Honam Mathematical J. 28 (2006), No. 2, pp. 241–259

INTERVAL-VALUED FUZZY SEMIOPEN, PREOPEN AND α-OPEN MAPPINGS

Young Bae Jun, Gi Chul Kang and Mehmet Ali Öztürk

Abstract. Using the concept of interval-valued fuzzy (IVF) sets, the notions of IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF α-open (α-closed) sets are introduced, and interrelations are investigated. Also, the concepts of IVF open mappings, IVF preopen mappings, IVF semiopen mappings and IVF α-open mappings are introduced, and interrelations are discussed.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [11], several researchers were concerned about the generalizations of the notion of fuzzy sets, e.g., fyzzy set of type n [12], intuitionistic fuzzy sets [1] and interval-valued fuzzy sets [4]. The concept of interval-valued fuzzy sets was introduced by Gorzałczany [4], and recently there has been progress in the study of such sets by several researchers (see [3], [5], [6], [7], [8], [9], [10]). Azad [2] introduced fuzzy semiopen (semiclosed) sets and fuzzy regular open (closed) sets, and then considered generalizations of semicontinuous mapping, semiopen mapping, semiclosed mapping, almost continuous mapping, and weakly continuous mapping in fuzzy setting. In [5], the topology of interval-valued fuzzy sets is defined, and some

Received April 23, 2006. Revised June 22, 2006.

2000 Mathematics Subject Classification : 54A40.

Key words and phrases :IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets, IVF α-open (α-closed) set, IVF open mappings, IVF preopen mappings, IVF semiopen mappings, IVF α-open mappings.

of its properties are discussed, and then Mondal et al. [6] studied the connectedness in the topology of interval-valued fuzzy sets. In this paper, using the concept of interval-valued fuzzy (IVF) sets, we introduced the notion of IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF α-open (α-closed) sets, and then we investigate relationships between IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF α-open (α-closed) sets. We also introduced the notion of IVF open mappings, IVF preopen mappings, IVF semiopen mappings and IVF α-open mappings. We provide relationships between IVF open mappings, IVF preopen mappings, IVF semiopen mappings and IVF α-open mappings.

2. Preliminaries

Let $D[0,1]$ be the set of all closed subintervals of the unit interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for every $a \in (0,1)$. We also note that

(i) $(\forall M, N \in D[0,1])$ $(M = N \Leftrightarrow M^L = N^L,\ M^U = N^U)$.

(ii) $(\forall M, N \in D[0,1])$ $(M \leq N \Leftrightarrow M^L \leq N^L,\ M^U \leq N^U)$.

For every $M \in D[0,1]$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$.

Let X be a nonempty set. A function $A : X \to D[0,1]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^L$ and $A(x)^U$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. In particular, for any $a \in [0,1]$, the IVF set whose value is $\mathbf{a} = [a,a]$ for all $x \in X$ is denoted by simply $\tilde{a}$. For a point $p \in X$ and for $[a,b] \in D[0,1]$ with $b > 0$, the IVF set which takes the value $[a,b]$ at p and $\mathbf{0}$ elsewhere in X is called an *interval-valued fuzzy point* (briefly, an *IVF point*) and is denoted by $[a,b]_p$. In particular, if $b=a$, then it is also denoted by a_p. Denote by $IVFS(X)$ the set of all IVF sets in X. In IVF point M_x, where $M \in D[0,1]$, is said to *belong* to an IVF set A in X, denoted by $M_x \tilde{\in} A$, if $A(x)^L \geq M^L$ and $A(x)^U \geq M^U$. It can be easily shown that $A = \cup\{M_x \mid M_x \tilde{\in} A\}$ (see [5]).

