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THE ORDERING OF CONDITIONALLY WEAK POSITIVE QUADRANT DEPENDENCE

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Abstract. In this paper, we introduced a new notion of conditionally weakly positive quadrant dependence($CWPQD$) between two random variables and the partial ordering of $CWPQD$ is developed to compare pairs of $CWPQD$ random vectors. Some properties and closure under certain statistical operations are derived

1. Introduction

Lehmann[11] introduced the concepts of positive(negative) dependence together with some other dependent concepts. Since then, a great many papers have been studied on the subject and its extensions and numerous multivariate inequalities have been obtained. In other words, a great many papers have been devoted to various generalizations of Lehmann's concepts to finite-dimensional distributions and this results have been extended in several directions, see Karlin and Rinott[9], Ebrahimi and Ghosh[7], Shaked[13], Sampson[12] and Baek[4]. Recently, Brady and Singpurwalla[6] introduced some new conditionally independent and positive(negative) quadrant dependence concepts of random variables. These concepts are qualitative form of dependence(i.e., it

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simply indicates whether the pair of random variables are mutually conditionally positive dependent or not) which has led to many applications in applied probability, reliability, and statistical inference such as analysis of variance, multivariate hypothesis test, sequential testing.

In this paper we introduced a new notion of $CWPQD$ between two random variables and the partial ordering of $CWPQD$ is developed to compare pairs of $CWPQD$ random vectors.

Like the above, since $CWPQD$ is a qualitative form of dependence, it would seen difficult or impossible to compare different pairs of random variables as to their "degree of $CWPQD$-ness." Therefore, the main purpose of this paper is to develop a partial ordering which permits us to compare pairs of the dependence structures of a new $CWPQD$ random vector of interest as to their degree of $CWPQD$-ness(the exact definition is given in Section 3).

In Section 2, we present some definitions and notations used throughout this paper. The definitions and basic properties of $CWPQD$ ordering are derived in Section 3. Also, we have considered a family of bivariate distributions with specified marginals, the numbers of the family depending on a certain parameter, say λ. As $\lambda \uparrow$, the corresponding distribution, say H_λ, becomes increasingly $CWPQD$. Certain closure properties of $CWPQD$ ordering are derived in Section 4. It is shown that the $CWPQD$ ordering is preserved under combination and limit in distribution.

2. Preliminaries

An important principle of probability theory is that the notions of dependence and independence are conditional, the conditioning being done on some observable or unobservable quantity, say $\Theta \in R$. Suppose that I_1, I_2 and I_3 partition of R such that $I_1 \cup I_2 \cup I_3 = R$. Brady and

Singpurwalla[6] introduced some concepts of conditional dependence between random variables. Let $\underline{X}$ and $\underline{Y}$ be two vector of random variables, of dimension p and q, respectively.

In this section we present definitions, notations, and properties used throughout this paper. We start by stating the definitions of conditionally independence and positive(negative) dependence due to Brady and Singpurwalla[6].

Definition 2.1. *The random vector $\underline{X} = (X_1, \cdots, X_p)$ is $\theta \in I_1$ conditionally independent of $\underline{Y} = (Y_1, \cdots, Y_q)$ and $\theta \in I_2 (\theta \in I_3)$ conditionally positive(negative) dependent on $\underline{Y}$, denoted by $\{(\underline{X} \coprod \underline{Y}) \mid \theta \in I_1, > \theta \in I_2, < \theta \in I_3\}$ if*

(a) $P(\underline{X} \in A \mid \underline{Y} \in B,\ \theta \in I_1) = P(\underline{X} \in A \mid \theta \in I_1\}$,

(b) $P(\underline{X} \in A \mid \underline{Y} \in B,\ \theta \in I_2) \geq P(\underline{X} \in A \mid \theta \in I_2\}$,

(c) $P(\underline{X} \in A \mid \underline{Y} \in B,\ \theta \in I_3) \leq P(\underline{X} \in A \mid \theta \in I_3\},\ \forall\ A, B, \theta$,

where A, B are open upper sets(U is an open upper set if $\underline{a} \in U$ and $\underline{a} < \underline{b}$ implies $\underline{b} \in U$(Shaked[13])).

