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Algebraic completeness results for sKD and its Extensions*

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This paper investigates algebraic semantics for sKD and its extensions $\mathrm{sKD}_{\triangle}$, $\mathrm{sKD}\forall$, and $\mathrm{sKD}\forall_{\triangle}$: sKD is a variant of the infinite -valued Kleene- Diense logic KD; $\mathrm{sKD}_{\triangle}$ is the sKD with the Baaz's projection $\triangle$; and $\mathrm{sKD}\forall$ and $\mathrm{sKD}\forall_{\triangle}$ are the first order extensions of sKD and $\mathrm{sKD}_{\triangle}$, respectively. I first provide algebraic completeness for each of sKD and $\mathrm{sKD}_{\triangle}$. Next I show that each $\mathrm{sKD}\forall$ and $\mathrm{sKD}\forall_{\triangle}$ is algebraically complete..

【주요어】 sKD, $\mathrm{sKD}_{\triangle}$, $\mathrm{sKD}\forall$, $\mathrm{sKD}\forall_{\triangle}$, algebraic semantics

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1. Introduction

Rescher [6] first considered the logic KD1) as an infinite-valued extension of the Kleene's three-valued logic [5] and its many-valued extension of Dienes [1]. But it gives a difficulty in showing semantic completeness of KD because it does have no tautologies in case it has the sole designated value the greatest, and in case it has as designated all the elements except for the least element, it may have the same tautologies as the classical propositional logic (CPL) by taking axioms and rules for CPL as those for KD.

Yang [7], [8] has recently investigated several logics with weak Boolean (wB) negation —. It is of interest that he [8] suggested in place of KD a variant of KD (sKD), and extended this to the sKD with — (wB-sKD) and the wB-sKD with quantifiers (wB-sKD∀). He gave algebraic soundness and completeness for each of them. In it he also suggested other extensions of sKD such as $\mathrm{sKD}_{\triangle}$ (the sKD with the [so called] Baaz's projection △) and $\mathrm{sKD}\forall_{\triangle}$ (the $\mathrm{sKD}_{\triangle}$ with quantifiers), together with the remark that each of $\mathrm{sKD}_{\triangle}$ and $\mathrm{sKD}\forall_{\triangle}$ is algebraically complete. He, however, did not give any exact proofs of them.

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1) By $S^{\supset}_{\aleph}$, Rescher expressed this logic. But we call it *KD* in honor of Kleene and Diense who first gave the idea of it as many-valued logic.

This paper verifies that his statement is correct. To do this, we shall first show that each sKD and sKD$'_{\triangle}$ is algebraically complete. Next we shall provide algebraic completeness for the first order extensions of sKD and sKD$_{\triangle}$ sKD$\forall$ and sKD$\forall_{\triangle}$, respectively. (Note that Yang [9] also gave algebraic completeness for sKD. But in this paper we provide its completeness in a little different style from [9].)

For convenience, by *sKD*$_{(\triangle)}$, we shall ambiguously express sKD and sKD$_{\triangle}$ together, if we do not need distinguish them, but context should determine which system is intended; and by *sKD*$\forall_{(\triangle)}$, sKD$\forall$ and sKD$\forall_{\triangle}$ together. Also we shall adopt the notation and terminology similar to those in [2], [3], [4], and assume familiarity with them.

2. Axiom Schemes and Rules for sKD$_{(\triangle)}$

For convenience, we present the axiomatic systems for sKD$_{(\triangle)}$ using the following axiom schemes and rules of inference. We shall use the biconditional ↔, where $A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$, and the falsity f. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

AXIOM SCHEMES

A1. $A \rightarrow A$ (self-implication)

A2. $(A \to B) \to ((C \to A) \to (C \to B))$ (prefixing)
A3. $(A \to (B \to C)) \to (B \to (A \to C))$ (permutation)
A4. $A \to (B \to A)$ (positive paradox)
A5. $(A \wedge B) \to A$, $(A \wedge B) \to B$ ($\wedge$-elimination)
A6. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
($\wedge$-introduction)
A7. $A \to (A \vee B)$, $B \to (A \vee B)$ ($\vee$-introduction)
A8. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
($\vee$-elimination)
A9. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(distributive law)
A10. $(A \to B) \vee (B \to A)$ (chain)
A11. ${\sim}{\sim}A \leftrightarrow A$ (double negation)
A12. $(A \to B) \to ({\sim}B \to {\sim}A)$ (contraposition)
A13. $({\sim}A \vee B) \to (A \to B)$
A14. $(A \to B) \vee ((A \to B) \to ({\sim}A \vee B))$
A15. $(A \to (A \to {\sim}A)) \to (A \to {\sim}A)$ (special contraction)
A16. $\triangle A \vee {\sim}\triangle A$
A17. $\triangle(A \vee B) \to (\triangle A \vee \triangle B)$
A18. $\triangle A \to A$
A19. $\triangle A \to \triangle\triangle A$
A20. $\triangle(A \to B) \to (\triangle A \to \triangle B)$

RULES
$A \to B, A \vdash B$ (modus ponens (MP))
$A, B \vdash A \wedge B$ (adjunction (AD))
From $\vdash A$ derive $\vdash \triangle A$ (necessitation (N))

DEFINITIONS
df1. $A \vee B := ((A \to B) \to B) \wedge ((B \to A) \to A)$
df2. ${\sim}A := A \to f$
df3. $A \mathbin{\&} B := {\sim}(A \to {\sim}B)$.

SYSTEMS
sKD: A1 to A15; MP, AD; df1 to df3.
$\mathrm{sKD}_{\triangle}$: sKD + A16 to A20, N.

Note that by df1 and df2 we may concern ourselves with

$\to$, $\wedge$, and f as propositional connectives for sKD, and $\to$, $\wedge$, f, and $\triangle$ for $\mathrm{sKD}_{\triangle}$. By df3, we can obtain

$$\text{(R)}\ (A \to (B \to C)) \leftrightarrow ((A \,\&\, B) \to C) \quad \text{(residuation)}$$

as a theorem of $\mathrm{sKD}_{(\triangle)}$.[2]

Note that in $\mathrm{sKD}_{\triangle}$ $\triangle$ can not be defined by $\neg$ and $\sim$ as in SBL (the strict basic (fuzzy) logic) of [3]. Note also that in sKD $\wedge$ can not be defined as $A \wedge B := A \,\&\, (A \to B)$ and thus the axiom $(A \,\&\, (A \to B)) \to (B \,\&\, (B \to A))$ of BL (the basic logic for residuated fuzzy logics) in [3], [4] is not valid in it. However, we can obtain in place of it $(A \wedge B) \to (B \wedge A)$ as a theorem of sKD.