Let $f : X \to Y$ be a mapping and let A be an IVF set in X. Then the *image* of A under f, denoted by $f(A)$, is defined as follows:

$$[f(A)(y)]^L = \begin{cases} \sup\limits_{y=f(x)} [A(x)]^L, & f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

$$[f(A)(y)]^U = \begin{cases} \sup\limits_{y=f(x)} [A(x)]^U, & f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for all $y \in Y$. Let B be an IVF set in Y. Then the it inverse image of B under f, denoted by $f^{-1}(B)$, is defined as follows:

$$(\forall x \in X)\,([f^{-1}(B)(x)]^L = [B(f(x))]^L,\ [f^{-1}(B)(x)]^U = [B(f(x))]^U).$$

Definition 2.1. [5] A family τ of IVF sets in X is called an *interval-valued fuzzy topology* (briefly, *IVF topology*) on X if it satisfies:

(i) $\tilde{0}, \tilde{1} \in \tau$,

(ii) $A, B \in \tau \Rightarrow A \cap B \in \tau$,

(iii) $A_i \in \tau, i \in \Delta \Rightarrow \bigcup\limits_{i\in\Delta} A_i \in \tau$.

Every member of τ is called an *IVF open set.* An IVF set A in X is called an *IVF closed set* if the complement of A is an IVF open set, that is, $A^c \in \tau$. Moreover, (X,τ) is called an *interval-valued fuzzy topological space* (briefly, *IVF topological space*).

Definition 2.2. [5] Let (X,τ) and (Y,κ) be IVF topological spaces. A function $f : X \to Y$ is said to be *continuous* if $f^{-1}(A) \in \tau$ for all $A \in \kappa$.

For an IVF set A in an IVF topological space (X, τ), the *closure* $\mathrm{cl}(A)$ and the *interior* $\mathrm{int}(A)$ of A are defined, respectively, as

$$\mathrm{cl}(A) = \cap\{B \in IVFS(X) \mid B \text{ is IVF closed and } A \subseteq B\},$$

$$\mathrm{int}(A) = \cup\{B \in IVFS(X) \mid B \text{ is IVF open and } B \subseteq A\}.$$

Then we have the following results (see [5]):

- $\mathrm{int}(A)$ is the largest IVF open set which is contained in A,
- A is IVF open if and only if $A = \mathrm{int}(A)$,
- $\mathrm{cl}(A) = (\mathrm{int}(A^c))^c$.

3. Interval-valued fuzzy semiopen sets

In what follows let (X, τ) denote an IVF topological space unless otherwise specified.

Definition 3.1. An IVF set A in (X, τ) is called an *IVF semiopen set* in (X, τ) if it satisfies:

$$(\exists B \in \tau)\,(B \subseteq A \subseteq \mathrm{cl}(B));$$

and an *IVF semiclosed set* in (X, τ) if it satisfies:

$$(\exists B^c \in \tau)\,(\mathrm{int}(B) \subseteq A \subseteq B).$$

Denote by $IVFSOS(X)$ (resp. $IVFSCS(X)$) the set of all IVF semiopen sets (resp. IVF semiclosed sets) in (X, τ).

Theorem 3.2. *Let A be an IVF set in (X, τ). Then*

(i) $A \in IVFSOS(X) \Leftrightarrow A \subseteq \mathrm{cl}(\mathrm{int}(A))$,
(ii) $A \in IVFSCS(X) \Leftrightarrow \mathrm{int}(\mathrm{cl}(A)) \subseteq A$.

Proof. (i) Let A be an IVF set in (X, τ) such that $A \subseteq \mathrm{cl}(\mathrm{int}(A))$. Note that $B := \mathrm{int}(A)$ is IVF open and $B \subseteq A \subseteq \mathrm{cl}(B)$. Hence A is IVF semiopen.

Conversely, let A be an IVF semiopen set in (X,τ). Then there exists $B\in\tau$ such that $B\subseteq A\subseteq \mathrm{cl}(B)$. But $B\subseteq \mathrm{int}(A)$ and thus $\mathrm{cl}(B)\subseteq \mathrm{cl}(\mathrm{int}(A))$. Hence $A\subseteq \mathrm{cl}(B)\subseteq \mathrm{cl}(\mathrm{int}(A))$.

