mathcal{B}(B_i)).\end{aligned}$$

Therefore τ is an L-FGO.

Conversely, suppose τ is an L-FGO. Then clearly τ is a gradation of openness on X. Let $\mu \in L^X$ and $\alpha \in L_{0,1}$. Define an L-fuzzy family $\mathcal{G}_\mu : L^X \to L$ by $\mathcal{G}_\mu(\mu) = \alpha$ and $\mathcal{G}_\mu(\nu) = 0$, if $\nu\ (\neq \mu) \in L^X$. Then

$$\begin{aligned}\tau(\vee\mathcal{G}_\mu) \geq \wedge_{C\in S(\mathcal{G}_\mu)}[\mathcal{G}_\mu(C) \wedge \tau(C)] &\Leftrightarrow \tau(\mu \wedge \widetilde{\mathcal{G}_\mu(\mu)}) \geq [\mathcal{G}_\mu(\mu) \wedge \tau(\mu)] \\ &\Leftrightarrow \tau(\mu \wedge \tilde{\alpha}) \geq \alpha \wedge \tau(\mu).\end{aligned}$$

Again,

$$\begin{aligned}\tau(\wedge\mathcal{G}_\mu) \geq \mathcal{G}_\mu(\mu) \wedge \tau(\mu) &\Leftrightarrow \tau(\mu \vee \widetilde{\mathcal{G}_\mu'(\mu)}) \geq \mathcal{G}_\mu(\mu) \wedge \tau(\mu) \\ &\Leftrightarrow \tau(\mu \vee \tilde{\alpha}') \geq \alpha \wedge \tau(\mu).\end{aligned}$$

Thus τ is a gradation of openness satisfying (c1) and (c2). □

Remark 3.8. From (c1) and (c2) we see that $\tau(\tilde{1} \wedge \tilde{\alpha}) \geq \tau(\tilde{1}) \wedge \alpha$ i.e., $\tau(\tilde{\alpha}) \geq \alpha$ and $\tau(\tilde{0} \vee \tilde{\alpha}) \geq \tau(\tilde{0}) \wedge \alpha'$ i.e., $\tau(\tilde{\alpha}) \geq \alpha'$. Thus $\tau(\tilde{\alpha}) \geq \alpha \vee \alpha'$, $\forall \alpha \in L_{0,1}$.

Proceeding in a similar way as Theorem 3.7 we have the following:

Theorem 3.9. *A gradation of closedness $\mathcal{F}$ on X is an L-fuzzy gradation of closedness on X iff*

(c3) $\mathcal{F}(A \vee \tilde{\alpha}) \geq \mathcal{F}(A) \wedge \alpha'$, $\forall \alpha \in L_{0,1}$.
(c4) $\mathcal{F}(A \vee \tilde{\alpha}) \geq \mathcal{F}(A) \wedge \alpha'$, $\forall \alpha \in L_{0,1}$.

Remark 3.10. From (c3) and (c4) we see that $\mathcal{F}(\tilde{1} \wedge \tilde{\alpha}) \geq \mathcal{F}(\tilde{1}) \wedge \alpha$ i.e., $\mathcal{F}(\tilde{\alpha}) \geq \alpha$ and $\mathcal{F}(\tilde{0} \vee \tilde{\alpha}) \geq \mathcal{F}(\tilde{0}) \wedge \alpha'$ i.e., $\mathcal{F}(\tilde{\alpha}) \geq \alpha'$. Thus $\mathcal{F}(\tilde{\alpha}) \geq \alpha \vee \alpha'$, $\forall \alpha \in L_{0,1}$.

Definition 3.11. Let $\tau : L^X \to L$ be a mapping. For $r \in L_0$, define $\tau_r = \{\lambda \in L^X;\ \tau(\lambda) \geq r\}$.

Theorem 3.12. *Let τ be an L-FGO on X. Then $\{\tau_r\}_{r\in L_0}$ is a descending family of L-fuzzy topologies (Chang-type) on X satisfying*

(1) $\tau_{\vee_{i\in\Delta}\alpha_i} = \cap_{i\in\Delta}\tau_{\alpha_i}$.

(2) $A \in \tau_r \Rightarrow A \wedge \tilde{\alpha} \in \tau_{\alpha\wedge r}$ *and* $A \vee \tilde{\alpha} \in \tau_{\alpha'\wedge r}$, $\forall \alpha \in L_{0,1}$.

Proof. Since τ is an L-FGO, it is a gradation of openness and hence $\{\tau_r\}_{r\in L_0}$ is a descending family of L-fuzzy topologies (Chang-type) on X.
To show condition (1), we observe that $\tau_{\vee_{i\in\Delta}\alpha_i} \subset \cap_{i\in\Delta}\tau_{\alpha_i}$ is obvious.
Further, $\lambda \in \cap_{i\in\Delta}\tau_{\alpha_i} \Rightarrow \tau(\lambda) \geq \alpha_i,\ \forall\ i \in \Delta \Rightarrow \tau(\lambda) \geq \vee_{i\in\Delta}\alpha_i \Rightarrow \lambda \in \tau_{\vee_{i\in\Delta}\alpha_i}$. So $\cap_{i\in\Delta}\tau_{\alpha_i} \subset \tau_{\vee_{i\in\Delta}\alpha_i}$. Hence $\tau_{\vee_{i\in\Delta}\alpha_i} = \cap_{i\in\Delta}\tau_{\alpha_i}$.

Next to show the condition (2), we see that $A \in \tau_r \Rightarrow \tau(A) \geq r$. Then by (c1) and (c2) of Theorem 3.7, we have $\tau(A\wedge\tilde{\alpha}) \geq \tau(A)\wedge\alpha \geq r\wedge\alpha \Rightarrow A\wedge\tilde{\alpha} \in \tau_{r\wedge\alpha}$, $\forall\alpha \in L_{0,1}$. and $\tau(A \vee \tilde{\alpha}) \geq \tau(A) \wedge \alpha' \geq r \wedge \alpha' \Rightarrow A \vee \tilde{\alpha} \in \tau_{r\wedge\alpha'}$, $\forall\alpha \in L_{0,1}$. Hence $\{\tau_r\}_{r\in L_0}$ is a descending family of L-fuzzy topologies (Chang-type) on X satisfying (1) and (2). □

Theorem 3.13. *Let $\{T_r; r \in L_0\}$ be a descending family of L-fuzzy topologies (Chang-type) on X satisfying conditions (1) and (2) of Theorem 3.12. Then the mapping $\tau : L^X \to L$ defined by*

$$\tau(A) = \vee\{r;\ A \in T_r\}$$

is an L-FGO on X such that $\tau_r = T_r,\ r \in L_0$.