We note that "$\sim$," "$\wedge$," "$\vee$," and "$\triangle$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear. Note also that with respect to any $\triangle$ formula A of the form $\triangle B$, the customary definitions of connectives, e.g., $A \to B = {\sim}A \vee B$, etc., in CPL can be applied to $\mathrm{sKD}_{\triangle}$ since such formulas have Boolean properties (see T5 in section 4).

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[2] We can easily prove this. We show left to right as an example: let $A \to (B \to C)$. Then by A12, transitivity, and MP, $A \to ({\sim}C \to {\sim}B)$. Thus by A3 and MP, ${\sim}C \to (A \to {\sim}B)$, and so by A12, transitivity, and MP, ${\sim}(A \to {\sim}B) \to C$. Hence by df3 $(A \,\&\, B) \to C$.

3. skd* and skd$^*_\triangle$ algebras

To prove algebraic completeness for sKD$_{(\triangle)}$, we must define an algebra, more exactly a matrix, that will characterize sKD$_{(\triangle)}$. Following [8], we shall call it a *strong Kleene-Diense*$^*_{(\triangle)}$ *(skd*$^*_{(\triangle)}$*) algebra*, more exactly, an skd* algebra for sKD and an skd$^*_\triangle$ algebra for sKD$_\triangle$, respectively.

Note that, for convenience, by an *skd*$^*_{(\triangle)}$ (algebra), we shall ambiguously express an skd* and an skd$^*_\triangle$ (algebra) together.

First, we define an *skd** *algebra* whose class will characterize sKD. An skd* algebra is a structure **A** = (A, $\top$, $\bot$, $\sim$, $\wedge$, $\vee$, $\rightarrow$) such that

(i) (A, $\top$, $\bot$, $\sim$, $\wedge$, $\vee$) is a bounded de Morgan (b-DM) lattice, i.e., (A, $\wedge$, $\vee$) is a distributive lattice with the greatest element $\top$ and the least $\bot$, and $\sim$ is a unary operation on A which is an involution.
(ii) let $a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a)$. The following conditions hold for all a, b, c: (with respect to lattice ordering $\leq$)
(1) $(a \rightarrow b) \vee (b \rightarrow a) = \top$
(2) $(a \rightarrow b) \vee ((a \rightarrow b) \rightarrow ({\sim} a \vee b)) = \top$
(3) $(a \rightarrow b) \leq ((a \rightarrow b) \leftrightarrow \top)$
(4) $((a \rightarrow b) \rightarrow ({\sim} a \vee b)) \leq ((a \rightarrow b) \leftrightarrow ({\sim} a \vee b))$
(5) $(a \rightarrow (a \rightarrow {\sim} a)) \leq (a \rightarrow {\sim} a)$
(6) $b \leq a \rightarrow c$ iff $a * b \leq c$, where (df4) $a * b := {\sim}(a \rightarrow {\sim} b)$.

Note that df4 corresponds to df3. Note also that an skd* algebra has as additional definitions algebraic counterparts corresponding to df1 and df2 (see below).

We shall call the implication satisfying (1) to (5) *strong Kleene-Diense (skd) implication* and its corresponding algebra, i.e., an algebra satisfying (i) and (1) to (5) in (ii), an *skd algebra*; call the implication satisfying (1) to (6) *skd* implication*. Thus, an skd* algebra can be regarded as an skd algebra with a residuation. Hence, an skd* algebra may be called (or regarded as) a *residuated skd algebra*.

We next define an *skd$^*_\triangle$ algebra* whose class will characterize sKD$_\triangle$. An skd$^*_\triangle$ algebra is a structure $(A, \top, \bot, {\sim}, \wedge, \vee, \rightarrow, \triangle)$ such that (i) $(A, \top, \bot, {\sim}, \wedge, \vee, \rightarrow)$ is an skd* algebra and (ii) $(A, \top, \bot, {\sim}, \vee, \triangle)$ satisfies the following conditions:

($\triangle$1) $\triangle a \vee {\sim}\triangle a = \top$;
($\triangle$2) $\triangle(a \vee b) \leq (\triangle a \vee \triangle b)$;
($\triangle$3) $\triangle a \leq a$;
($\triangle$4) $\triangle a \leq \triangle\triangle a$;
($\triangle$5) $\triangle a * \triangle(a \rightarrow b) \leq \triangle b$, where $a * b := {\sim}(a \rightarrow {\sim}b)$; and
($\triangle$6) $\triangle\top = \top$.

We call an algebra satisfying (ii) a *$\triangle$ algebra*. Thus, an skd$^*_\triangle$ algebra may be regarded as an skd* algebra plus a $\triangle$ algebra.

An skd$^*_{(\triangle)}$ algebra is *linearly ordered* if the ordering of its algebra is linear, i.e., $a \leq b$ or $b \leq a$ (equivalently, $a \wedge b = a$ or $a \wedge b = b$) for each pair a, b.

We note that in an skd$^*_{(\triangle)}$ algebra $*$ is a *left-continuous* t-norm (but not a *continuous* one) and → is its residual (see Definition 2.1.1 in [4]), and $\sim$ is the precomplement in the sense that $\sim$ can be defined as $\sim a := a \to \bot$ (see [4]). $(A, \top, \bot, *, \vee, \to)$ is a *residuated lattice* in the sense that it satisfies the definition of a residuated lattice (see Definition 2.3.2 in [4]).

Since $\top$ is the dual of $\bot$, i.e., $\top = \sim\bot$, join $\vee$ can be defined by using $\to$ and meet $\wedge$ (see df1), and $\sim$ by $\to$ and $\bot$ (see df2), an skd$^*_{(\triangle)}$ algebra $(A, \top, \bot, \sim, \wedge, \vee, \to, (\triangle))$ may be abbreviated to $(A, \bot, \wedge, \to, (\triangle))$.

Examples of skd$^*_{(\triangle)}$ algebras are:

1) The algebras $([0, 1], \max, \min, *, \to, \sim, (\triangle,)\ 0, 1)$ of rationals/reals between the unit interval 0 and 1, with a (left-continuous) t-norm $*$, its corresponding residuated implication $\to$ satisfying T4 below, and with an involutive negation function $\sim : [0, 1] \to [0, 1]$ (see T1 to T4 below with respect to skd* algebras), (and with any necessitation function $\triangle$ satisfying T5 below with respect to skd$^*_{\triangle}$ algebras).
2) The quotient algebra sKD$_{(\triangle)}/\equiv$ of provably equivalent formulas (see Proposition 5 below).

4. Algebraic completeness for sKD$_{(\triangle)}$

For any left-continuous t-norm $*$, we can define a propositional calculus *skd*$^*_{(\triangle)}$, in an analogy to the way that Hájek [7] does for continuous t-norms, i.e., taking $*$ and its residuum $\to$ as the truth functions for the (strong) conjunction & and the implication $\to$, respectively.