(ii) Sufficiency is clear. Assume that $A\in IVFSCS(X)$. Then there exists an IVF set B in X such that $B^c\in\tau$ and $\mathrm{int}(B)\subseteq A\subseteq B$. From $A\subseteq B$ and $B^c\in\tau$, we have $\mathrm{cl}(A)\subseteq \mathrm{cl}(B)=B$, and so $\mathrm{int}(\mathrm{cl}(A))\subseteq \mathrm{int}(B)\subseteq A$. □

Theorem 3.3. (1) *The closure of an IVF open set is an IVF semiopen set.*
(2) *The interior of an IVF closed set is an IVF semiclosed set.*

Proof. (1) Let A be an IVF open set in (X,τ). Then $A=\mathrm{int}(A)$, and so

$$\mathrm{cl}(A)=\mathrm{cl}(\mathrm{int}(A))\subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))).$$

Hence $\mathrm{cl}(A)$ is an IVF semiopen set in (X,τ).

(2) Follows immediately from (1) by taking complements. □

Lemma 3.4. *For a family $\{A_k\}_{k\in\triangle}$ of IVF sets in (X,τ), we have*

(i) $\bigcup_{k\in\triangle}\mathrm{cl}(A_k)\subseteq \mathrm{cl}\Big(\bigcup_{k\in\triangle}A_k\Big)$.

(ii) $\bigcup_{k\in\triangle}\mathrm{int}(A_k)\subseteq \mathrm{int}\Big(\bigcup_{k\in\triangle}A_k\Big)$.

(iii) *If $\triangle$ is finte, then* $\bigcup_{k\in\triangle}\mathrm{cl}(A_k)=\mathrm{cl}\Big(\bigcup_{k\in\triangle}A_k\Big)$.

Proof. It is simple. □

Theorem 3.5. *Let $\{A_k\mid k\in\triangle\}$ be a collection of IVF semiopen sets in (X,τ). Then $\bigcup_{k\in\triangle}A_k$ is an IVF semiopen set in (X,τ), where $\triangle$ is any index set.*

Proof. For each $k\in\triangle$, we have $B_k\in\tau$ such that $B_k\subseteq A_k\subseteq \mathrm{cl}(B_k)$. Then

$$\bigcup_{k\in\triangle}B_k\subseteq\bigcup_{k\in\triangle}A_k\subseteq\bigcup_{k\in\triangle}\mathrm{cl}(B_k)\subseteq \mathrm{cl}\Big(\bigcup_{k\in\triangle}B_k\Big).$$

Putting $B=\bigcup_{k\in\triangle}B_k$, the desired result is obtained. □

Similarly we have the following theorem.

Theorem 3.6. *Any intersection of IVF semiclosed sets is an IVF semiclosed set.*

Theorem 3.7. *Let A and B be IVF sets in (X, τ) such that $A \subseteq B \subseteq \mathrm{cl}(A)$. If A is an IVF semiopen set in (X, τ), then so is B.*

Proof. If A is an IVF semiopen set in (X, τ), then there exists $G \in \tau$ such that $G \subseteq A \subseteq \mathrm{cl}(G)$. Then $G \subseteq B$. But $\mathrm{cl}(A) \subseteq \mathrm{cl}(G)$ and thus $B \subseteq \mathrm{cl}(G)$. Hence $G \subseteq B \subseteq \mathrm{cl}(G)$ and B is an IVF semiopen set in (X, τ). □

Remark 3.8. If A is an IVF open set in (X, τ), then A is an IVF semiopen set in (X, τ). The converse is false as seen in the following example.

Example 3.9. Let A and D be IVF sets in $I = [0, 1]$ given by

$$A(x) = \begin{cases} [\frac{3}{4}x, \frac{3}{4}x + \frac{1}{8}], & 0 \leq x \leq \frac{1}{2}, \\ [\frac{7}{8} - x, 1 - x], & \frac{1}{2} \leq x \leq \frac{7}{8}, \\ [0, 1 - x], & \frac{7}{8} \leq x \leq 1, \end{cases}$$

$$D(x) = \begin{cases} [x, x + \frac{1}{8}], & 0 \leq x \leq \frac{7}{8}, \\ [x, 1], & \frac{7}{8} \leq x \leq 1 \end{cases}$$

for all $x \in I$. Then $\tau = \{\tilde{0}, A, \tilde{1}\}$ is an IVF topology in $I = [0, 1]$. It is easy to show that D is an IVF semiopen set in (I, τ). But we know that D is not an IVF open set in (I, τ).