Proof. From Proposition 2.2 of [5] τ is a gradation of openness and it satisfies $\tau_r = T_r,\ r \in L_0$.
Let $\tau(A) = p$. If $p = 0$, then obviously (c1) and (c2) of Theorem 3.7 hold. If $p > 0$, Then

$$\begin{aligned} A \in \tau_p &\Rightarrow A \in T_p \\ &\Rightarrow A \wedge \tilde{\alpha} \in T_{\alpha\wedge p},\ \forall\alpha \in L_{0.1} \\ &\Rightarrow A \wedge \tilde{\alpha} \in \tau_{\alpha\wedge p},\ \forall\alpha \in L_{0,1},\ [\text{as } \tau_{\alpha\wedge p} = T_{\alpha\wedge p}] \\ &\Rightarrow \tau(A \wedge \tilde{\alpha}) \geq \alpha \wedge p,\ \forall\alpha \in L_{0,1} \\ &\Rightarrow \tau(A \wedge \tilde{\alpha}) \geq \alpha \wedge \tau(A),\ \forall\alpha \in L_{0,1}. \end{aligned}$$

Again,

$$\begin{aligned} A \in \tau_p &\Rightarrow A \in T_p \\ &\Rightarrow A \vee \tilde{\alpha} \in T_{\alpha'\wedge p},\ \forall\alpha \in L_{0,1} \\ &\Rightarrow A \vee \tilde{\alpha} \in \tau_{\alpha'\wedge p},\ \forall\alpha \in L_{0,1} \end{aligned}$$

$$\Rightarrow \tau(A \vee \tilde{\alpha}) \geq \alpha' \wedge p, \ \forall \alpha \in L_{0,1}$$
$$\Rightarrow \tau(A \vee \tilde{\alpha}) \geq \alpha' \wedge \tau(A), \ \forall \alpha \in L_{0,1}.$$

Hence by Theorem 3.7, τ is an L-FGO on X. □

Theorem 3.14. *A gradation of closedness $\mathcal{F}$ is an L-FGC iff $A \in \mathcal{F}_r \Rightarrow A \vee \tilde{\alpha}' \in \mathcal{F}_{\alpha \wedge r}$, $\forall \alpha \in L_{0,1}$ and $A \wedge \tilde{\alpha}' \in \mathcal{F}_{\alpha' \wedge r}$, $\forall \alpha \in L_{0,1}$.*

Proof. Let a gradation of closedness $\mathcal{F}$ be an L-FGC. Now $A \in \mathcal{F}_r \Rightarrow \mathcal{F}(A) \geq r$. By Theorem 3.9, $\mathcal{F}(A \vee \tilde{\alpha}') \geq \mathcal{F}(A) \wedge \alpha \geq r \wedge \alpha \Rightarrow A \vee \tilde{\alpha}' \in \mathcal{F}_{r \wedge \alpha}$, $\forall \alpha \in L_{0,1}$ and $\mathcal{F}(A \wedge \tilde{\alpha}') \geq \mathcal{F}(A) \wedge \alpha' \geq r \wedge \alpha' \Rightarrow A \wedge \tilde{\alpha}' \in \mathcal{F}_{r \wedge \alpha'}$, $\forall \alpha \in L_{0,1}$.
Conversely, let a gradation of closedness $\mathcal{F}$ satisfies the given condition.
Let $\mathcal{F}(A) = r$. Then

$$A \in \mathcal{F}_r \Rightarrow A \vee \tilde{\alpha}' \in \mathcal{F}_{r \wedge \alpha} \text{ and } A \wedge \tilde{\alpha}' \in \mathcal{F}_{r \wedge \alpha'}$$
$$\Rightarrow \mathcal{F}(A \vee \tilde{\alpha}') \geq \alpha \wedge r \text{ and } \mathcal{F}(A \wedge \tilde{\alpha}') \geq \alpha' \wedge r$$
$$\Rightarrow \mathcal{F}(A \vee \tilde{\alpha}') \geq \alpha \wedge \mathcal{F}(A) \text{ and } \mathcal{F}(A \wedge \tilde{\alpha}') \geq \alpha' \wedge \mathcal{F}(A), \ \forall \alpha \in L_{0,1}.$$

Hence by Theorem 3.9, $\mathcal{F}$ is an L-FGC. □

Definition 3.15. Let τ be an L-FGO and $\mathcal{F}$ (i.e., $\mathcal{F}_\tau$) be the associated L-FGC on X. For $M \in L^X$, $r \in L_0$ define the r-closure of M by

$$cl(M, r) = \wedge\{N \geq M; \ N \in \mathcal{F}_r\}.$$

Theorem 3.16 *Let $\mathcal{F}$ be an L-FGC on X and for $M \in L^X$, $r \in L_0$, $cl(M,r)$ be the r-closure of M, then $M = cl(M,r) \Leftrightarrow M \in \mathcal{F}_r$.*

Proof. $M = cl(M,r) \Rightarrow M = \wedge\{N \geq M; \ N \in \mathcal{F}_r\} \Rightarrow M \in \mathcal{F}_r$.
Again $M \in \mathcal{F}_r \Rightarrow$by definition of $cl(M,r)$, $M = cl(M,r)$.

Theorem 3.17. *Let L be an order dense lattice, $\mathcal{F}$ be an L-FGC on X and $cl(M,r)$ be the r-closure of M, $M \in L^X$, $r \in L_0$. Then for $M_1, M_2 \in L^X$, $s \in L_0$*

(1) $cl(\tilde{0}, r) = \tilde{0}$, $cl(\tilde{1}, r) = \tilde{1}$,
(2) $cl(M, r) \geq M$, $\forall M \in L^X$,
(3) $cl(M, r) \subset cl(M, s)$, *if* $r \leq s$,
(4) $cl(M_1 \vee M_2, r) = cl(M_1, r) \vee cl(M_2, r)$,
(5) $cl(cl(M, s), s) = cl(M, s)$,
(6) $r = \vee\{s \in L_0; \ cl(M, s) = M\} \Rightarrow cl(M, r) = M$,

(7) $M = cl(M,r) \Rightarrow cl(M \vee \tilde{t}', r \wedge t) = cl(M,r) \vee \tilde{t}'$, $\forall t \in L_{0,1}$ *and*

$$cl(M \wedge \tilde{t}', r \wedge t') = cl(M,r) \wedge \tilde{t}', \ \forall t \in L_{0,1}.$$

Proof. From Proposition 2.12 of [5], proofs of (1)-(6) follow.

(7) By Theorem 3.16,

$$\begin{aligned} M &= cl(M,r) \\ &\Leftrightarrow M \in \mathcal{F}_r \\ &\Rightarrow M \vee \tilde{t}' \in \mathcal{F}_{t\wedge r} \ \& \ M \wedge \tilde{t}' \in \mathcal{F}_{t'\wedge r}, \ \forall t \in L_{0,1} \\ &\Rightarrow cl(M \vee \tilde{t}', t \wedge r) = M \vee \tilde{t}' \ \& \ cl(M \wedge \tilde{t}', t' \wedge r) = M \wedge \tilde{t}', \ \forall t \in L_{0,1} \end{aligned}$$

$$\Rightarrow cl(M \vee \tilde{t}', t \wedge r) = cl(M,r) \vee \tilde{t}' \ \& \ cl(M \wedge \tilde{t}', t' \wedge r) = cl(M,r) \wedge \tilde{t}', \ \forall t \in L_{0,1}.$$

□

Theorem 3.18. *Let $cl : L^X \times L_0 \to L^X$ be a mapping satisfying (1)-(7) of Theorem 3.17. Then the mapping $\mathcal{F} : L^X \to L$ defined by*

$$\mathcal{F}(M) = \vee\{r \in L_0; \ cl(M,r) = M\}$$

is an L-FGC on X such that $cl_{\mathcal{F}} = cl$.

Proof. By Proposition 2.13 of [5] $\mathcal{F}$ is a gradation of closedness on X and $cl_{\mathcal{F}} = cl$. To show $\mathcal{F}$ is an L-FGC let for $r \in L_0$, $A \in \mathcal{F}_r$. Then $\mathcal{F}(A) \geq r$ i.e., $l = \vee\{t \in L_0; \ cl(A,t) = A\} \geq r$. By (6), $cl(A,l) = A$. So, $cl(A,r) = A$.
By (7) of Theorem 3.17, we get

$$cl(A \vee \tilde{t}', r \wedge t) = cl(A,r) \vee \tilde{t}' = A \vee \tilde{t}', \ \forall t \in L_{0,1}$$

and

$$cl(A \wedge \tilde{t}', r \wedge t') = cl(A,r) \wedge \tilde{t}' = A \wedge \tilde{t}', \ \forall t \in L_{0,1}.$$

Hence $A \vee \tilde{t}' \in \mathcal{F}_{r\wedge t}$ & $A \wedge \tilde{t}' \in \mathcal{F}_{r\wedge t'}$, $\forall t \in L_{0,1}$. Therefore $\mathcal{F}$ is an L-FGC (by Theorem 3.14). □

4. GRADED FUZZY FILTER AND GRADED FUZZY GRILL

In this section definitions of graded fuzzy filters and graded fuzzy grills are given and some of their properties are studied.