The language of skd$^*_{(\triangle)}$ is defined as usual from a countable set of propositional variables p_0, p_1, ⋯, three(four) connectives &, →, ∧, (△,) and the truth constant f. Further connectives can be defined as in section 2. An *evaluation* for $sKD_{(\triangle)}$ is a function v: PV → [0, 1] that is extended to all well-formed formulas of $L({\sim}, \rightarrow, \wedge, \vee, (\triangle,)\ p_0, p_1, \cdots)$ by the following tables, T1 to T4 for sKD and T1 to T5 for $sKD_{\triangle}$: (PV: set of propositional variables, [0, 1]: the unit interval)

TABLES
T1. $v({\sim}A) = 1 - v(A)$,
T2. $v(A \wedge B) = \min (v(A), v(B))$,
T3. $v(A \vee B) = \max (v(A), v(B))$,
T4. $v(A \rightarrow B) = 1$ if $v(A) \leq v(B)$
$\max (v({\sim}A), v(B))$ otherwise,
T5. $v(\triangle A) = 1$ if $v(A) = 1$
0 otherwise.

Note that in fact T1 and T3 are redundant because the former can be defined by T4 and v(f) = 0 and the latter by both T2 and T4 (see df1 and df2). We define a formula A to be an *1-tautology* of $sKD_{(\triangle)}$, briefly an *$sKD_{(\triangle)}$-tautology*, if v(A) = 1, i.e., ⊤, for each $sKD_{(\triangle)}$-evaluation v.

We next define several notions. A *theory* over $sKD_{(\triangle)}$ is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of $sKD_{(\triangle)}$ or a member of T or follows from some preceding members of the sequence using the rules above. T ⊢ A, more exactly $T \vdash_{sKD(\triangle)} A$, means that A is *provable* in T, i.e., there is an $sKD_{(\triangle)}$-proof of A in T. The deduction theorem for

$sKD_{(\triangle)}$ is as follows:

Proposition 1 let T be a theory and let A, B be formulas.

(i) $T \cup \{A\} \vdash_{sKD} B$ if and only if (iff) $T \vdash_{sKD} A^2 \to B$ where A^2 is A & A, 2 factors.

(ii) $T \cup \{A\} \vdash_{sKD\triangle} B$ iff $T \vdash_{sKD\triangle} \triangle A \to B$.

Proof For (i), Corollary 2 in [9].

For (ii), see Theorem 4 in [3]. □

Note that (R) ensures that A^n may be also regarded as $A \to (A \to \cdots (A \to$, n copies of A, and thus $T \vdash_{sKD} A^n \to B$ as $T \vdash_{sKD} A \to (A \to \cdots (A \to B)) \cdots)$. A theory is *inconsistent* if $T \vdash f$; otherwise it is *consistent.*

Let **A** be an $skd^*_{(\triangle)}$ algebra. In an analogy to the above, we define an ***A****-evaluation* of propositional variables to be any mapping v assigning to each propositional variable p an element v(p) of **A**. In the obvious way, this extends to an evaluations of all formulas using the operations on **A** as truth functions, i.e., $v(f) = 0$, i.e., $\bot$, $v(A \wedge B) = v(A) \wedge v(B)$, $v(A \to B) = v(A) \to v(B)$, (and $v(\triangle A) = \triangle v(A)$). (Thus, $v({\sim}A) = v(A) \to \bot$, $v(t) = 1$, i.e., $\top$, $v(A \vee B) = v(A) \vee v(B)$, and $v(A \,\&\, B) = v(A) * v(B)$.) We define a formula A to be an ***A****-tautology* if $v(A) = 1$ (or $\top$) for each **A**-evaluation v. Then, we can easily show that

Proposition 2 (Soundness) The logic $\mathrm{sKD}_{(\triangle)}$ is sound with respect to $\mathrm{sKD}_{(\triangle)}$-tautologies; if A is provable in $\mathrm{sKD}_{(\triangle)}$, then A is an A-tautology for each $\mathrm{skd}^*_{(\triangle)}$ algebra A.

We note that in each $\mathrm{skd}^*_{(\triangle)}$ algebra the equations (9) to (13), the (equational) conditions for adjointness (6), of Lemma 2.3.10 in [4] hold. Note also that each condition (1) to (5) for skd implication has a form of equation or can be defined in equation. Thus, since the class of (bounded) de Morgan lattices is a variety, (and each condition $\triangle 1$ to $\triangle 6$ also has a form of equation or can be defined in equation,) the class of all $\mathrm{skd}^*_{(\triangle)}$ algebras is a variety.

Proposition 3 The class of all $\mathrm{skd}^*_{(\triangle)}$ algebras is a variety of algebras.

Next, we show that classes of provably equivalent formulas form an $\mathrm{skd}^*_{(\triangle)}$ algebra. Let T be a fixed theory over $\mathrm{sKD}_{(\triangle)}$. For each formula A, let $[A]_T$ be the set of all formulas B such that $T \vdash A \leftrightarrow B$ (formulas T-provably equivalent to A). A_T is the set of all the classes $[A]_T$. We define that $[A]_T \rightarrow [B]_T = [A \rightarrow B]_T$, ${\sim}[A]_T = [{\sim}A]_T$, i.e., $[A]_T \rightarrow [f]_T = [A \rightarrow f]_T$, $[A]_T \wedge [B]_T = [A \wedge B]_T$, $[A]_T \vee [B]_T = [A \vee B]_T$, $[A]_T * [B]_T = [A \,\&\, B]_T$, $\triangle[A]_T = [\triangle A]_T$ (w.r.t $\mathrm{sKD}_\triangle$), $\perp$ (or 0) $= [f]_T$, and $\top$ (or 1) $= [t]_T$.[3]

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[3] It can be ensured that this definition is correct due to the provabilities as follows (we just need to check that $\leftrightarrow$ is a congruence with respect to $\wedge$, $\rightarrow$, (and $\triangle$)): we check just one

By A_T, we denote this algebra.

Note that to define A_T algebra we need just the definitions of $\rightarrow$, $\wedge$, $\perp$, (and $\triangle$) because we can define other operations and special element by using these.

Proposition 4 A_T is an $\mathrm{skd}^*_{(\triangle)}$ algebra.