Assume that $p = q = 1$. Then Definition 2.1 is equivalent to

Definition 2.2. *The pair (X, Y) or its distribution H is $\theta \in I_1$ conditionally independent and $\theta \in I_2 (\theta \in I_3)$ conditionally positive(negative) quadrant dependent($CPQD$ ($CNQD$)), denoted by $\{(\underline{X} \coprod \underline{Y}) \mid \theta \in I_1, > \theta \in I_2, < \theta \in I_3\}$ if*

(a) $P(X \leq x,\, Y \leq y \,|\, \theta \in I_1) = P(X \leq x \,|\, \theta \in I_1) P(Y \leq y \,|\, \theta \in I_1)$,

(b) $P(X \leq x,\, Y \leq y \,|\, \theta \in I_2) \geq P(X \leq x \,|\, \theta \in I_2) P(Y \leq y \,|\, \theta \in I_2)$,

(c) $P(X \leq x,\, Y \leq y \,|\, \theta \in I_3) \leq P(X \leq x \,|\, \theta \in I_3) P(Y \leq y \,|\, \theta \in I_3)$.

Lemma 2.1. *If conditions (a), (b) and (c) of Definition 2.2 hold and if the conditional expectations $E(XY \,|\, \theta)$, $E(X \,|\, \theta)$ and $E(Y \,|\, \theta)$ exist,* *then Definition 2.2 implies that*

$$(a)\ E(XY\,|\,\theta\in I_1)=E(X\,|\,\theta\in I_1)E(Y\,|\,\theta\in I_1),$$

$$(b)\ E(XY\,|\,\theta\in I_2)\geq E(X\,|\,\theta\in I_2)E(Y\,|\,\theta\in I_2),$$

$$(c)\ E(XY\,|\,\theta\in I_3)\leq E(X\,|\,\theta\in I_3)E(Y\,|\,\theta\in I_3).$$

We close this section by stating the definition of weakly positive quadrant dependence due to Alzaid[3].

Definition 2.3. *The pair (X,Y) or its distribution H is weakly positive quadrant dependent of the first type$(WPQD1)$(second type$(WPQD2)$ if for all $x,\ y$,*

$$\int_x^\infty\int_y^\infty P(X\leq u,Y\leq t)-P(X\leq u)P(Y\leq t)dtdu\geq 0$$

$$\left(\int_{-\infty}^x\int_{-\infty}^y P(X\leq u,Y\leq t)-P(X\leq u)P(Y\leq t)dtdu\geq 0\right).$$

A pair (X,Y) or its distribution H is weakly positive quadrant dependent $(WPQD)$ if it is $(WPQD1)$ and $(WPQD2)$.

Definition 2.4. *The pair (X,Y) or its distribution H is $\theta\in I_2$ conditionally weakly positive quadrant dependent of the first type$(CWPQD1)$ (second type$(CWPQD2)$) if for all $x,\ y$,*

$$\int_x^\infty\int_y^\infty\Big[P(X\leq u,Y\leq t\,|\,\theta\in I_2)$$
$$-P(X\leq u\,|\,\theta\in I_2)P(Y\leq t\,|\,\theta\in I_2)\Big]dtdu\geq 0$$

$$\Bigg(\int_{-\infty}^x\int_{-\infty}^y\Big[P(X\leq u,Y\leq t\,|\,\theta\in I_2)$$
$$-P(X\leq u\,|\,\theta\in I_2)P(Y\leq t\,|\,\theta\in I_2)\Big]dtdu\geq 0\Bigg).$$

A pair (X,Y) or its distribution H is $\theta \in I_2$ conditionally weakly positive quadrant dependent $(CWPQD)$ if it is $(CWPQD1)$ and $(CWPQD2)$.