Combining Theorem 3.7 and Remark 3.8, we know that in an IVF topological space (X, τ), the following are valid.

- $\tau \subseteq IVFSOS(X)$.
- $(\forall A \in IVFSOS(X))\ (\forall B \in IVFS(X))\ (A \subseteq B \subseteq \mathrm{cl}(A) \Rightarrow B \in IVFSOS(X))$.

Theorem 3.10. *Let $\mathscr{B} = \{B_k\}$ be a collection of IVF sets in (X, τ) such that*

(i) $\tau \subseteq \mathscr{B}$,

(ii) $(\forall B \in \mathscr{B})\ (\forall D \in IVFS(X))\ (B \subseteq D \subseteq \mathrm{cl}(B) \Rightarrow D \in \mathscr{B})$.

Then $IVFSOS(X) \subseteq \mathscr{B}$, thus $IVFSOS(X)$ is the smallest class of IVF sets in X satisfying (i) *and* (ii).

Proof. Let $A \in IVFSOS(X)$. Then $B \subseteq A \subseteq \mathrm{cl}(B)$ for some $B \in \tau$. Then $B \in \mathscr{B}$ by (i) and so $A \in \mathscr{B}$ by (ii). This completes the proof. □

Theorem 3.11. *Let A be an IVF sets in (X, τ). Then A is IVF semiopen if and only if for every IVF point $M_x \tilde{\in} A$, there exists an IVF semiopen set B_{M_x} such that $M_x \tilde{\in} B_{M_x} \subseteq A$.*

Proof. If A is an IVF semiopen set in (X, τ), then we may take $B_{M_x} = A$ for every $M_x \tilde{\in} A$. Conversely, we have

$$A = \bigcup_{M_x \tilde{\in} A} \{M_x\} \subseteq \bigcup_{M_x \tilde{\in} A} B_{M_x} \subseteq A$$

and hence $A = \bigcup_{M_x \tilde{\in} A} B_{M_x}$ which is an IVF semiopen set in (X, τ) by Theorem 3.5. □

4. Interval-valued fuzzy preopen sets

Definition 4.1. An IVF set A in (X, τ) is called an *IVF preopen set* in (X, τ) if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$; and is called an *IVF preclosed set* in (X, τ) if $\mathrm{cl}(\mathrm{int}(A)) \subseteq A$.

Denote by $IVFPOS(X)$ (resp. $IVFPCS(X)$) the set of all IVF preopen sets (resp. IVF preclosed sets) in (X, τ).

Remark 4.2. It is obvious that every IVF open (resp. closed) set is an IVF preopen (resp. preclosed) set, but the converse may not be true as seen in the following examples. Example 4.4 also shows that an IVF preopen set need not be an IVF semiopen set, and vice versa.

Example 4.3. Let $X = \{a, b\}$ and consider an IVF topology $\tau = \{\tilde{0}, A, \tilde{1}\}$ on X, where A is an IVF set in X defined by $A(a) = [\frac{3}{4}, \frac{7}{8}]$ and $A(b) = \mathbf{1}$. Then an IVF set B in X given by $B(a) = \mathbf{1}$ and $B(b) = \mathbf{0}$ is IVF preopen which is not IVF open.

Example 4.4. Let B, C and D be IVF sets in $I = [0, 1]$ defined by

$$B(x) = \begin{cases} [\frac{7}{8} - 2x, 1 - 2x], & 0 \le x \le \frac{7}{16}, \\ [0, 1 - 2x], & \frac{7}{16} \le x < \frac{1}{2}, \\ [\frac{3}{8}, \frac{1}{2}], & \frac{1}{2} \le x \le 1, \end{cases}$$

$$C(x) = \begin{cases} \mathbf{0}, & 0 \le x < \frac{1}{2}, \\ [\frac{3}{8}, \frac{1}{2}], & \frac{1}{2} \le x \le 1, \end{cases}$$

$$D(x) = \begin{cases} [0, x], & 0 \le x < \frac{1}{8}, \\ [x - \frac{1}{8}, x], & \frac{1}{8} \le x \le 1, \end{cases}$$

for all $x \in I$. Then $\kappa = \{\tilde{0}, B, C, \tilde{1}\}$ is an IVF topology for I. It is easy to show that D is an IVF preopen set in (I, κ). But it is clearly not IVF open. Since $D \not\subseteq B^c = \text{cl}(\text{int}(D))$, D is not IVF semiopen in (I, κ).