Proof We first note that the lattice ordering $\leq$ satisfies the following (see the proof of Lemma 2.3.12 [4]):

$$[A]_T \leq [B]_T \text{ iff } T \vdash A \rightarrow B.$$

The axiom schemes A5 to A9, A11, and A12 ensure that $\wedge$, $\vee$, and $\sim$ satisfy de Morgan lattice properties, i.e., (i) in section 3. A10, A14, A15, and (R) together with the theorems (7) $(A \rightarrow B) \rightarrow ((A \rightarrow B) \leftrightarrow t)$, (8) $((A \rightarrow B) \rightarrow ({\sim}A \vee B)) \rightarrow ((A \rightarrow B) \leftrightarrow ({\sim}A \vee B))$ ensure that $*$ and $\rightarrow$ together with $\vee$, $\wedge$, and $\sim$ satisfy (ii) in section 3. That is, A10, A14, A15, (R), (7), and (8) ensure that (1), (2), (5), (6), (3), and (4), respectively, can be satisfied by these operations. Thus A_T (of sKD) is an skd^* algebra. Moreover, with respect to $\mathrm{sKD}_\triangle$, the axiom schemes A16 to A20 together with A4, N, and MP ensure that $\triangle$ is a $\triangle$

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direction. Let $\vdash A \rightarrow B$. With respect to $\wedge$, by A5 and transitivity, $(A \wedge C) \rightarrow B$, and thus $(A \wedge C) \rightarrow (B \wedge C)$ by A5, A6, AD, and MP; with respect to $\rightarrow$, by transitivity, it is almost immediate that $(B \rightarrow C) \rightarrow (A \rightarrow C)$ and $(C \rightarrow A) \rightarrow (C \rightarrow B)$; with respect to $\triangle$, by N, $\vdash \triangle(A \rightarrow B)$, and thus $\vdash \triangle A \rightarrow \triangle B$ by A20 and MP.

algebra, i.e., each A16 to A20 satisfies △1 to △5, respectively, and A18, A4, N, and MP ensure that △6 can be satisfied. Thus, A_T (of $sKD_{\triangle}$) is an $skd^{*}_{\triangle}$ algebra. □

Now we show how filters on residuated lattices determine homomorphisms and characterize homomorphisms to linearly ordered algebras. Let A be a residuated lattice. A *filter* on A is a non-empty set $F \subseteq A$ such that for each $x, y \in A$,

(F1) $x \in F$ and $y \in F$ imply $x \wedge y \in F$,
(F2) $x \in F$ and $x \leq y$ (or $x \rightarrow y \in F$) imply $y \in F$,
(F3) $x \in F$ implies $\triangle x \in F$.

F is a *prime filter* iff it is a filter and for each $x, y \in A$,

(PF) $(x \rightarrow y) \in F$ or $(y \rightarrow x) \in F$.

We note that this definition is just for a filter of $skd^{*}_{\triangle}$ algebras. With respect to skd^{*} algebras, we need only (F1) and (F2) as the definition of a filter because an skd^{*} algebra does not have △. Note also that with respect to a filter of $skd^{*}_{(\triangle)}$ algebras (PF) implies the usual definition of a prime filter (and vice versa) as follows:

Lemma 1 A filter F (of an $skd^{*}_{(\triangle)}$ algebra) is prime iff (PF′) for each pair of elements x, y such that $x \vee y \in F$, $x \in F$ or $y \in F$.

Proof Left to right. Let F be prime and $x \vee y \in F$. By primeness, either $x \to y \in F$ or $y \to x \in F$. Let $x \to y \in F$. Then, A1 and A8 ensure that $(x \vee y) \to y \in F$. Thus, $y \in F$. Let $y \to x \in F$. Similarly, $x \in F$.

Right to left. A10 ensures that $(x \to y) \vee (y \to x) \in F$. Then, by primeness $(x \to y) \in F$ or $(y \to x) \in F$, as desired. □

Proposition 5 Let A be an $\mathrm{skd}^*_{(\triangle)}$ algebra and let F be a filter. Put $x \equiv_F y$ iff $(x \to y) \in F$ and $(y \to x) \in F$. Then,

(i) $\equiv_F$ is a congruence relation over an $\mathrm{skd}^*_{(\triangle)}$ algebra.
(ii) The quotient of algebra $A/\equiv_F$ is an $\mathrm{skd}^*_{(\triangle)}$ algebra.
(iii) $A/\equiv_F$ is linearly ordered iff F is a prime filter.
(iv) Linearly ordered $\mathrm{skd}^*_{\triangle}$ algebras A are simple, i.e., the only filters of a linearly ordered $\mathrm{skd}^*_{\triangle}$ algebra A are $\{1\}$ and A itself.

Proof For (i), we first observe that $\equiv_F$ is transitive to show that $\equiv_F$ is an equivalence: it follows from the fact that the formula $(A \to B) \to ((B \to C) \to (A \to C))$ is an 1-tautology over A, and thus $(a \to b) \leq ((b \to c) \to (a \to c))$: let $(a \to b), (b \to c) \in F$. If $(a \to b) \in F$, then $((b \to c) \to (a \to c)) \in F$. Hence, since $(b \to c) \in F$, $(a \to c) \in F$. Thus, we may define equivalence classes $[x]_F = \{y: x \equiv_F y\}$. We next verify that $\equiv_F$ is a congruence, i.e., preserves operations. Analogously to the proof of the statement that $[A]_T \leq [B]_T$ iff $T \vdash A \to B$ in Proposition 4, we can show that $[x]_F \leq [y]_F$ iff $x \to y \in F$: as an example we prove right to left. Let $x \rightarrow y \in F$. Then, since $x \rightarrow x \in F$, A6 ensures that $x \rightarrow (x \wedge y) \in F$. Thus, since A5 ensures that $(x \wedge y) \rightarrow x \in F$, $(x \wedge y) \equiv_F x$. Hence, $[x \wedge y]_F$ (= $[x]_F \wedge [y]_F$) = $[x]_F$, i.e., $[x]_F \leq [y]_F$. Note that, analogously to the footnote 3, we can verify that $[x]_F = [y]_F$ implies $[x \wedge z]_F = [y \wedge z]_F$, $[x \rightarrow z]_F = [y \rightarrow z]_F$, $[z \rightarrow x]_F = [z \rightarrow y]_F$, and $[\triangle x]_F = [\triangle y]_F$: we show as an example that $[x]_F = [y]_F$ implies $[\triangle x]_F = [\triangle y]_F$. Let $[x]_F \leq [y]_F$. Then, $x \rightarrow y \in F$. Thus, by (F3), $\triangle(x \rightarrow y) \in F$. Since $\triangle(x \rightarrow y) \leq \triangle x \rightarrow \triangle y$ by $\triangle 5$ and residuation, $\triangle x \rightarrow \triangle y \in F$ by (F2). Hence, $[\triangle x]_F \leq [\triangle y]_F$. Analogously, $[\triangle y]_F \leq [\triangle x]_F$ follows from $[y]_F \leq [x]_F$. So $[x]_F = [y]_F$ implies $[\triangle x]_F = [\triangle y]_F$. Therefore, $\equiv_F$ is a congruence.

(ii) and (iii) are analogous to those of Lemma 2.3.14 and those in proofs of Theorem 2.4.12 in [4].

The proof of (iv) reduces to showing that the only filters of a linearly ordered $\text{skd}^*_\triangle$ algebra **A** are $\{1\}$ and the full algebra **A** itself. This is true because if a filter F has an element $x \neq 1$, then $\triangle x = 0$ and thus $0 \in F$. Hence $F = A$. □

Proposition 6 Let **A** be an $\text{skd}^*_{(\triangle)}$ algebra and let $a \in A$, $a \neq 1$ (the greatest element). Then there is a prime filter F on A not containing a.