Definition 4.1. Let X be a set. A mapping $S : L^X \to L$ satisfying $S(\lambda_1) \geq S(\lambda_2)$ for $\lambda_1 \geq \lambda_2$ is called a graded fuzzy stack on X.

Definition 4.2. Let $\mathcal{F} : L^X \to L$ be a mapping satisfying

$$\mathcal{F}\Big(\wedge\Big\{\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2}\Big\}\Big) = [p_1' \vee \mathcal{F}(\lambda_1)] \wedge [p_2' \vee \mathcal{F}(\lambda_2)], \ p_1, p_2 \in L_0.$$

Then $\mathcal{F}$ is called a graded fuzzy filter on X. If $\mathcal{F}$ satisfies the condition $\mathcal{F}(\tilde{0}) = 0$ then $\mathcal{F}$ is called a proper graded fuzzy filter on X. Let $\phi(X)$ denote the set of all graded fuzzy filters on X.

Remark 4.3. From the condition of Definition 4.2, we have, in particular,

$$\begin{aligned}\mathcal{F}\Big(\wedge\Big\{\frac{\tilde{1}}{p}\Big\}\Big) &= p' \vee \mathcal{F}(\tilde{1}), \ \forall p \in L_{0,1}, \\ &\text{i.e., } \mathcal{F}(\tilde{p}' \vee \tilde{1}) = p' \vee \mathcal{F}(\tilde{1}), \ \forall p \in L_{0,1} \\ &\text{i.e., } \mathcal{F}(\tilde{1}) \geq p', \ \forall p \in L_{0,1} \\ &\text{i.e., } \mathcal{F}(\tilde{1}) \geq \vee\{p';\ p \in L_{0,1}\}.\end{aligned}$$

Hence $\mathcal{F}(\tilde{1}) = 1$.

Definition 4.4. Let $\mathcal{G} : L^X \to L$ be a mapping satisfying

$$\mathcal{G}\Big(\vee\Big\{\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2}\Big\}\Big) = [p_1 \wedge \mathcal{G}(\lambda_1)] \vee [p_2 \wedge \mathcal{G}(\lambda_2)], \ p_1, p_2 \in L_0.$$

Then $\mathcal{G}$ is called a graded fuzzy grill on X. If $\mathcal{G}$ satisfies $\mathcal{G}(\tilde{1}) = 1$ then $\mathcal{G}$ is called a proper graded fuzzy grill on X. Let $\Gamma(X)$ denote the set of all graded fuzzy grills on X.

Remark 4.5. From Definition 4.4, we have, in particular,

$$\begin{aligned}\mathcal{G}\Big(\vee\Big\{\frac{\tilde{0}}{p}\Big\}\Big) &= p \wedge \mathcal{G}(\tilde{0}), \ \forall p \in L_{0,1}, \\ &\text{i.e., } \mathcal{G}(\tilde{p} \wedge \tilde{0}) = p \wedge \mathcal{G}(\tilde{0}), \ \forall p \in L_{0,1} \\ &\text{i.e., } \mathcal{G}(\tilde{0}) \leq p, \ \forall p \in L_{0,1}\end{aligned}$$

therefore, $\mathcal{G}(\tilde{0}) \leq \wedge\{p;\ p \in L_{0,1}\}$. Hence $\mathcal{G}(\tilde{0}) = 0$.

Definition 4.6. A mapping $\mathcal{U} : L^X \to L$ is called a graded fuzzy prime filter on X if $\mathcal{U}$ is a graded fuzzy filter as well as a graded fuzzy grill on X.

Theorem 4.7. *A fuzzy filter $\mathcal{F}$ on X is a graded fuzzy filter on X iff*

$$\mathcal{F}(\lambda \vee \tilde{\alpha}) = \mathcal{F}(\lambda) \vee \alpha, \ \forall \alpha \in L_{0,1}.$$

Proof. Let $\mathcal{F}$ be a fuzzy filter satisfying the given condition. Suppose $\{\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2}\}$ is an L-fuzzy family of X. Then

$$\begin{aligned} \mathcal{F}\Big(\wedge \Big\{\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2}\Big\}\Big) &= \mathcal{F}[(\tilde{p}_1' \vee \lambda_1) \wedge (\tilde{p}_2' \vee \lambda_2)] \\ &= \mathcal{F}(\tilde{p}_1' \vee \lambda_1) \wedge \mathcal{F}(\tilde{p}_2' \vee \lambda_2), \text{ since } \mathcal{F} \text{ is a fuzzy filter} \\ &= [\mathcal{F}(\lambda_1) \vee p_1'] \wedge [\mathcal{F}(\lambda_2) \vee p_2'], \text{ by the given condition.} \end{aligned}$$

Hence $\mathcal{F}$ is a graded fuzzy filter.

Conversely, let $\mathcal{F}$ be a graded fuzzy filter on X. Clearly, $\mathcal{F}$ is a fuzzy filter. Since $\{\frac{\lambda}{\alpha}, \frac{\tilde{1}}{1}\}$ is an L-fuzzy family, we have

$$\begin{aligned} \mathcal{F}\Big(\wedge \Big\{\frac{\lambda}{\alpha}, \frac{\tilde{1}}{1}\Big\}\Big) &= [\alpha' \vee \mathcal{F}(\lambda)] \wedge [1' \vee \mathcal{F}(\tilde{1})] \\ &\Leftrightarrow \mathcal{F}[(\tilde{\alpha}' \vee \lambda) \wedge (\tilde{0} \vee \tilde{1})] = [\alpha' \vee \mathcal{F}(\lambda)] \wedge \mathcal{F}(\tilde{1}) \\ &\Leftrightarrow \mathcal{F}(\tilde{\alpha}' \vee \lambda) = \alpha' \vee \mathcal{F}(\lambda). \end{aligned}$$

Remark 4.8. We have from above theorem

$$\mathcal{F}(\tilde{0} \vee \tilde{\alpha}) = \mathcal{F}(\tilde{0}) \vee \alpha, \ \forall \alpha \in L_{0,1}.$$

So, for a proper graded fuzzy filter $\mathcal{F}$, $\mathcal{F}(\tilde{\alpha}) = \alpha$, $\forall \alpha \in L_{0,1}$.

Definition 4.9. Let $\mathcal{F} : L^X \to L$ be a graded fuzzy filter on X. Define

$$\mathcal{F}_m = \{\lambda \in L^X;\ \mathcal{F}(\lambda) \not\leq m'\}, \ m \in L_{0,1}.$$

Theorem 4.10. *Let $\mathcal{F} : L^X \to L$ be a graded fuzzy filter on X. Then $\{\mathcal{F}_m\}_{m \in L_{0,1}}$ is an ascending family of stacks of fuzzy sets on X such that*

(1) *$\mathcal{F}_m$ is a filter of fuzzy sets on X, $\forall m \in M(L)$,*
(2) $\cup_{n \in \Delta} \mathcal{F}_{\alpha_n} = \mathcal{F}_{\vee_n \alpha_n}$, $\alpha_n \in L_{0,1}$,
(3) $\lambda \notin \mathcal{F}_m \Leftrightarrow \lambda \vee \tilde{\alpha} \notin \mathcal{F}_{m \wedge \alpha'}$, $\forall m, \alpha \in L_{0,1}$.