Example 4.5. Let τ and D be as in Example 3.9. Note that D is an IVF semiopen set in (I, τ) but not an IVF preopen set in (I, τ). Also, $B \cap D$ is not an IVF preopen set in (I, κ) where κ is the IVF topology described in Example 4.4 and B is the IVF set given in Example 4.4.

In Theorem 3.3, we showed that the closure of an IVF open set is an IVF semiopen set. The following example shows that the closure of an IVF open set may not be an IVF preopen set.

Example 4.6. Let A and B be IVF sets in $I = [0, 1]$ defined by

$$A(x) = \begin{cases} [\frac{1}{16}, \frac{1}{8}], & 0 \le x < \frac{1}{2}, \\ [\frac{3}{16}, \frac{1}{4}], & \frac{1}{2} \le x \le 1, \end{cases}$$

and $B = \widetilde{[\frac{5}{16}, \frac{3}{8}]}$. Then $\tau = \{\tilde{0}, \tilde{1}, A, B\}$ is an IVF topology for I. Now we get $\mathrm{cl}(A) = B^c$, and so $\mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(B^c) = B$. It follows that

$$\mathrm{cl}(A) = B^c \nsubseteq B = \mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(\mathrm{cl}(\mathrm{cl}(A)))$$

so that $\mathrm{cl}(A)$ is not IVF preopen.

Theorem 4.7. *Let A be an IVF set in an IVF topological space (X, τ). Then*

(i) $A \in IVFPOS(X) \Leftrightarrow (\exists B \in \tau)\,(A \subseteq B \subseteq \mathrm{cl}(A))$,

(ii) $A \in IVFPCS(X) \Leftrightarrow (\exists B^c \in \tau)\,(\mathrm{int}(A) \subseteq B \subseteq A)$.

Proof. (i) If A is IVF preopen, then $A \subseteq \mathrm{int}(\mathrm{cl}(A))$. Putting $B = \mathrm{int}(\mathrm{cl}(A))$, we know that B is IVF open and $A \subseteq B \subseteq \mathrm{cl}(A)$.

Conversely, let B be an IVF open set in (X, τ) such that $A \subseteq B \subseteq \mathrm{cl}(A)$. Then $A \subseteq \mathrm{int}(B) \subseteq \mathrm{int}(\mathrm{cl}(A))$, and so A is an IVF preopen set in (X, τ).

(ii) Similar to the proof of (i). □

Theorem 4.8. *An arbitrary union* (resp. *intersection*) *of IVF preopen* (resp. *preclosed*) *sets is an IVF preopen* (resp. *preclosed*) *set.*

Proof. Let $\{A_k\}_{k \in \Delta}$ be a collection of IVF preopen sets of an IVF topological space (X, τ). Then $A_k \subseteq \mathrm{int}(\mathrm{cl}(A_k))$ for all $k \in \Delta$. It follows that

$$\bigcup_{k \in \Delta} A_k \subseteq \bigcup_{k \in \Delta} \mathrm{int}(\mathrm{cl}(A_k)) \subseteq \mathrm{int}\Big(\mathrm{cl}\Big(\bigcup_{k \in \Delta} A_k\Big)\Big)$$

so that $\bigcup_{k \in \Delta} A_k$ is an IVF preopen set in (X, τ). Taking complements, we get the desired result for the case of IVF preclosed. □

Theorem 4.9. *Let A be an IVF sets in (X, τ). Then A is IVF preopen if and only if for every IVF point $M_x \tilde{\in} A$, there exists an IVF preopen set B such that $M_x \tilde{\in} B \subseteq A$.*

Proof. Similar to the proof of Theorem 3.11. □

5. Interval-valued fuzzy α-open sets

Definition 5.1. An IVF set A in (X, τ) is called an *IVF α-open set* in (X, τ) if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$; and is called an *IVF α-closed set* in (X, τ) if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

Denote by $IVF\alpha OS(X)$ the set of all IVF α-open sets in (X, τ).