Proof Note that we may use the primeness (PF′) in place of (PF) by Lemma 1, and thus by Prime Filter Separation Principle in a distributive lattice, it is immediate. The proof is very analogous to that of Lemma 8.6.2 in [2].

Let A, a, 1 be as in the hypothesis. Then, $F_0 = [1)$, $= \{x \in A: 1 \leq x\}$, is a filter separating 1 from a. We let E be the family of filters of **A**, which have 1 as a member but not a. E is non-empty, since it contains F_0. Now let C be any non-empty chain of E. Then, $\cup C \in E$. For clearly $1 \in \cup C$, and $a \notin \cup C$. Then, it remains to show that $\cup C$ is a filter. Suppose $b, c \in \cup C$, but then there are $F', F'' \in C$ such that $b \in F'$ and $c \in F''$. But either $F' \subseteq F''$ or $F'' \subseteq F'$, and so either $(b \wedge c) \in F''$ or $(b \wedge c) \in F'$ because both F' and F'' are filters. But then in either case $(b \wedge c) \in \cup C$, and thus (F1) is satisfied. Similarly, we can show that (F2) is satisfied. The interesting point to check is that (F3) can be satisfied (with respect to an skd$^*_{\triangle}$ algebra): let $a \in \cup C$. Then, there is $F' \in C$ such that $a \in F'$. So, since F' is a filter, $\triangle a \in F'$. Hence, $\triangle a \in \cup C$, as desired.

By Zorn's Lemma, we may conclude that E has some maximal member F, which is a filter such that $1 \in F$ and $a \notin F$. It remains to show that F is prime. Its proof is as usual (see the proof of Lemma 8.6.2 in [2] for it). □

Proposition 7 Each skd$^*_{(\triangle)}$ algebra is a subdirect product of linearly ordered skd$^*_{(\triangle)}$ algebras.

Proof Its proof is as usual (see the proof of Lemma 2.3.16 in [4]). □

Note that with respect to skd$^*_\triangle$ algebras this theorem is a subdirect decomposition theorem because by Proposition 5 (iv) linearly ordered skd$^*_\triangle$ algebras are simple, and thus subdirectly irreducible, which is not the case for skd* algebras.

Let us associate with each formula A of sKD$_{(\triangle)}$ a term A^t of the language of skd$^*_{(\triangle)}$ algebras by replacing the connectives and constants $\sim$, $\rightarrow$, &, $\wedge$, $\vee$, ($\triangle$,) f, t by function symbols and special elements $\sim$, $\rightarrow$, $*$, $\wedge$, $\vee$, ($\triangle$,) 1 (or $\top$), 0 (or $\perp$), respectively, and replacing each propositional variable p_i by a corresponding object variable x_i.

Proposition 8 (i) Each formula which is an A-tautology for all linearly ordered skd$^*_{(\triangle)}$ algebras is an A-tautology for all skd$^*_{(\triangle)}$ algebras.

(ii) A is an A-tautology iff the identity $A^t = 1$, i.e., $\top$, is true in A.

Proof (i) follows from (ii) and the subdirect product representation. (ii) is evident since the value of the term A^t given by an evaluation v is $v_A(A)$. □

Theorem 1 (Weak completeness) sKD$_{(\triangle)}$ is complete with respect to the class of $\mathrm{skd}^*_{(\triangle)}$ algebras, i.e., for each formula A the following are equivalent:

(i) A is provable in $\mathrm{sKD}_{(\triangle)}$, i.e., $\vdash_{\mathrm{sKD}(\triangle)} A$,
(ii) For each linearly ordered $\mathrm{skd}^*_{(\triangle)}$ algebra **A**, A is an **A**-tautology;
(iii) For each $\mathrm{skd}^*_{(\triangle)}$ algebra **A**, A is an **A**-tautology.

Proof The implications of (i) to (ii) and (ii) to (iii) have been established. Thus, it suffices to show that (iii) to (i) holds:

Note that Proposition 4 says that the algebra $\mathbf{A}_{\mathrm{sKD}(\triangle)}$ of classes of equivalent formulas of $\mathrm{sKD}_{(\triangle)}$ is an $\mathrm{skd}^*_{(\triangle)}$ algebra. Thus, an A satisfying (iii) is an $\mathbf{A}_{\mathrm{sKD}(\triangle)}$-tautology. Now let $v(p_i) = [p_i]_{\mathrm{sKD}(\triangle)}$ for all propositional variables. Then $v(A) = [A]_{\mathrm{sKD}(\triangle)} = [t]_{\mathrm{sKD}(\triangle)}$, and thus $\vdash_{\mathrm{sKD}(\triangle)} A \leftrightarrow t$. Hence $\vdash_{\mathrm{sKD}(\triangle)} A$. □

To achieve strong completeness for $\mathrm{sKD}_{(\triangle)}$, we add more definitions on a theory T to the definitions above. Let **A** be an $\mathrm{skd}^*_{(\triangle)}$ algebra. Note that elements of T are axioms of T. An **A**-evaluation v is an *A-model* of T if $v_{\mathbf{A}}(\alpha) = 1_{\mathbf{A}}$, i.e., $\top_{\mathbf{A}}$, for each axioms $\alpha \in T$. T is *complete* if for each pair A, B of formulas, $T \vdash A \to B$ or $T \vdash B \to A$. Note that corresponding to Lemma 1 it can be ensured that T is complete iff for each pair of A, B such that $T \vdash A \vee B$, $T \vdash A$ or $T \vdash B$ (see Lemma 5.2.3 in [4]). We call this, i.e., the T of the second statement, also complete.

Lemma 2 Let A^n be $A \& \cdots \& A$, n factors. sKD proves:

(i) $(A \vee B)^n \leftrightarrow (A^n \vee B^n)$.
(ii) $(A \wedge B)^n \leftrightarrow (A^n \wedge B^n)$.
sKD$_\triangle$ proves:
(iii) $\triangle(A \wedge B) \leftrightarrow (\triangle A \wedge \triangle B)$.

Proof Let A^n be $A \& \cdots \& A$, n factors. For (i) and (ii), we first note that sKD proves:

(a) $(A \& A) \rightarrow (A \& A \& A)$; $A^n \rightarrow A^{n+1}$, $2 \leq n$,
(b) $(A \& A) \rightarrow A$; $A^n \rightarrow A^{n-1}$, $2 \leq n$,
(c) $A^n \leftrightarrow A^m$, $2 \leq n, m$; $A^2 \leftrightarrow A^n$, $2 \leq n$.

We can easily prove these.