3. *CWPQD* Ordering random variables

Let $\beta = \beta(F(x\,|\,\theta),\, G(y\,|\,\theta))$ be the class of bivariate distribution functions H on R^2 having F and G as marginals distribution functions given θ. We consider, β^+ a subclass of β, defined by

$$\begin{aligned}\beta^+ \;=\; & \Big\{H(x,y\,|\,\theta \in I_2) : H \text{ is } CWPQD,\; H(x,\infty\,|\,\theta \in I_2) = F(x\,|\,\theta \in I_2),\\ & H(\infty,y\,|\,\theta \in I_2) = G(y\,|\,\theta \in I_2)\Big\}.\end{aligned}$$

Definition 3.1. *Let H_1 and H_2 belong to β^+. The bivariate random vector (X_1,X_2) or its distribution H_1 is more $\theta \in I_2$ conditionally weakly positive quadrant dependent of the first type($(CWPQD1)$ (second type($CWPQD2$)) than (Y_1,Y_2) or H_2 if for all x, y*

$$\begin{aligned}&\int_x^\infty \int_y^\infty P(X_1 \le u, X_2 \le t\,|\,\theta \in I_2)dtdu\\ &\ge \int_x^\infty \int_y^\infty P(Y_1 \le u, Y_2 \le t\,|\,\theta \in I_2)dtdu \qquad (3.1)\end{aligned}$$

$$\begin{aligned}&\int_{-\infty}^x \int_{-\infty}^y P(X_1 \le u, X_2 \le t\,|\,\theta \in I_2)dtdu\\ &\ge \int_{-\infty}^x \int_{-\infty}^y P(Y_1 \le u, Y_2 \le t\,|\,\theta \in I_2)dtdu \qquad (3.2)\end{aligned}$$

where the integrals exist.

We write $(X_1,X_2) > (CWPQD1)(Y_1,Y_2)(H_1 > (CWPQD1)H_2)$ $[(X_1,X_2) > (CWPQD2)(Y_1,Y_2)(H_1 > (CWPQD2)H_2)]$. We say that the bivariate distribution H_1 is more $\theta \in I_2$ conditionally weakly positive quadrant dependent $(CWPQD)$ than H_2 if $H_1 > (CWPQD1)H_2$ and $H_1 > (CWPQD2)H_2$. We write $H_1 > (CWPQD)H_2$.

Property 3.1. *Let H_1, H_2 and H_3 belong to β^+. Assume that $H_1 >$ (CWPQD) H_2 and $H_2 > (CWPQD)H_3$, then $H_1 > (CWPQD)H_3$.*

We now turn our attention to a simple but important property of class β^+.

Property 3.2. *The class β^+ is convex.*

Proof. Let H_1, H_2 belong to β^+ and for $0 < \alpha < 1$,

$$H = \alpha H_1 + (1-\alpha)H_2 \tag{3.3}$$

i.e., a convex combination of H_1 and H_2. Since each of H_1 and $H_2 \in \beta^+$, (3.3) may be written as

$$\begin{aligned}
&\int_x^\infty \int_y^\infty H(u,t\,|\,\theta \in I_2)dtdu \\
&= \int_x^\infty \int_y^\infty \Big[\alpha H_1(u,t\,|\,\theta \in I_2) + (1-\alpha)H_2(u,t\,|\,\theta \in I_2)\Big] dtdu \\
&\geq \int_x^\infty \int_y^\infty \alpha F(u\,|\,\theta \in I_2)G(t\,|\,\theta \in I_2)dtdu \\
&+ \int_x^\infty \int_y^\infty (1-\alpha)F(u\,|\,\theta \in I_2)G(t\,|\,\theta \in I_2)dtdu \\
&= \int_x^\infty \int_y^\infty F(u\,|\,\theta \in I_2)G(t\,|\,\theta \in I_2)dtdu,
\end{aligned}$$

so that H is $CWPQD1$. The proof of the $CWPQD2$ is similar to the $CWPQD1$.