Example 5.2. Let A, B, C and D be IVF sets in $I = [0, 1]$ defined by

$$A(x) = \begin{cases} [x, x + \frac{1}{8}], & 0 \leq x \leq \frac{1}{2}, \\ [1 - x, \frac{9}{8} - x], & \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$B(x) = \begin{cases} \mathbf{1}, & 0 \leq x \leq \frac{1}{2}, \\ \mathbf{\frac{5}{8}}, & \frac{1}{2} < x \leq 1, \end{cases}$$

$$D(x) = \begin{cases} [x, x + \frac{1}{8}], & 0 \leq x < \frac{7}{8}, \\ \mathbf{1}, & \frac{7}{8} \leq x \leq 1. \end{cases}$$

The collection $\tau = \{\tilde{0}, A, \tilde{1}\}$ is an IVF topology for I. The IVF sets B and D are IVF α-open sets in (I, τ), and $B \cap D$ is also an IVF α-open sets in (I, τ).

If A is an IVF open set in (X, τ), then $\text{int}(A) = A$. Since $A \subseteq \text{cl}(A)$, it follows that

$$A = \text{int}(A) \subseteq \text{int}(\text{cl}(A)) = \text{int}(\text{cl}(\text{int}(A)))$$

so that A is an IVF α-open set in (X, τ). Now if A is an IVF α-open set in (X, τ), then

$$\text{int}(A) \subseteq A \subseteq \text{int}(\text{cl}(\text{int}(A))) \subseteq \text{cl}(\text{int}(A)).$$

Since $\text{int}(A)$ is an IVF open set in (X, τ), it follows that A is an IVF semiopen set in (X, τ). Assume that A is an IVF α-open set in (X, τ). Then

$$A \subseteq \text{int}(\text{cl}(\text{int}(A))) \subseteq \text{int}(\text{cl}(A)),$$

and thus A is an IVF preopen set in (X, τ).

Example 5.3. Let A, B, and C be IVF sets in $I = [0, 1]$ defined by

$$A(x) = \begin{cases} [0, \frac{1}{8}], & 0 \leq x \leq \frac{1}{2}, \\ [2x-1, 2x-\frac{7}{8}], & \frac{1}{2} \leq x \leq \frac{15}{16}, \\ [2x-1, 1], & \frac{15}{16} < x \leq 1, \end{cases}$$

$$B(x) = \begin{cases} \mathbf{1}, & 0 \leq x \leq \frac{1}{4}, \\ [-4x+2, 1], & \frac{1}{4} \leq x \leq \frac{9}{32}, \\ [-4x+2, -4x+\frac{17}{8}], & \frac{9}{32} \leq x \leq \frac{1}{2}, \\ \mathbf{0}, & \frac{1}{2} < x \leq 1, \end{cases}$$

$$C(x) = \begin{cases} [0, \frac{1}{8}], & 0 \leq x \leq \frac{1}{4}, \\ [\frac{1}{3}(4x-1), \frac{1}{3}(4x-1)+\frac{1}{8}], & \frac{1}{4} < x \leq \frac{29}{32}, \\ [\frac{1}{3}(4x-1), 1], & \frac{29}{32} \leq x \leq 1. \end{cases}$$

Then $\tau_1 = \{\tilde{0}, A, \tilde{1}\}$, $\tau_2 = \{\tilde{0}, A, B, A \cup B, A \cap B, \tilde{1}\}$ and $\tau_3 = \{\tilde{0}, C, \tilde{1}\}$ are IVF topologies for $I = [0, 1]$. We know that $A \cup B$ and $A \cap B$ are IVF sets in X given by

$$(A \cup B)(x) = \begin{cases} \mathbf{1}, & 0 \leq x \leq \frac{1}{4}, \\ [-4x+2, 1], & \frac{1}{4} \leq x \leq \frac{9}{32}, \\ [-4x+2, -4x+\frac{17}{8}], & \frac{9}{32} \leq x \leq \frac{1}{2}, \\ [2x-1, 2x-\frac{7}{8}], & \frac{1}{2} \leq x \leq \frac{15}{16}, \\ [2x-1, 1], & \frac{15}{16} < x \leq 1, \end{cases}$$

$$(A \cap B)(x) = \begin{cases} [0, \frac{1}{8}], & 0 \leq x \leq \frac{1}{2}, \\ \mathbf{0}, & \frac{1}{2} < x \leq 1, \end{cases}$$

and that

(i) The IVF set C is IVF α-open which is not IVF open in (I, τ_1).