For (i) and (ii), we just show that $(A \vee B)^3 \leftrightarrow (A^3 \vee B^3)$ as an example. Note that $(A \vee B)^2 \leftrightarrow (A^2 \vee B^2)$ (see Lemma 2.2.24 in [4]):

For left to right, since $(A \vee B)^3 \leftrightarrow (A^3 \vee (A^2 \& B) \vee (A \& B^2) \vee B^3)$, we need to show that (*) $(A \& B^2) \rightarrow (A^3 \vee B^3)$ and (**) $(A^2 \& B) \rightarrow (A^3 \vee B^3)$. For (*), $(A \rightarrow B^2) \rightarrow ((A \& B^2) \rightarrow B^4)$ by Lemma 2.2.8 (6), which is also a theorem of sKD, in [4]. Since $B^4 \rightarrow B^3$ by (b), $(A \rightarrow B^2) \rightarrow ((A \& B^2) \rightarrow B^3)$ by transitivity, A3, and MP. Similarly, since $B^3 \rightarrow (A^3 \vee B^3)$ by A7, $(A \rightarrow B^2) \rightarrow ((A \& B^2) \rightarrow (A^3 \vee B^3))$. Analogously, we can show that $(B^2 \rightarrow A) \rightarrow ((A \& B^2) \rightarrow (A^3 \vee B^3))$ (we just note that by (a), $A^2 \rightarrow A^3$). Thus, by AD, A8, and MP, $((A \rightarrow B^2) \vee (B^2 \rightarrow A)) \rightarrow ((A \& B^2) \rightarrow (A^3 \vee B^3))$. Hence, by A10 and MP, $(A \& B^2) \rightarrow (A^3 \vee B^3)$. Analogously to (*), we can prove (**). Right to left is immediate by A7.

Analogously, just by iterating and using (c), we can show $(A \vee B)^n \leftrightarrow (A^n \vee B^n)$.

For right to left of (iii), let $(A \wedge B) \rightarrow A$ by A5. Then, by N, A20, and MP, $\triangle(A \wedge B) \rightarrow \triangle A$. Analogously, $\triangle(A \wedge B) \rightarrow \triangle B$. Thus, $\triangle(A \wedge B) \rightarrow (\triangle A \wedge \triangle B)$ by AD, A6, and MP.

For left to right of (iii), let $(\triangle A \wedge \triangle B) \rightarrow \triangle A$ by A5. Then, by A18, transitivity, and MP, $(\triangle A \wedge \triangle B) \rightarrow A$. Analogously, $(\triangle A \wedge \triangle B) \rightarrow B$. Thus, $\triangle(A \wedge B) \rightarrow (A \wedge B)$ by AD, A6, and MP, and so by N and A20, $\triangle(\triangle A \wedge \triangle B) \rightarrow \triangle(A \wedge B)$. Note that since $\triangle$ formulas satisfy Boolean properties, $\triangle(\triangle A \wedge \triangle B) \leftrightarrow (\triangle\triangle A \wedge \triangle\triangle B)$. Thus, $(\triangle\triangle A \wedge \triangle\triangle B) \rightarrow \triangle(A \wedge B)$. Now by A5, A19, transitivity, and MP, $(\triangle A \wedge \triangle B) \rightarrow \triangle\triangle A$. Analogously, $(\triangle A \wedge \triangle B) \rightarrow \triangle\triangle B$. Hence, by AD, A6, and MP, $(\triangle A \wedge \triangle B) \rightarrow (\triangle\triangle A \wedge \triangle\triangle B)$. Therefore, by transitivity and MP, $(\triangle A \wedge \triangle B) \rightarrow \triangle(A \wedge B)$, as desired. □

Proposition 9 (i) T is complete iff the $\mathrm{skd}^*_{(\triangle)}$ algebra A_T is linearly ordered.

(ii) If T is a theory and $T \nvdash A$, then there is a consistent complete supertheory $T' \supseteq T$ such that $T' \nvdash A$.

Proof (i) Left to right. Let T be complete and A, B be the pair of formulas of its language. We note that (*) $[A]_T \leq [B]_T$ iff $T \vdash A \rightarrow B$. Since T is complete, either $T \vdash$

$A \rightarrow B$ and thus by (*) $[A]_T \leq [B]_T$, or $T \vdash B \rightarrow A$ and thus by (*) $[B]_T \leq [A]_T$. Hence $\leq$ is linear and thus A_T is linearly ordered.

Right to left. Let A_T be linearly ordered and A, B be as above. Then, either $[A]_T \leq [B]_T$ and $T \vdash A \rightarrow B$, or $[B]_T \leq [A]_T$ and $T \vdash B \rightarrow A$. Hence, T is complete.

(ii) We shall use the completeness property of T, which corresponds to (PF′) in Lemma 1. Where Δ is a set of formulas not necessarily a theory, $\Delta \vdash A$ can be thought of as saying that A is deducible from the 'axioms' Δ. The set of $\{A: \Delta \vdash A\}$ is intuitively the smallest theory containing the axioms Δ, and we shall label it as *Th(Δ)*.

Now take an enumeration $\{A_n: n \in \omega\}$ of the well-formed formulas of $sKD_{(\Delta)}$. We define a sequence of sets by induction as follows:

$$T_0 = \{A' : T \vdash_{sKD(\Delta)} A'\}.$$

$$\begin{array}{lll} T_{i+1} = & Th(T_i \cup \{A_{i+1}\}) & \text{if it is not the case that } T_i, A_{i+1} \vdash_{sKD(\Delta)} A, \\ & T_i & \text{otherwise.} \end{array}$$

Let T' be the union of all these T_n's. It is easy to see that T' is a theory not containing A (and thus it is consistent). Also we can show that it is complete.

Suppose toward contradiction that $B \vee C \in T'$ and $B, C \notin T'$. Then the theories obtained from $T' \cup B$ and $T' \cup C$ must both contain A. It follows that there is a conjunction of members of T' T'' such that $T'' \wedge B \vdash_{sKD} A$ and $T'' \wedge C \vdash_{sKD} A$. Then, by A8, Proposition 1 (i), AD, and MP, $\vdash_{sKD} ((T'' \wedge B)^2 \vee (T'' \wedge C)^2) \to A$. Hence, since (i) and (ii) of Lemma 2 ensure that $\vdash_{sKD} ((T''^2 \wedge B^2) \vee (T''^2 \wedge C^2)) \to A$, by A2, A9, and MP, $\vdash_{sKD} (T''^2 \wedge (B^2 \vee C^2)) \to A$, and thus $\vdash_{sKD} (T'' \wedge (B \vee C))^2 \to A$. Therefore, $T'' \wedge (B \vee C) \vdash_{sKD} A$ by Proposition 1 (i). From this we get that $T' \vdash A$, which is contrary to our supposition.