Moreover,

$$\begin{aligned}
\lim_{t\to\infty} H(u,t\,|\,\theta \in I_2) &= \alpha F(u\,|\,\theta \in I_2) + (1-\alpha)F(u\,|\,\theta \in I_2) \\
&= F(u\,|\,\theta \in I_2)
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\lim_{t\to\infty} H(u,t\,|\,\theta \in I_2) &= \alpha G(t\,|\,\theta \in I_2) + (1-\alpha)G(t\,|\,\theta \in I_2) \\
&= G(t\,|\,\theta \in I_2).
\end{aligned} \tag{3.5}$$

It follows from (3.3), (3.4) and (3.5) that $H \in \beta^+$. Thus β^+ is convex. □

Property 3.3. *Let* β_c^+ *be the class of all convex combinations* $\{H|(1-\alpha)H_0 + \alpha H^*,\ 0 \le \alpha \le 1\}$, *where* $H_0 = F(x)G(y)$, $H^* = F(x) \wedge G(y)$, $x \wedge y = min(x,y)$. *Then for* $0 \le \alpha \le 1$, $H \in \beta^+$ *and* $\beta_c^+ \subset \beta^+$.

Proof. Suppose $H = (1-\alpha)H_0 + \alpha H^*$. For $\alpha = 0,\ 1$, it is clear that $H \in \beta^+$. For $0 < \alpha < 1$,

$$
\begin{aligned}
& \int_x^\infty \int_y^\infty H(u,t\,|\,\theta \in I_2)dtdu \\
=\ & \int_x^\infty \int_y^\infty (1-\alpha)F(u|\theta \in I_2)G(t|\theta \in I_2)dtdu \\
& + \int_x^\infty \int_y^\infty \alpha\Big(F(u|\theta \in I_2) \wedge G(t|\theta \in I_2)\Big)dtdu \\
=\ & \begin{cases} \int_x^\infty \int_y^\infty F(u|\theta \in I_2)G(t|\theta \in I_2)(1+\alpha\frac{\bar{G}(t|\theta \in I_2)}{G(t|\theta \in I_2)})dtdu\ if\ t \ge u \\ \\ \int_x^\infty \int_y^\infty F(u|\theta \in I_2)G(t|\theta \in I_2)(1+\alpha\frac{\bar{F}(t|\theta \in I_2)}{F(t|\theta \in I_2)})dtdu\ if\ t < u \end{cases} \\
\ge\ & \int_x^\infty \int_y^\infty F(u|\theta \in I_2)G(t|\theta \in I_2)dtdu.
\end{aligned}
$$

Therefore $H \in \beta^+$. Thus $\beta_c^+ \subset \beta^+$.

□

Definition 3.2. *A family of distributions* $H = \{H_\lambda(x,y\,|\,\theta \in I_2) : \lambda \in \Lambda \subset R\}$ *is increasingly CWPQD in* λ *if*

$$\lambda' > \lambda \implies H'_\lambda > (CWPQD)H_\lambda.$$

Example 3.1. *A bivariate family of* $H_\lambda(x,y\,|\,\theta \in I_2)$, $0 < \lambda < 1$ *is increasingly CWPQD in* λ, *where* $H_\lambda(x,y\,|\,\theta \in I_2) = \lambda H(x,y\,|\,\theta \in I_2) + (1-\lambda)F(x\,|\,\theta \in I_2) \wedge G(y\,|\,\theta \in I_2)$ *and* $H \in \beta^+$. *It is clear that* $H_\lambda \subset \beta^+$ *by Property 3.3. For* $0 < \lambda_1 < \lambda_2 < 1$, *by Property 3.3, it may be easily seen that* $H_\lambda(x,y\,|\,\theta \in I_2)$ *is increasingly CWPQD in* λ.

4. Closure Properties of $(\beta^+, > (CWPQD))$

In this section we establish preservation of the $CWPQD$ ordering under combination and limit in distribution. Below, we show that $CWPQD$ ordering is preserved under combination. For proof of Theorem 4.1 we need the following lemma which is of independent interest given θ.