(ii) The IVF set C is IVF semiopen which is not IVF α-open in (I, τ_2).

(iii) The IVF set A is IVF preopen which is not IVF α-open in (I, τ_3).

(iv) The IVF set A is IVF preopen which is not IVF semiopen in (I, τ_3).

(v) The IVF set C is IVF semiopen which is not IVF preopen in (I, τ_2).

(vi) The IVF set C^c is IVF α-closed which is not IVF closed in (I, τ_1).

(vii) The IVF set C^c is IVF semiclosed which is not IVF α-closed in (I, τ_2).

(viii) The IVF set A^c is IVF preclosed which is not IVF α-closed in (I, τ_3).

(ix) The IVF set A^c is IVF preclosed which is not IVF semiclosed in (I, τ_3).

(x) The IVF set C^c is IVF semiclosed which is not IVF preclosed in (I, τ_2).

(xi) Consider the IVF topology $\kappa = \{\tilde{0}, B, \tilde{1}\}$ for I. The IVF set $A \cup B$ is an IVF α-open set in (I, κ) which is not IVF open.

We have the following diagram described relations between IVF open set (IVF closed set), IVF α-open set (IVF α-closed set), IVF semiopen set (IVF semiclosed set), and IVF preopen set (IVF preclosed set) in which reverse implications do not valid.

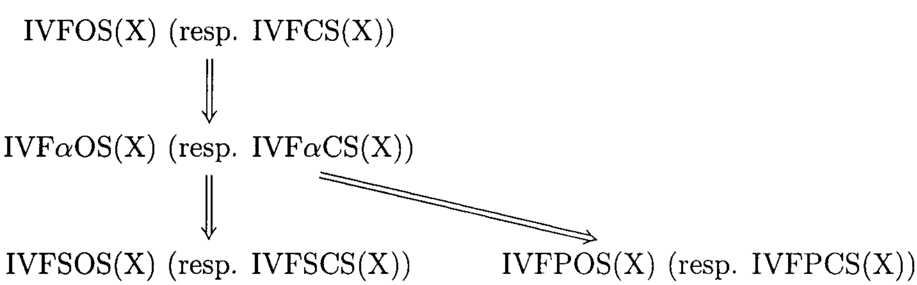

Diagram 5.1

where $IVFCS(X)$ (resp. $IVF\alpha CS(X)$, $IVFSCS(X)$, $IVFPCS(X)$) is the set of all IVF closed sets (resp. IVF α-closed sets, IVF semiclosed sets, IVF preclosed sets) in (X, τ).

Theorem 5.4. *Let A and B be any IVF sets in X and consider an IVF topology $\tau = \{\tilde{0}, B, \tilde{1}\}$ for X. If $B \nsubseteq B^c$, then $A \cup B$ is an IVF α-open set in (X, τ).*

Proof. Note that $B \subseteq A \cup B$. If $A \cup B = \tilde{1}$, then obviously $A \cup B$ is an IVF α-open set in (X, τ). Assume that $A \cup B \neq \tilde{1}$. Then $\text{int}(A \cup B) = B$, and so $\text{cl}(\text{int}(A \cup B)) = \text{cl}(B) = \tilde{1}$ since $B \nsubseteq B^c$. It follows that

$$A \cup B \subseteq \tilde{1} = \text{int}(\text{cl}(\text{int}(A \cup B)))$$

so that $A \cup B$ is an IVF α-open set in (X, τ). □

Theorem 5.5. (i) *An arbitrary union of IVF α-open sets is an IVF α-open set.*
(ii) *An arbitrary intersection of IVF α-closed sets is an IVF α-closed set.*

Proof. (i) Let $\{A_k \mid k \in \Delta\}$ be a collection of IVF α-open sets of an IVF topological space (X, τ). Then $A_k \subseteq \text{int}(\text{cl}(\text{int}(A_k)))$ for all $k \in \Delta$. Thus

$$\bigcup_{k\in\Delta} A_k \subseteq \bigcup_{k\in\Delta} \text{int}(\text{cl}(\text{int}(A_k))) \subseteq \text{int}\Big(\text{cl}\Big(\text{int}\Big(\bigcup_{k\in\Delta} A_k\Big)\Big)\Big),$$

and so $\bigcup_{k\in\Delta} A_k$ is an IVF α-open set in (X, τ).