In an analogy to the above, we can show this completeness with respect to sKD$_\triangle$. Suppose toward contradiction that $B \vee C \in T'$ and $B, C \notin T'$. Then the theories obtained from $T' \cup B$ and $T' \cup C$ must both contain A. It follows that there is a conjunction of members of T' T'' such that $T'' \wedge B \vdash_{sKD\triangle} A$ and $T'' \wedge C \vdash_{sKD\triangle} A$. Then, by A8, Proposition 1 (ii), AD, and MP, $\vdash_{sKD\triangle} (\triangle(T'' \wedge B) \vee \triangle(T'' \wedge C)) \to A$. Hence, since A17 and (iii) of Lemma 2 ensure that $\vdash_{sKD\triangle} ((\triangle T'' \wedge \triangle B) \vee (\triangle T'' \wedge \triangle C)) \to A$, by A2, A9, and MP $\vdash_{sKD\triangle} (\triangle T'' \wedge (\triangle B \vee \triangle C)) \to A$, and thus $\vdash_{sKD\triangle} \triangle(T'' \wedge (B \vee C)) \to A$. Therefore, $T'' \wedge (B \vee C) \vdash_{sKD\triangle} A$ by Proposition 1 (ii). From this we get that $T' \vdash A$, which is contrary to our supposition. ☐

By using Proposition 9 (and Soundness as usual), we can easily show that

Theorem 2 (Strong completeness) Let T be a theory over sKD$_{(\triangle)}$ and let A be a formula. Then the following are equivalent:

(i) $T \vdash_{sKD(\triangle)} A$.

(ii) For each linearly ordered $skd^{*}_{(\triangle)}$ algebra **A** and each **A**-model v of T, $v_{\mathbf{A}}(A) = 1_{\mathbf{A}}$, i.e., $\top_{\mathbf{A}}$.

(iii) For each $skd^{*}_{(\triangle)}$ algebra **A** and each **A**-model v of T, $v_{\mathbf{A}}(A) = 1_{\mathbf{A}}$, i.e., $\top_{\mathbf{A}}$.

5. $sKD\forall_{(\triangle)}$: the first order extension of $sKD_{(\triangle)}$

The completeness theorems for fuzzy predicate logics presented in [3], [4] may generalize for the present situation.

A trivial generalization of those of section 6 in [3] and Chapter V in [4] gives the notions of a language, its interpretations, and formulas for $sKD\forall_{(\triangle)}$ as follows:

Given a linearly ordered $skd^{*}_{(\triangle)}$ algebra **A**, an *A-interpretation*, i.e., an *A-structure*, of a language consisting of some predicates $P \in Pred$ and constants $c \in Const$ is a structure **M** = $(M, (r_P)_{P \in Pred}, (m_c)_{c \in Const})$, where $M \neq \varnothing$, r_P: $M^{ar(P)} \rightarrow A$, and $m_c \in M$ (for each $P \in Pred$, $c \in Const$).

Let L be a predicate language and let **M** be an **A**-structure for L. An *M-evaluation* of object variables is a mapping e assigning to each object variable x an element $e(x) \in M$. Let e, e′ be two evaluations. $e \equiv_x e'$ means that $e(y) = e'(y)$ for each variable y distinct from x.

The value of a term given by M,e is defined as follows: $|x|_{M,e} = e(x)$ and $|c|_{M,e} = m_c$. The *(truth) value* $|A|^{A}_{M,e}$ of a formula (where $e(x) \in M$ for each variable x) is defined inductively: for A being $P(x, \cdots, c, \cdots)$,

$|P(x, \cdots, c, \cdots)|^{A}_{M,e} = r_P(e(x), \cdots, m_c, \cdots)$,

the value commutes with connectives, and

$|(\forall x)A|^{A}_{M,e} = \inf\{|A|^{A}_{M,e'}: e \equiv_x e'\}$

if this infimum exists, otherwise undefined, and similarly for $\exists x$ and sup. M is *A-safe* if all infs and sups needed for definition of the value of any formula exist in A, i.e., $|A|^{A}_{M,e}$ is defined for all A, e.

Let A be a formula of a language L and let M be a safe A-structure for L. The *truth value* of A in M is

$|A|^{A}_{M} = \inf\{|A|^{A}_{M,e}$: e M-evaluation.

A formula A of a language L is an *A-tautology* if $|A|_M = 1_A$ for each safe A-structure M, i.e., $|A|^{A}_{M,e} = 1$ for each safe A-structure M and each M-evaluation of object variables.

The axioms of $sKD\forall_{(\triangle)}$ are those of $sKD_{(\triangle)}$ plus the following set of axioms for quantifiers (taken by Hájek [4] as those of the basic predicate logic $BL\forall$):

($\forall$1) $(\forall x)A(x) \rightarrow A(t)$ (t substitutable for x in A(x))
($\exists$1) $A(t) \rightarrow (\exists x)A(x)$ (t substitutable for x in A(x))
($\forall$2) $(\forall x)(A \rightarrow B) \rightarrow (A \rightarrow (\forall x)B)$ (x not free in A)
($\exists$2) $(\forall x)(A \rightarrow B) \rightarrow ((\exists x)A \rightarrow B)$ (x not free in B)
($\forall$3) $(\forall x)(A \vee B) \rightarrow ((\forall x)A \vee B)$ (x not free in B)

Rules of inference for $sKD\forall_{(\triangle)}$ are MP, AD, N, and generalization (GN), i.e., from A infer $(\forall x)A$. More exactly, for $sKD\forall$ MP, AD, and GN, and for $sKD\forall_{\triangle}$ MP, AD, N, and GN all. Note that in $sKD\forall_{(\triangle)}$ one quantifier is definable from the other one and the negation, for instance, $(\exists x)A := {}^{\sim}(\forall x)^{\sim}A$. Thus the above set of axioms for quantifiers could be simplified, i.e., $(\forall 3)$, $(\exists 1)$, and $(\exists 2)$ become provable as in the Łukasiewicz predicate logic Ł$\forall$ (cf. see Remark 5.4.2 in [4]).

Proposition 12 (i) The axioms $(\forall 1)$ and $(\forall 2)$ are **A**-tautologies for each linearly ordered $\mathrm{skd}^{*}_{(\Delta)}$ algebra **A**. (ii) The rules MP, AD, N, and GN preserve **A**-tautologyhood.

Proof (i) By Lemmas 5.1.9 in [4].

(ii) MP and GN are by Lemma 5.1.10 in [4]. Thus, for $\mathrm{sKD}\forall_{(\Delta)}$ we need just to consider that the rules AD, N preserve **A**-tautologyhood. For AD, we show that

(1) for any formulas A, B, safe **A**-structure **M**, and evaluation e,

$|A|^{A}_{M,e} * |B|^{A}_{M,e} \leq |A \wedge B|^{A}_{M,e}$;

thus, if $|A|^{A}_{M,e} = |B|^{A}_{M,e} = 1_{A}$, then $|A \wedge B|^{A}_{M,e} = 1_{A}$, and

(2) consequently,

$|A|^{A}_{M} * |B|^{A}_{M} \leq |A \wedge B|^{A}_{M}$;

thus if A, B are 1_A-true in **M**, then $A \wedge B$ is.