Lemma 4.1. *Let $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ have distributions H_1 and H_2 respectively, where H_1 and H_2 belong to β^+ such that $H_1 > (CWPQD) H_2$, and let $\underline{Z} = (Z_1, Z_2)$ with an arbitrary $CWPQD$ distribution function H independent of both $\underline{X}$ and $\underline{Y}$ given θ. Then*

$$\underline{X} + \underline{Z} > (CWPQD) \underline{Y} + \underline{Z}.$$

Proof. First we will show that for each $(a_1, a_2) \in R^2$,

$$\int_x^\infty \int_y^\infty P(X_1 + Z_1 \le a_1, X_2 + Z_2 \le a_2 \,|\, \theta \in I_2) da_2 da_1 \tag{4.1}$$

$$\ge \int_x^\infty \int_y^\infty P(X_1 + Z_1 \le a_1 \,|\, \theta \in I_2) P(X_2 + Z_2 \le a_2 \,|\, \theta \in I_2) da_2 da_1.$$

Note that the left side of (4.1)

$$= \int_x^\infty \int_y^\infty \int\int P(X_1 \le a_1 - z_1, X_2 \le a_2 - z_2 \,|\, \theta \in I_2) dH_{Z_1, Z_2 \,|\, \theta \in I_2}$$

$$(z_1, z_2 \,|\, \theta \in I_2) da_2 da_1$$

$$\ge \int_x^\infty \int_y^\infty \int\int \prod_{i=1}^2 P(X_i \le a_i - z_i \,|\, \theta \in I_2) dH_{Z_1, Z_2 \,|\, \theta \in I_2}$$

$$(z_1, z_2 \,|\, \theta \in I_2) da_2 da_1$$

$$\ge \int_x^\infty \int_y^\infty \int\int \prod_{i=1}^2 P(X_i \le a_i - z_i \,|\, \theta \in I_2) dH_{Z_i \,|\, \theta \in I_2}(z_i \,|\, \theta \in I_2) da_2 da_1$$

$$= \int_x^\infty \int_y^\infty \prod_{i=1}^2 P(X_i + Z_i \le a_i \,|\, \theta \in I_2) da_2 da_1.$$

So $\underline{X}+\underline{Z}$ is $CWPQD1$, similarly $\underline{Y}+\underline{Z}$ is $CWPQD1$.
Next we need to show that for each $(a_1, a_2) \in R^2$,

$$\int_x^\infty \int_y^\infty P(X_1+Z_1 \le a_1, X_2+Z_2 \le a_2 \,|\, \theta \in I_2) da_2 da_1$$
$$\ge \int_x^\infty \int_y^\infty P(Y_1+Z_1 \le a_1, Y_2+Z_2 \le a_2 \,|\, \theta \in I_2) da_2 da_1. \tag{4.2}$$

Note that the left side of (4.2)

$$= \int_x^\infty \int_y^\infty \int\int P(X_1 \le a_1 - z_1, X_2 \le a_2 - z_2 \,|\, \theta \in I_2) dH_{Z_1,Z_2\,|\,\theta\in I_2}$$
$$(z_1, z_2 \,|\, \theta \in I_2) da_2 da_1$$
$$\ge \int_x^\infty \int_y^\infty \int\int P(Y_1 \le a_1 - z_1, Y_2 \le a_2 - z_2 \,|\, \theta \in I_2) dH_{Z_1,Z_2\,|\,\theta\in I_2}$$
$$(z_1, z_2 \,|\, \theta \in I_2) da_2 da_1$$
$$= \int_x^\infty \int_y^\infty P(Y_1+Z_1 \le a_1, Y_2+Z_2 \le a_2 \,|\, \theta \in I_2) da_2 da_1.$$

The above inequality follows from the assumption that $\underline{X} > (CWPQD)\underline{Y}$ Thus $\underline{X}+\underline{Z} > (CWPQD1)\underline{Y}+\underline{Z}$. Similarly $\underline{X}+\underline{Z} > (CWPQD2)\underline{Y}+\underline{Z}$. □