Proof. Clearly $\mathcal{F}_m$, $\forall m \in L_{0,1}$ is a stack of fuzzy sets on X. Let $m \geq n$. Then

$$\lambda \in \mathcal{F}_n \Rightarrow \mathcal{F}(\lambda) \not\leq n' \Rightarrow \mathcal{F}(\lambda) \not\leq m' \Rightarrow \lambda \in \mathcal{F}_m.$$

So, $\mathcal{F}_n \subset \mathcal{F}_m$ if $m \geq n$, where $m, n \in L_{0,1}$.

Since $\mathcal{F}(\tilde{1}) = 1 \not\leq m'$, $\forall m \in M(L)$, therefore $\tilde{1} \in \mathcal{F}_m$, $\forall m \in M(L)$.

Let $\lambda_1, \lambda_2 \in \mathcal{F}_m$. Then

$$\mathcal{F}(\lambda_1) \not\leq m',\ \mathcal{F}(\lambda_2) \not\leq m'.$$

So,

$$\mathcal{F}(\lambda_1 \wedge \lambda_2) = \mathcal{F}(\lambda_1) \wedge \mathcal{F}(\lambda_2) \not\leq m',$$

as

$$m \in M(L) \Rightarrow \lambda_1 \wedge \lambda_2 \in \mathcal{F}_m,\ m \in M(L).$$

Clearly

$$\cup_{n\in\Delta}\mathcal{F}_{\alpha_n} \subset \mathcal{F}_{\vee_n \alpha_n}.$$

Next

$$\begin{aligned} \lambda \in \mathcal{F}_{\vee_n \alpha_n} &\Rightarrow \mathcal{F}(\lambda) \not\leq \wedge_{n\in\Delta}\alpha_n' \\ &\Rightarrow \mathcal{F}(\lambda) \not\leq \alpha_{n_o}',\ \text{for some } n_o \in \Delta \\ &\Rightarrow \lambda \in \mathcal{F}_{\alpha_{n_o}} \subset \cup_{n\in\Delta}\mathcal{F}_{\alpha_n}. \end{aligned}$$

Therefore,

$$\mathcal{F}_{\vee_n \alpha_n} \subset \cup_{n\in\Delta}\mathcal{F}_{\alpha_n}.$$

So,

$$\cup_{n\in\Delta}\mathcal{F}_{\alpha_n} = \mathcal{F}_{\vee_n \alpha_n},\ \alpha_n \in L_{0,1}.$$

Lastly, $\lambda \notin \mathcal{F}_m \Rightarrow \mathcal{F}(\lambda) \leq m'$. From Theorem 4.7,

$$\begin{aligned} \mathcal{F}(\lambda \vee \tilde{\alpha}) &= \mathcal{F}(\lambda) \vee \alpha \leq m' \vee \alpha \\ &\Rightarrow \lambda \vee \tilde{\alpha} \notin \mathcal{F}_{m\wedge\alpha'},\ \forall m, \alpha \in L_{0,1}. \end{aligned}$$

Conversely, $\forall m, \alpha \in L_{0,1}$,

$$\begin{aligned} \lambda \vee \tilde{\alpha} \notin \mathcal{F}_{m\wedge\alpha'} &\Rightarrow \mathcal{F}(\lambda \vee \tilde{\alpha}) \leq m' \vee \alpha \\ &\Rightarrow \mathcal{F}(\lambda) \vee \alpha \leq m' \vee \alpha \\ &\Rightarrow \wedge_{\alpha\in L_{0,1}}(\mathcal{F}(\lambda) \vee \alpha) \leq \wedge_{\alpha\in L_{0,1}}(m' \vee \alpha) \\ &\Rightarrow \mathcal{F}(\lambda) \vee (\wedge_{\alpha\in L_{0,1}}\alpha) \leq m' \vee (\wedge_{\alpha\in L_{0,1}}\alpha) \\ &\Rightarrow \mathcal{F}(\lambda) \vee 0 \leq m' \vee 0 \\ &\Rightarrow \mathcal{F}(\lambda) \leq m' \\ &\Rightarrow \lambda \notin \mathcal{F}_m. \end{aligned}$$

So,

$$\lambda \notin \mathcal{F}_m \Leftrightarrow \lambda \vee \tilde{\alpha} \notin \mathcal{F}_{m\wedge\alpha'},\ \forall m, \alpha \in L_{0,1}.$$

Hence, $\{\mathcal{F}_m\}_{m\in M(L)}$ is a ascending family of filters of fuzzy sets on X satisfying (1) and (2). □

Theorem 4.11. *Let $\{\mathcal{F}_m\}_{m\in L_{0,1}}$ be an ascending family of stacks of fuzzy sets on X satisfying conditions*

(1) *$\mathcal{F}_m$ is a filter of fuzzy sets on X, $\forall m \in M(L)$,*
(2) $\cup_{n\in\Delta}\mathcal{F}_{\alpha_n} = \mathcal{F}_{\vee_n\alpha_n}$, $\alpha_n \in L_{0.1}$
(3) $\lambda \notin \mathcal{F}_m \Leftrightarrow \lambda \vee \tilde{\alpha} \notin \mathcal{F}_{m\wedge\alpha'}$, $\forall m, \alpha \in L_{0,1}$. *then the mapping $f : L^X \to L$ defined by*
$$f(\lambda) = \wedge\{m' \in Pr(L);\ \lambda \notin \mathcal{F}_m\}$$
is a fuzzy filter on X with $f_m = \mathcal{F}_m$, $m \in L_{0,1}$.

If, further, the condition

(4) *For $\beta \in M(L)$ and $\alpha \in L_{0,1}$, $\beta \not\leq \alpha' \Rightarrow \tilde{\alpha} \in \mathcal{F}_\beta$ is satisfied then f is a graded fuzzy filter on X.*

Proof. Since $\mathcal{F}_m$, $\forall m \in L_{0,1}$ is a non-empty stack of fuzzy sets, so $\tilde{1} \in \mathcal{F}_m$, $\forall m \in L_{0,1}$. Therefore, $\mathcal{F}(\tilde{1}) \notin m'$, $\forall m \in L_{0,1}$. Therefore, $f(\tilde{1}) = 1$.

Let $\lambda \wedge \mu \notin \mathcal{F}_m$. Then $\lambda \notin \mathcal{F}_m$ or $\mu \notin \mathcal{F}_m$ (since $\mathcal{F}_m$ is a filter of fuzzy sets for $m \in M(L)$), i.e., $f(\lambda \wedge \mu) \geq f(\lambda) \wedge f(\mu)$.

Again $\lambda \notin \mathcal{F}_m \Rightarrow \lambda \wedge \mu \notin \mathcal{F}_m$ (since $\mathcal{F}_m$ is a stack of fuzzy sets), i.e., $f(\lambda) \geq f(\lambda \wedge \mu)$. Similarly, $f(\mu) \geq f(\lambda \wedge \mu)$. So, $f(\lambda \wedge \mu) \leq f(\lambda) \wedge f(\mu)$. Hence
$$f(\lambda \wedge \mu) = f(\lambda) \wedge f(\mu).$$

Now, $\lambda \notin \mathcal{F}_m \Leftrightarrow f(\lambda) \leq m' \Leftrightarrow \lambda \notin f_m$. Therefore $\mathcal{F}_m = f_m$, $m \in M(L)$.