(1) is as in propositional calculus. To prove (2) put $|A|_w = a_w$, $\inf_w a_w = a$, $|B|_w = b_w$, and $\inf_w b_w = b$. We have to show that $\inf_w a_w * \inf_w b_w \leq \inf_w(a_w \wedge b_w)$

(indices **A**, **M** deleted, w runs over all evaluations $\equiv_x$ e).

Since $sKD\forall_{(\triangle)}$ proves $(A\&B) \rightarrow (A\wedge B)$ and $(\forall x)(A\wedge B) \leftrightarrow ((\forall x)A\wedge(\forall x)B)$ (see Corollary 5.1.22 (17) [4]), it can be obtained that $((\forall x)A \ \& \ (\forall x)B) \rightarrow (\forall x)(A \wedge B)$. This ensures that $(\inf_w a_w * \inf_w b_w) \leq \inf_w(a_w \wedge b_w)$.

For N, we show that (3) for any formulas A, B, safe A-structure M, and evaluation e,

if $|A|^{A}_{M,e} = 1_A$, then $|\triangle A|^{A}_{M,e} = 1_A$, and

(4) consequently,

if $|A|^{A}_{M} = 1_A$, then $|\triangle A|^{A}_{M} = 1_A$,

thus if A is 1_A-true in M, then $\triangle A$ is.

(3) is as in propositional calculus with $\triangle$ from the property $1 = \triangle 1$ of $\triangle$. To prove (4) put $|A|_w = a_w$, $\inf_w a_w = a$. We have to show that

$\inf_w a_w = 1$ implies $\inf_w \triangle a_w = 1$

(indices A, M deleted, w runs over all evaluations $\equiv_x e$). Since by Lemma 8 in [8] $(\forall x)\triangle A \leftrightarrow \triangle(\forall x)A$, and thus $\inf_w \triangle a_w = \triangle \inf_w a_w$, this is just to show that $a = 1$ implies $\triangle a = 1$. It is immediate as in (1). □

Definitions of a theory T over $sKD\forall_{(\triangle)}$ are almost the same as $sKD_{(\triangle)}$. We need just to consider such definitions in M. Let A be a linearly ordered $skd^*_{(\triangle)}$ algebra and let M be a safe A-structure for the language of T. M is an *A-model* of T if all axioms of T are 1_A-true in M, i.e., $|A|^{A}_{M} = 1_A$ in each $A \in T$. Then, Proposition 12 ensures that $sKD\forall_{(\triangle)}$ is sound with respect to linearly ordered $skd^*_{(\triangle)}$ algebras as follows.

Proposition 13 (Soundness) Let T be a theory in the language of T over $sKD\forall_{(\triangle)}$ and let A be a formula of T. If $T \vdash A$, then $|A|^{\mathbf{A}}_{M} = 1_{\mathbf{A}}$ for each linearly ordered $skd^{*}_{(\triangle)}$ algebra $\mathbf{A}$ and each $\mathbf{A}$-model M of T.

Proof By induction on the length of a proof. □

To investigate completeness for $sKD\forall_{(\triangle)}$, we have the same definitions on "consistency" and "completeness" of a theory T as in $sKD_{(\triangle)}$. We, moreover, define the Henkinness of T (over $sKD\forall_{(\triangle)}$) as follows: T is *Henkin* if for each closed formula of the form $(\forall x)A(x)$ unprovable in T, i.e., $T \nvdash (\forall x)A(x)$, there is a constant c in the language of T such that A(c) is unprovable in T, i.e., $T \nvdash A(c)$.

For each theory T over $sKD\forall_{(\triangle)}$, let $\mathbf{A}_T$ be the algebra of classes of T-equivalent closed formulas with the usual operations. It is clear that $\mathbf{A}_T$ is an $skd^{*}_{(\triangle)}$ algebra.

Lemma 5 For each theory T and each closed formula A, if $T \nvdash A$, then there is a complete Henkin supertheory T' of T such that $T' \nvdash A$.

Proof See the proofs of Proposition 9 (ii) above and Lemma 5.2.7 in [4]. □

Lemma 6 For each complete Henkin theory T and each closed formula A, if $T \nvdash A$, then there is a linearly ordered $\mathrm{skd}^{*}_{(\triangle)}$ algebra **A** and **A**-model **M** of T such that $|A|^{\mathbf{A}}_{\mathbf{M}} < 1_T$.

Proof By Lemma 5.2.8 in [4]. □

By using Lemmas 5 and 6, we can show the completeness for $\mathrm{sKD}\forall_{(\triangle)}$ as follows.

Theorem 4 (Completeness) Let T be a theory over $\mathrm{sKD}\forall_{(\triangle)}$ and let A be a formula. T proves A over $\mathrm{sKD}\forall_{(\triangle)}$ iff $|A|^{\mathbf{A}}_{\mathbf{M}} = 1_{\mathbf{A}}$ for each linearly ordered $\mathrm{skd}^{*}_{(\triangle)}$ algebra **A**, each safe **A**-model **M** of T.

Remark 2 Note that Yang [8] proved that wB-sKD is equivalent to $\mathrm{sKD}_{\triangle}$. Thus, since wB-sKD∀ is obtained by adding to wB-sKD the same additional axioms and deduction rule for quantifiers as $\mathrm{sKD}\forall_{\triangle}$, i.e., (∀1), (∀2), (∀3), (∃1), (∃2), and GN, it can be ensured that $\mathrm{sKD}\forall_{\triangle}$ is equivalent to wB-sKD∀.

References

Dienes, Z. P.(1949), "On an Implication function in Many-valued Systems of Logic," *The Journal of symbolic Logic*, 14, pp.95-97.

Dunn, J. M. and Hardegree, G.(2001), *Algebraic Methods in Philosophical Logic*, Oxford, Oxford Univ Press.

Estiva, F., Godo, L., Hájek, P., and Navara, M.(2000), "Residuated fuzzy logics with an involutive logics," *Archive for Mathematical Logic*, 39, pp.103-124.

Hájek, P.(1998), *Metamathematics of Fuzzy Logic*, Amsterdam, Kluwer.

Kleene, S. C.(1952), *Introduction to metamathematics*, North - Holland, Amsterdam.

Rescher, N.(1969), *Many-valued Logic*, New York, McGraw-Hill.

Yang, E.(2004), "Routley-Meyer semantics for some weak Boolean logics, and some Translations," *Logic Journal of the IGPL*, 12, pp.355-369.

Yang, E.(2004), "Many-valued semantics for some weak Boolean Logics," 2004 logic summer conference in Korea.

Yang, E.(2005), "Strong Kleene-Diense Logic: a variant of the infinite-valued Kleene-Diense Logic," This journal, 8/2, pp.85-107.

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