Theorem 4.1. *Suppose (X_i, Y_i) and (U_i, V_i) are such that $(X_i, Y_i) > (CWPQD)$ (U_i, V_i) for $i = 1,\ 2$. Further, let (X_1, Y_1) be independent of (X_2, Y_2) given θ and (U_1, V_1) be independent of (U_2, V_2) given θ. Then*

$$(X_1+X_2, Y_1+Y_2) > (CWPQD)(U_1+U_2, V_1+V_2).$$

Proof. By assumption $(X_1, Y_1) > (CWPQD)(U_1, V_1)$. Specifying $\underline{Z}$ to be (X_2, Y_2), we apply Lemma 4.1 to obtain

$$(X_1+X_2, Y_1+Y_2) > (CWPQD)(U_1+X_2, V_1+Y_2). \tag{4.3}$$

Next, we use the assumption $(X_2, Y_2) > (CWPQD)(U_2, V_2)$. Specify $\underline{Z}$ to be (U_1, V_1) and again use Lemma 4.1 yielding

$$(U_1+X_2, V_1+Y_2) > (CWPQD)(U_1+U_2, V_1+V_2). \tag{4.4}$$

By combining (4.3) and (4.4), $(X_1+X_2, Y_1+Y_2) > (CWPQD)(U_1+U_2, V_1+V_2)$. □

Corollary 4.1. *Let* $\underline{X} = (X_1, X_2)$ *and* $\underline{Y} = (Y_1, Y_2)$ *have distributions* H_1 *and* H_2*, where* H_1 *and* H_2 *belong to* β^+ *such that* $(X_1, X_2) > (CWPQD)(Y_1, Y_2)$ *and let* $\underline{Z} = (Z_1, Z_2)$ *with an arbitrary CWPQD distribution function* H *independent of both* $\underline{X}$ *and* $\underline{Y}$ *given* θ*. Then* $(f(X_1) + Z_1, f(X_2) + Z_2) > (CWPQD)(f(Y_1) + Z_1, f(Y_2) + Z_2)$ *for increasing function* f*.*

The next theorem demonstrates the preservation of the $CWPQD$ ordering under limits.

Theorem 4.2. *Let (a)* $\underline{Xn} = (X_{1n}, X_{2n})$ *and* $\underline{Yn} = (Y_{1n}, Y_{2n})$ *have distributions* H_n *and* H'_n *such that* $H_n > (CWPQD)H'_n$ *for every* n*, (b)* (X_1, X_2) *and* (Y_1, Y_2) *have distribution* H_1 *and* H'_1*, and (c)* H_n, H'_n *converge weakly to* H_1*,* H'_1 *respectively. Then* $H_1 > (CWPQD)H'_1$*.*

Proof.

$$\begin{aligned}
&\int_x^\infty \int_y^\infty P(X_1 \le u, X_2 \le t \,|\, \theta \in I_2)dtdu \\
&= \int_x^\infty \int_y^\infty \lim_{n\to\infty} P(X_{1n} \le u, X_{2n} \le t \,|\, \theta \in I_2)dtdu \\
&\ge \int_x^\infty \int_y^\infty \lim_{n\to\infty} P(Y_{1n} \le u, Y_{2n} \le t \,|\, \theta \in I_2)dtdu \\
&= \int_x^\infty \int_y^\infty P(Y_1 \le u, Y_2 \le t \,|\, \theta \in I_2)dtdu.
\end{aligned}$$

Similarly $H_1 > (CWPQD2)H'_1$. Thus $H_1 > (CWPQD)H'_1$. □

Theorem 4.3. *Let (a)* $\underline{Xn} = (X_{1n}, X_{2n})$ *and* $\underline{Yn} = (Y_{1n}, Y_{2n})$ *be sequence of nonnegative two random vectors with distributions* H_n *and* H'_n *such that* $H_n > (CWPQD1)H'_n$ *for every* n*, (b)* (X_1, X_2) *and* (Y_1, Y_2) *have distribution* H_1 *and* H'_1 *such that* H_n*,* H'_n *converge weakly to* H_1*,* H'_1 *respectively. If* $E(XY \,|\, \theta \in I_2)$*,* $E(X \,|\, \theta \in I_2)$*,* $E(Y \,|\, \theta \in I_2)$

are finite and $Cov(X_{1n}, X_{2n} \,|\, \theta \in I_2) \longrightarrow Cov(X_1, X_2 \,|\, \theta \in I_2)$ *and* $Cov(Y_{1n}, Y_{2n} \,|\, \theta \in I_2) \longrightarrow Cov(Y_1, Y_2 \,|\, \theta \in I_2)$. *Then* $H_1 > (CWPQD1)H_1'$.