Next, let $m \in L_{0,1}$. Since $M(L)$ is join-generating $\exists\ \{\alpha_i;\ i \in \Delta\} \subset M(L)$ such that $\vee_{i\in\Delta}\alpha_i = m$. Then
$$\begin{aligned}\lambda \notin \mathcal{F}_m &\Leftrightarrow \lambda \notin \mathcal{F}_{\alpha_i},\ \forall i \in \Delta\\ &\Rightarrow f(\lambda) \leq \alpha_i',\ \forall i \in \Delta\\ &\Rightarrow f(\lambda) \leq \wedge_{i\in\Delta}\alpha_i' = (\vee_{i\in\Delta}\alpha_i)' = m'\\ &\Rightarrow \lambda \notin f_m\\ &\Rightarrow \lambda \notin f_{\vee_i\alpha_i}\\ &\Rightarrow \lambda \notin f_{\alpha_i},\ \forall i \in \Delta\\ &\Rightarrow \lambda \notin \mathcal{F}_{\alpha_i},\ \forall i \in \Delta\\ &\Rightarrow \lambda \notin \cup_i\mathcal{F}_{\alpha_i} = \mathcal{F}_{\vee_i\alpha_i} = \mathcal{F}_m.\end{aligned}$$

Hence

$$\mathcal{F}_m = f_m, \ \forall m \in L_{0,1}.$$

Let $f(\lambda) = p$ and let $p = 0$.Then $\lambda \notin \mathcal{F}_m$, $\forall m \in L_{0,1}$, therefore, $\lambda \vee \tilde{\alpha} \notin \mathcal{F}_{m \wedge \alpha'} = f_{m\wedge\alpha'}$, $\forall m, \alpha \in L_{0,1} \Rightarrow f(\lambda \vee \tilde{\alpha}) \leq m' \vee \alpha \Rightarrow f(\lambda \vee \tilde{\alpha}) \leq (\wedge_{m \in M(L)} m') \vee \alpha = 0 \vee \alpha$. By condition (4), $0 \vee \alpha = \alpha \leq f(\tilde{\alpha}) \leq f(\lambda \vee \tilde{\alpha}) \leq 0 \vee \alpha$, i.e., $f(\lambda \vee \tilde{\alpha}) = 0 \vee \alpha = f(\lambda) \vee \alpha$.

Next let, $f(\lambda) = p = 1$. Then $f(\lambda \vee \tilde{\alpha}) \geq f(\lambda) = 1$. So,

$$f(\lambda \vee \tilde{\alpha}) = 1 = 1 \vee \alpha = f(\lambda) \vee \alpha.$$

Lastly, let, $f(\lambda) = p$, where $0 < p < 1$. Then

$$\begin{aligned} \lambda \notin f_{p'} &\Leftrightarrow \lambda \vee \tilde{\alpha} \notin f_{p' \wedge \alpha'}, \ \forall p, \ \alpha \in L_{0,1} \\ &\Leftrightarrow f(\lambda \vee \tilde{\alpha}) \leq (p' \wedge \alpha')' = p \vee \alpha, \ \forall p, \ \alpha \in L_{0,1} \\ &\Leftrightarrow f(\lambda \vee \tilde{\alpha}) \leq f(\lambda) \vee \alpha, \ \forall \alpha \in L_{0,1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i) \end{aligned}$$

Now, $f(\lambda \vee \tilde{\alpha}) \geq f(\tilde{\alpha}) \geq \alpha$ (by condition (4)) and $f(\lambda \vee \tilde{\alpha}) \geq f(\lambda)$.
So,

$$f(\lambda \vee \tilde{\alpha}) \geq f(\lambda) \vee \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

By (i) and (ii),

$$f(\lambda \vee \tilde{\alpha}) = f(\lambda) \vee \alpha, \ \forall \alpha \in L_{0,1}.$$

Therefore f is a graded fuzzy filter on X. □

Theorem 4.12. *A fuzzy grill $\mathcal{G}$ on X is a graded fuzzy grill on X iff*

$$\mathcal{G}(\lambda \wedge \tilde{\alpha}) = \mathcal{G}(\lambda) \wedge \alpha, \ \forall \alpha \in L_{0,1}.$$

Proof. Let $\mathcal{G}$ be a fuzzy grill satisfying the given condition. Suppose $\{\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2}\}$ is an L-fuzzy family of X. Then

$$\begin{aligned} \mathcal{G}\left(\vee \left\{\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2}\right\}\right) &= \mathcal{G}[(\tilde{p}_1 \wedge \lambda_1) \vee (\tilde{p}_2 \wedge \lambda_2)] \\ &= \mathcal{G}(\tilde{p}_1 \wedge \lambda_1) \vee \mathcal{G}(\tilde{p}_2 \wedge \lambda_2), \text{ since } \mathcal{G} \text{ is a fuzzy grill} \\ &= [\mathcal{G}(\lambda_1) \wedge p_1] \vee [\mathcal{G}(\lambda_2) \wedge p_2], \text{ by the given condition.} \end{aligned}$$

Hence $\mathcal{G}$ is a graded fuzzy grill.

Conversely, let $\mathcal{G}$ be a graded fuzzy grill on X. Then clearly, $\mathcal{G}$ is a fuzzy grill. Since $\{\frac{\lambda}{\alpha}\}$ is an L-fuzzy family, we have

$$\mathcal{G}\left(\vee \left\{\frac{\lambda}{\alpha}\right\}\right) = [\alpha \wedge \mathcal{G}(\lambda)]$$

$$\Leftrightarrow \mathcal{G}(\tilde{\alpha} \wedge \lambda) = \alpha \wedge \mathcal{G}(\lambda), \ \forall \alpha \in L_{0,1}.$$

□

Remark 4.13. We have from above theorem

$$\mathcal{G}(\tilde{1} \wedge \tilde{\alpha}) = \mathcal{G}(\tilde{1}) \wedge \alpha, \ \forall \alpha \in L_{0,1}.$$

So for a proper graded fuzzy grill $\mathcal{G}$, $\mathcal{G}(\tilde{\alpha}) = \alpha, \ \forall \alpha \in L_{0,1}$.

Definition 4.14. Let $\mathcal{G} : L^X \to L$ be a graded fuzzy grill on X. Define

$$\mathcal{G}_m = \{\lambda \in L^X; \ \mathcal{G}(\lambda) \geq m\}, \ m \in L_{0,1}.$$

Theorem 4.15. *Let $\mathcal{G} : L^X \to L$ be a graded fuzzy grill on X. Then $\{\mathcal{G}_m\}_{m \in L_{0,1}}$ is a descending family of stacks of fuzzy sets on X such that*

(1) *$\mathcal{G}_m$ is a grill of fuzzy sets on X, $\forall m \in M(L)$,*
(2) *$\cap_{n \in \Delta} \mathcal{G}_{\alpha_n} = \mathcal{G}_{\vee_n \alpha_n}$, where $\alpha_n \in L_{0,1}$,*
(3) *$\lambda \in \mathcal{G}_m \Leftrightarrow \lambda \wedge \tilde{\alpha} \in \mathcal{G}_{m \wedge \alpha}$, $\forall m, \ \alpha \in L_{0,1}$.*

Proof. Let $m \geq n$. Then

$$\lambda \in \mathcal{G}_m \Rightarrow \mathcal{G}(\lambda) \geq m \geq n \Rightarrow \lambda \in \mathcal{G}_n \Rightarrow \mathcal{G}_m \subset \mathcal{G}_n.$$

Next

$$\lambda \supset \mu \in \mathcal{G}_m \Rightarrow \mathcal{G}(\mu) \geq m \Rightarrow \mathcal{G}(\lambda) \geq m \Rightarrow \lambda \in \mathcal{G}_m.$$

Hence $\{\mathcal{G}_m\}_{m \in L_{0,1}}$ is a descending family of stacks of fuzzy sets on X.