Proof. Observe that

$$\begin{aligned}
&Cov(X_{1n}, X_{2n} \,|\, \theta \in I_2) - \int_0^x \int_0^y \Big[P(X_{1n} \le s, X_{2n} \le t \,|\, \theta \in I_2) \\
&-P(X_{1n} \le s \,|\, \theta \in I_2)P(X_{2n} \le t \,|\, \theta \in I_2)\Big] dtds \\
&= \int_x^\infty \int_y^\infty \Big[P(X_{1n} \le s, X_{2n} \le t \,|\, \theta \in I_2) \\
&-P(X_{1n} \le s \,|\, \theta \in I_2)P(X_{2n} \le t \,|\, \theta \in I_2)\Big] dtds \\
&\ge \int_x^\infty \int_y^\infty \Big[P(Y_{1n} \le s, Y_{2n} \le t \,|\, \theta \in I_2) \\
&-P(Y_{1n} \le s \,|\, \theta \in I_2)P(Y_{2n} \le t \,|\, \theta \in I_2)\Big] dtds \\
&= Cov(Y_{1n}, Y_{2n} \,|\, \theta \in I_2) - \int_0^x \int_0^y \Big[P(Y_{1n} \le s, Y_{2n} \le t \,|\, \theta \in I_2) \\
&-P(Y_{1n} \le s \,|\, \theta \in I_2)P(Y_{2n} \le t \,|\, \theta \in I_2)\Big] dtds.
\end{aligned}$$

Taking the limit and using the dominated convergence theorem and using the assumption of the theorem concerning the convergence of $Cov(X_n, Y_n \,|\, \theta \in I_2)$, we get the required result. □

References

[1] Abdul-Hadi, N. A., Naftali, A. L., Ram n, V. L. and Frank, P. (1979). Partial Ordering of Positive Quadrant Dependence with Applications. *Technical Report 78-5, Florida State University.*

[2] Ahmed, A., Langberg, N., Leon, R. and Proschan, F. (1978). Two Concepts of Positive Dependence with Applications in Multivariate Analysis. *Technical Report AFOSR 78-6, Department of Statistics, Florida State University.*

[3] Alzaid, A. A. (1990). A weak quadrant dependence concept with applications. *Comm. Statist. Stochastic. Models*, **6** (2), 353-363.

[4] Baek, J. I. (1997). A weakly dependence structure of multivariate processes. *Stat. & Probab. Lett.* **34**, 355-363

[5] Barlow, R. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. *Holt, Rinehart and Winston, Inc., New York.*

[6] Brady, B. M. and Singpurwalla, N. D. (1990). Stochastically Monotone Dependence Topics in Statistical Dependence,(H. W Block, A. R. Sampson, T. H. Savits Ed.) *Inst. Math. Statist.,* **16**, 93-102.

[7] Ebrahimi, N. and Ghosh, M. (1981). Multivariate Negative Dependence. *Commun. Statist.,* **10**, 307-339.

[8] Holland, P. W. and Rosenbaum, P. R. (1986). Conditional Association and Unidimensionality in Monotone Latent Variable Models. *Ann. Statist.,* **14**, 1523-1543.

[9] Karlin, S. and Rinott, Y. (1980). Classes of orderings of measures and related correlation inequalities. *J. Multivariate. Anal.* **10**, 467-498.

[10] Kimeldorf, G. and Sampson, A. R. (1978). Monotone Dependence. *Ann. Statist.*, **6**, 895-903.

[11] Lehmann, E. (1966). Some Concepts of Dependence. *Ann. Math. Statist.,* **37**, 1137-1153.

[12] Sampson, A. R. (1983). Positive dependence properties of elliptically symmetric distributions. *J. of Multivariate Analysis.* **13**, 375-381.

[13] Shaked, M. (1982). A General Theory of Some Positive Dependence Notions. *J. of Multivariate Analysis.* **12**, 199-218.

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