Since $\mathcal{G}$ is a graded fuzzy grill, so $\mathcal{G}(\tilde{0}) = 0$. Therefore $\tilde{0} \notin \mathcal{G}_m, \ \forall m \in L_{0,1}$. Let $\lambda_1 \vee \lambda_2 \in \mathcal{G}_m$. Then

$$\begin{aligned} \mathcal{G}(\lambda_1 \vee \lambda_2) &\geq m \\ &\Rightarrow \mathcal{G}(\lambda_1) \vee \mathcal{G}(\lambda_2) \geq m, \text{ since } \mathcal{G} \text{ is a graded fuzzy grill} \\ &\Rightarrow \mathcal{G}(\lambda_1) \geq m \text{ or } \mathcal{G}(\lambda_2) \geq m, \text{ as } m \in M(L) \\ &\Rightarrow \lambda_1 \in \mathcal{G}_m \text{ or } \lambda_2 \in \mathcal{G}_m. \end{aligned}$$

Hence $\mathcal{G}_m$ is a grill of fuzzy sets on X, $\forall m \in M(L)$.

Clearly

$$\cap_{n \in \Delta} \mathcal{G}_{\alpha_n} \supset \mathcal{G}_{\vee_n \alpha_n}.$$

Let $\lambda \in \cap_{n\in\Delta}\mathcal{G}_{\alpha_n}$. Then

$$\begin{aligned}\lambda \in \mathcal{G}_{\alpha_n},\ \forall n \in \Delta &\Rightarrow \mathcal{G}(\lambda) \geq \alpha_n,\ \forall n \in \Delta\\ &\Rightarrow \mathcal{G}(\lambda) \geq \vee_{n\in\Delta}\alpha_n\\ &\Rightarrow \lambda \in \mathcal{G}_{\vee_{n\in\Delta}\alpha_n}\\ &\Rightarrow \cap_{n\in\Delta}\mathcal{G}_{\alpha_n} \subset \mathcal{G}_{\vee_n\alpha_n}.\end{aligned}$$

Therefore

$$\cap_{n\in\Delta}\mathcal{G}_{\alpha_n} = \mathcal{G}_{\vee_n\alpha_n}.$$

Lastly, $\lambda \in \mathcal{G}_m \Rightarrow \mathcal{G}(\lambda) \geq m$. From Theorem 4.12,

$$\begin{aligned}\mathcal{G}(\lambda \wedge \tilde{\alpha}) &= \mathcal{G}(\lambda) \wedge \alpha\\ &\Rightarrow \mathcal{G}(\lambda \wedge \tilde{\alpha}) \geq m \wedge \alpha\\ &\Rightarrow \lambda \wedge \tilde{\alpha} \in \mathcal{G}_{m\wedge\alpha},\ \forall m,\ \alpha \in L_{0,1}.\end{aligned}$$

Conversely, $\forall m,\ \alpha \in L_{0.1}$

$$\begin{aligned}\lambda \wedge \tilde{\alpha} \in \mathcal{G}_{m\wedge\alpha} &\Rightarrow \mathcal{G}(\lambda \wedge \tilde{\alpha}) \geq m \wedge \alpha\\ &\Rightarrow \mathcal{G}(\lambda) \wedge \alpha \geq m \wedge \alpha, \text{ by Theorem 4.12}\\ &\Rightarrow \vee_{\alpha\in L_{0,1}}(\mathcal{G}(\lambda) \wedge \alpha) \geq \vee_{\alpha\in L_{0,1}}(m \wedge \alpha)\\ &\Rightarrow \mathcal{G}(\lambda) \wedge (\vee_{\alpha\in L_{0,1}}\alpha) \geq m \wedge (\vee_{\alpha\in L_{0,1}}\alpha)\\ &\Rightarrow \mathcal{G}(\lambda) \wedge 1 \geq m \wedge 1\\ &\Rightarrow \mathcal{G}(\lambda) \geq m\\ &\Rightarrow \lambda \in \mathcal{G}_m.\end{aligned}$$

So, $\lambda \in \mathcal{G}_m \Leftrightarrow \lambda \wedge \tilde{\alpha} \in \mathcal{G}_{m\wedge\alpha},\ \forall m,\ \alpha \in L_{0,1}$. □

Theorem 4.16. *Let $\{\mathcal{G}_m\}_{m\in L_{0,1}}$ be a descending family of stacks of fuzzy sets on X satisfying*

(1) *$\mathcal{G}_m$ is a grill of fuzzy sets on X, $\forall m \in M(L)$,*
(2) $\cap_{n\in\Delta}\mathcal{G}_{\alpha_n} = \mathcal{G}_{\vee_n\alpha_n},\ \alpha_n \in L_{0,1}$,
(3) $\lambda \in \mathcal{G}_m \Leftrightarrow \lambda \wedge \tilde{\alpha} \in \mathcal{G}_{m\wedge\alpha},\ \forall m,\ \alpha \in L_{0.1}$.

Then the mapping $g : L^X \to L$ defined by

$$g(\lambda) = \vee\{m \in M(L);\ \lambda \in \mathcal{G}_m\}$$

is a fuzzy grill on X with $g_m = \mathcal{G}_m,\ \forall m \in L_{0.1}$.

If, further, the condition

(4) $\tilde{\alpha} \in \mathcal{G}_\beta \Rightarrow \beta \leq \alpha,\ \alpha \in L_{0.1},\ \beta \in M(L)$ *is satisfied, then* g *is a graded fuzzy grill on* X.

Proof. Since $\tilde{0} \notin \mathcal{G}_m$, $\forall m \in M(L)$, it follows that $g(\tilde{0}) = 0$. Now,

$$\mu \supset \lambda \in \mathcal{G}_m \Rightarrow \mu \in \mathcal{G}_m \text{ (by the stack property of } \mathcal{G}_m)$$

So,

$$\{m;\ \mu \in \mathcal{G}_m\} \supset \{m;\ \lambda \in \mathcal{G}_m\}.$$

Thus $g(\mu) \geq g(\lambda)$. Then $g(\lambda \vee \mu) \geq g(\lambda) \vee g(\mu)$. Now,

$$\lambda \vee \mu \in \mathcal{G}_m \Rightarrow \lambda \in \mathcal{G}_m \text{ or } \mu \in \mathcal{G}_m \Rightarrow g(\lambda) \vee g(\mu) \geq m.$$

Therefore

$$g(\lambda \vee \mu) = \vee\{m;\ \lambda \vee \mu \in \mathcal{G}_m\} \leq g(\lambda) \vee g(\mu).$$

So,

$$g(\lambda \vee \mu) = g(\lambda) \vee g(\mu).$$

For $m \in L_{0,1}$, let $g_m = \{\lambda \in L^X;\ g(\lambda) \geq m\}$. Then, clearly, $\{g_m\}_{m \in L_{0,1}}$ is a descending family.

Next, let $B \in g_m$. Then $g(B) \geq m \Rightarrow \vee\{s \in M(L); B \in \mathcal{G}_s\} \geq m$. Let $S = \{s \in M(L); B \in \mathcal{G}_s\}$. Then $B \in \cap_{s \in S}\mathcal{G}_s = \mathcal{G}_{\vee\{s;s \in S\}} \subset \mathcal{G}_m$, since $\vee\{s; s \in S\} \geq m$ and $\{\mathcal{G}_m\}_{m \in L_{0.1}}$ is a descending family. So, $g_m \subset \mathcal{G}_m$. Again,

$$A \in \mathcal{G}_m \Rightarrow g(A) \geq m \Rightarrow A \in g_m \Rightarrow \mathcal{G}_m \subset g_m.$$

Hence

$$g_m = \mathcal{G}_m,\ \forall m \in M(L).$$

Next, let $m \in L_{0,1}$. Since $M(L)$ is join-generating $\exists\ \{\alpha_i;\ i \in \Delta\} \subset M(L)$ such that $\vee_{i \in \Delta}\alpha_i = m$. Now

$$\begin{aligned}
\lambda \in \mathcal{G}_m &\Leftrightarrow \lambda \in \mathcal{G}_{\alpha_i},\ \forall i \in \Delta \\
&\Rightarrow g(\lambda) \geq \alpha_i,\ \forall i \in \Delta \\
&\Rightarrow g(\lambda) \geq \vee_{i \in \Delta}\alpha_i = m \\
&\Rightarrow \lambda \in g_m \\
&\Rightarrow \lambda \in g_{\alpha_i},\ \forall i \in \Delta, \text{ since } \{g_m\}_{m \in L_{0,1}} \text{ is a descending family} \\
&\Rightarrow \lambda \in \mathcal{G}_{\alpha_i},\ \forall i \in \Delta, \text{ since } \alpha_i \in M(L) \\
&\Rightarrow \lambda \in \cap_{i \in \Delta}\mathcal{G}_{\alpha_i} = \mathcal{G}_{\vee_i \alpha_i} = \mathcal{G}_m \\
&\Rightarrow \lambda \in \mathcal{G}_m
\end{aligned}$$

Hence

$$\mathcal{G}_m = g_m, \ \forall m \in L_{0,1}.$$

Hence g is a fuzzy grill on X with $g_m = \mathcal{G}_m, \ \forall m \in L_{0,1}$.

Next, let $g(\lambda) = p$. If $p = 0$, then $g(\lambda) = 0 = \vee\{m \in M(L); \ \lambda \in \mathcal{G}_m\}$. Then $g(\lambda \wedge \tilde{\alpha}) \leq g(\lambda) = 0 = 0 \wedge \alpha = g(\lambda) \wedge \alpha$. Again $g(\lambda) \wedge \alpha = 0 \leq g(\lambda \wedge \tilde{\alpha})$. Therefore, in this case, $g(\lambda \wedge \tilde{\alpha}) = g(\lambda) \wedge \alpha, \ \forall \alpha \in L_{0,1}$.

If $p = 1$, then

$$\begin{aligned} g(\lambda) = 1 &\Rightarrow \lambda \in g_m, \ \forall m \in M(L) \\ &\Rightarrow \lambda \wedge \tilde{\alpha} \in g_{\alpha \wedge m}, \ \forall m \in M(L) \text{ (by (2))} \\ &\Rightarrow g(\lambda \wedge \tilde{\alpha}) \geq \alpha \wedge m, \ \forall m \in M(L) \\ &\Rightarrow g(\lambda \wedge \tilde{\alpha}) \geq \alpha \wedge (\vee_{m \in M(L)} m) = \alpha \wedge 1 \\ &\Rightarrow g(\lambda \wedge \tilde{\alpha}) \geq g(\lambda) \wedge \alpha, \ \forall \alpha \in L_{0,1}. \end{aligned}$$

Again,

$$\begin{aligned} g(\lambda \wedge \tilde{\alpha}) &\leq g(\lambda) \wedge g(\tilde{\alpha}) \text{ (by fuzzy stack property)} \\ &\leq g(\lambda) \wedge \alpha \text{ (by condition (4))}. \end{aligned}$$

Therefore,

$$g(\lambda \wedge \tilde{\alpha}) = g(\lambda) \wedge \alpha, \ \forall \alpha \in L_{0,1}.$$

Lastly, if $\alpha, \ p \in L_{0,1}$, then

$$\begin{aligned} \lambda \in g_p &\Leftrightarrow \lambda \wedge \tilde{\alpha} \in g_{\alpha \wedge p} \\ &\Leftrightarrow g(\lambda \wedge \tilde{\alpha}) \geq \alpha \wedge p \\ &\Leftrightarrow g(\lambda \wedge \tilde{\alpha}) \geq \alpha \wedge g(\lambda) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i) \end{aligned}$$

Next,

$$g(\lambda \wedge \tilde{\alpha}) \leq g(\tilde{\alpha}) \leq \alpha, \text{ (by (4)) and } g(\lambda \wedge \tilde{\alpha}) \leq g(\lambda).$$

So,

$$g(\lambda \wedge \tilde{\alpha}) \leq g(\lambda) \wedge \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

From (i) and (ii) we have $g(\lambda \wedge \tilde{\alpha}) = g(\lambda) \wedge \alpha$. Hence g forms a graded fuzzy grill on X. □

5. GRADED FUZZY PROXIMITIES

In [7], definition of a fuzzy proximity was introduced and some of its properties were studied in I-fuzzy setting. Further works on fuzzy proximities were done in [6]. In this section we introduce definitions of graded fuzzy preproximities and graded fuzzy proximities on X in L-fuzzy setting and study some of their properties.

Definition 5.1. A mapping $\delta : L^X \times L^X \to L$ satisfying

(1) $\delta(\lambda, \tilde{0}) = 0$,

(2) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$,

(3) $\delta\big(\vee \{\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2}\}, \mu\big) = [p_1 \wedge \delta(\lambda_1, \mu)] \vee [p_2 \wedge \delta(\lambda_2, \mu)]$, $p_1, p_2 \in L_0$

is called a graded fuzzy preproximity on X. Set of all graded fuzzy preproximities on X is denoted by $m(X)$.

Definition 5.2. For $\delta \in m(X)$, $\lambda \in L^X$, define $\delta(\lambda) : L^X \to L$ by

$$\delta(\lambda)(\mu) = \delta(\lambda, \mu), \ \forall \mu \in L^X.$$

Theorem 5.3. *A mapping $\delta : L^X \times L^X \to L$ is a graded fuzzy preproximity on X iff the following conditions hold:*

(1) $\delta(\lambda, \mu) = \delta(\mu, \delta)$,

(2) $\forall \lambda \in L^X$, $\delta(\lambda) \in \Gamma(X)$.

Proof. Let δ be a graded fuzzy preproximity on X. Then

$$\begin{aligned} \delta(\lambda)\left(\vee \left\{ \frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2} \right\} \right) &= \delta\left(\lambda, \vee \left\{ \frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2} \right\} \right) \\ &= [p_1 \wedge \delta(\lambda, \lambda_1)] \vee [p_2 \wedge \delta(\lambda, \lambda_2)] \\ &= [p_1 \wedge \delta(\lambda)(\lambda_1)] \vee [p_2 \wedge \delta(\lambda)(\lambda_2)] \end{aligned}$$

Thus

$$\delta(\lambda) \in \Gamma(X).$$

Conversely, let the conditions hold. Then $\delta = \delta^{-1}$ and since $\delta(\lambda) \in \Gamma(X)$, therefore we have

$$\begin{aligned} \delta(\lambda)\left(\vee \left\{ \frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2} \right\} \right) &= [p_1 \wedge \delta(\lambda)(\lambda_1)] \vee [p_2 \wedge \delta(\lambda)(\lambda_2)] \\ &\Rightarrow \delta\left(\lambda, \vee \left\{ \frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2} \right\} \right) \\ &= [p_1 \wedge \delta(\lambda, \lambda_1)] \vee [p_2 \wedge \delta(\lambda, \lambda_2)]. \end{aligned}$$

Also we have from Remark 4.5,

$$\delta(\lambda)(\tilde{0}) = 0 \Rightarrow \delta(\lambda, \tilde{0}) = 0.$$

Thus δ forms a graded fuzzy preproximity. □

Definition 5.4. A mapping $\rho : L^X \times L^X \to L$ having the following properties is called a graded fuzzy proximity on X:

(P1) $\rho(\tilde{\alpha}, \tilde{\alpha}') = 0$,

(P2) $\rho(\lambda, \mu) = \rho(\mu, \lambda)$,

(P3) $\rho(\vee\{\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2}\}, \mu) = [p_1 \wedge \rho(\lambda_1, \mu)] \vee [p_2 \wedge \rho(\lambda_2, \mu)]$, $p_1, p_2 \in L_0$

(P4) $\rho(\lambda, \mu) \not\geq r' \Rightarrow \rho(cl(\lambda, r), \mu) \not\geq r'$, where

$$cl(\lambda, r) = \wedge\{\eta' \geq \lambda;\ \rho(\lambda, \eta) \not\geq r'\},\ r \in L_1.$$

Remark 5.5. Evidently a graded fuzzy proximity is a fuzzy proximity.

Theorem 5.6. *Let ρ be a graded fuzzy proximity on X. Then $cl(\lambda, r) : L^X \times L_1 \to L^X$ defined in (P4) satisfies the following:*

(cl1) $cl(\tilde{\alpha}, r) = \tilde{\alpha},\ \forall \alpha \in L$

(cl2) $cl(\lambda, r) \geq \lambda$,

(cl3) $cl(\lambda, r) \leq cl(\lambda, s),\ if\ r \leq s$,

(cl4) $cl(\lambda, r) \leq cl(\mu, r),\ if\ \lambda \leq \mu$,

(cl5) $cl(\lambda_1 \vee \lambda_2, r) = cl(\lambda_1, r) \vee cl(\lambda_2, r)$, *if* $r \in Pr(L)$,

(cl6) $cl(cl(\lambda, r), r) = cl(\lambda, r)$,

(cl7) $\lambda = cl(\lambda, r) \Rightarrow cl(\lambda \vee \tilde{\alpha}', r \wedge \alpha) = cl(\lambda, r) \vee \tilde{\alpha}'$, *if* $r \in Pr(L)$ *and*

$$cl(\lambda \wedge \tilde{\alpha}', r \wedge \alpha') = cl(\lambda, r) \wedge \tilde{\alpha}',\ \forall \alpha \in L_{0.1}.$$

Proof. Proofs of (cl1)-(cl4) are obvious.

(cl5): $cl(\lambda_1 \vee \lambda_2, r) = \wedge\{\eta' \geq \lambda_1 \vee \lambda_2;\ \rho(\lambda_1 \vee \lambda_2, \eta) \not\geq r'\},\ r \in L_1$.
Obviously,

$$cl(\lambda_1 \vee \lambda_2, r) \geq cl(\lambda_1, r) \vee cl(\lambda_2, r).$$

If possible, let, $cl(\lambda_1 \vee \lambda_2, r) \not\leq cl(\lambda_1, r) \vee cl(\lambda_2, r)$. Then $\exists\ x_0$ such that

$$cl(\lambda_1 \vee \lambda_2, r)(x_0) \not\leq cl(\lambda_1, r)(x_0) \vee cl(\lambda_2, r)(x_0).$$

Since $M(L)$ is join generating $\exists\ s \in M(L)$ such that

$$cl(\lambda_1 \vee \lambda_2, r)(x_0) \geq s \not\leq cl(\lambda_1, r)(x_0) \vee cl(\lambda_2, r)(x_0).$$

Again

$$s \not\leq cl(\lambda_1, r)(x_0) \vee cl(\lambda_2, r)(x_0) \Rightarrow s \not\leq cl(\lambda_1, r)(x_0)$$

and

$$s \not\leq cl(\lambda_2, r)(x_0) \Rightarrow \eta_i \leq \lambda_i' \text{ such that } \rho(\lambda_i, \eta_i) \not\geq r' \text{ and } s \not\leq \eta_i'(x_0),\ i = 1, 2$$

Put $\eta = \eta_1 \wedge \eta_2$. Then $\eta \leq (\lambda_1 \vee \lambda_2)'$ and

$$\begin{aligned}\rho(\lambda_1 \vee \lambda_2, \eta) &= \rho(\lambda_1, \eta) \vee \rho(\lambda_2, \eta) \\ &\leq \rho(\lambda_1, \eta_1) \vee \rho(\lambda_2, \eta_2) \not\geq r' \text{ (since } r \in Pr(L)).\end{aligned}$$

Now,

$$\eta'(x_0) = \eta_1'(x_0) \vee \eta_2'(x_0) \not\geq s \text{ (since } s \in M(L) \text{ and } s \not\leq \eta_i'(x_0),\ i = 1, 2).$$

Therefore

$$cl(\lambda_1 \vee \lambda_2, r)(x_0) \leq \eta'(x_0) \not\geq s \Rightarrow cl(\lambda_1 \vee \lambda_2, r)(x_0) \not\geq s, \text{ a contradiction.}$$

Hence (cl5) holds.

(cl6): We have

$$(cl(\lambda, r))' = \vee\{\eta;\ \eta \leq \lambda',\ \rho(\lambda, \eta) \not\geq r'\}.$$

Then

$$\begin{aligned}&(\vee\{\eta;\ \eta \leq \lambda',\ \rho(\lambda, \eta) \not\geq r'\}) \wedge (cl(\lambda, r))' \\ &\leq \vee\{\eta \leq (cl(\lambda, r))';\ \rho(cl(\lambda, r), \eta) \not\geq r'\}, \text{ (by p4).}\end{aligned}$$

So,

$$(cl(\lambda, r))' \wedge (cl(\lambda, r))' \leq (cl(cl(\lambda, r), r))'$$

i.e.,

$$cl(\lambda, r) \geq cl(cl(\lambda, r), r).$$

So, by (cl2),

$$cl(cl(\lambda, r), r) = cl(\lambda, r).$$

(cl7): Let $\lambda = cl(\lambda, r)$. Then

$$\begin{aligned}cl(\lambda \vee \tilde{\alpha}', r \wedge \alpha) &\geq \lambda \vee \tilde{\alpha}', \text{ (by (cl2))} \\ &= cl(\lambda, r) \vee \tilde{\alpha}'\end{aligned}$$

Now,

$$\begin{aligned} cl(\lambda \vee \tilde{\alpha}', r \wedge \alpha) &\leq cl(\lambda \vee \tilde{\alpha}', r), \text{ (by (cl3))} \\ &= cl(\lambda, r) \vee cl(\tilde{\alpha}', r), \text{ (by (cl5))} \\ &= cl(\lambda, r) \vee \tilde{\alpha}', \text{ (by (cl1))}. \end{aligned}$$

Hence,

$$cl(\lambda \vee \tilde{\alpha}', r \wedge \alpha) = cl(\lambda, r) \vee \tilde{\alpha}'.$$

Again,

$$\begin{aligned} cl(\lambda \wedge \tilde{\alpha}', r \wedge \alpha') &\geq \lambda \wedge \tilde{\alpha}', \text{ by (cl2))} \\ &= cl(\lambda, r) \wedge \tilde{\alpha}'. \end{aligned}$$

Also,

$$cl(\lambda \wedge \tilde{\alpha}', r \wedge \alpha') \leq cl(\lambda, r), \text{ (by (cl3) and (cl4))}.$$

Similarly,

$$cl(\lambda \wedge \tilde{\alpha}', r \wedge \alpha') \leq cl(\tilde{\alpha}', r) = \tilde{\alpha}', \text{ (by cl1)}$$

Therefore,

$$cl(\lambda \wedge \tilde{\alpha}', r \wedge \alpha') \leq cl(\lambda, r) \wedge \tilde{\alpha}'.$$

Hence,

$$cl(\lambda \wedge \tilde{\alpha}', r \wedge \alpha') = cl(\lambda, r) \wedge \tilde{\alpha}'.$$

□

Conclusion

In this paper, by using the concept of L-fuzzy family, the ideas of graded fuzzy filters, graded fuzzy grills, graded fuzzy preproximities and graded fuzzy proximities are introduced and some of their properties are studied. By doing so the involvement of fuzziness in those structures are enhanced. In our next paper, further studies on graded fuzzy proximities will be done.

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