(ii) Follows immediately from (i) by taking complements. □

The following example shows that the intersection of two IVF α-open sets may not be an IVF α-open set.

Example 5.6. Let $X = \{a, b, c\}$. Consider IVF sets A, B, and D in X defined by

$A(a) = [0.3, 0.4]$, $A(b) = [0.2, 0.3]$, $A(c) = [0.7, 0.8]$,
$B(a) = [0.8, 0.9]$, $B(b) = [0.9, 1]$, $B(c) = [0.4, 0.5]$,
$D(a) = [0.8, 0.9]$, $D(b) = [0.9, 1]$, $D(c) = [0.5, 0.6]$.

Then $\tau = \{\tilde{0}, \tilde{1}, A, B, A \cap B, A \cup B\}$ is an IVF topology on X. It is easy to check that A and D are IVF α-open sets in X. But $A \cap D$ is not an IVF α-open set in X.

We provide condition(s) for an IVF set to be an IVF α-open set.

Theorem 5.7. *If A is an IVF set in (X, τ) and B is an IVF semiopen set in (X, τ) such that $B \subseteq A \subseteq \text{int}(\text{cl}(B))$, then A is an IVF α-open set in (X, τ).*

Proof. Since B is IVF semiopen, we have $B \subseteq \text{cl}(\text{int}(B))$. It follows from hypothesis that

$$A \subseteq \text{int}(\text{cl}(B)) \subseteq \text{int}(\text{cl}(\text{cl}(\text{int}(B)))) = \text{int}(\text{cl}(\text{int}(B))) \subseteq \text{int}(\text{cl}(\text{int}(A)))$$

so that A is an IVF α-open set in (X, τ). □

We provide characterizations of an IVF α-open set.

Theorem 5.8. *Let A be an IVF sets in (X, τ). Then A is IVF α-open if and only if it is both IVF semiopen and IVF preopen.*

Proof. Necessity is by the Diagram 5.1. Suppose that A is both an IVF semiopen set and an IVF preopen set in (X, τ). Then $A \subseteq \text{cl}(\text{int}(A))$, and so

$$\text{cl}(A) \subseteq \text{cl}(\text{cl}(\text{int}(A))) = \text{cl}(\text{int}(A)).$$

It follows that $A \subseteq \text{int}(\text{cl}(A)) \subseteq \text{int}(\text{cl}(\text{int}(A)))$, so that A is an IVF α-open set in (X, τ). □

Theorem 5.9. *Let A be an IVF sets in (X, τ). Then A is IVF α-open if and only if for every IVF point $M_x \tilde{\in} A$, there exists an IVF α-open set B such that $M_x \tilde{\in} B \subseteq A$.*

Proof. Similar to the proof of Theorem 3.11.

□

6. Interval-valued fuzzy preopen, semiopen, and α-open mappings

Definition 6.1. Let (X, τ) and (Y, κ) be IVF topological spaces and let $f : X \to Y$ be a mapping. Then f is called

(i) an *IVF open mapping* if it satisfies:

$$(\forall A \in IVFS(X))\,(A \in \tau \Rightarrow f(A) \in \kappa),$$

(ii) an *IVF preopen mapping* if it satisfies:

$$(\forall A \in IVFS(X))\,(A \in \tau \Rightarrow f(A) \in IVFPOS(Y)),$$

(iii) an *IVF semiopen mapping* if it satisfies:

$$(\forall A \in IVFS(X))\,(A \in \tau \Rightarrow f(A) \in IVFSOS(Y)),$$

(iv) an *IVF α-open mapping* if it satisfies:

$$(\forall A \in IVFS(X))\,(A \in \tau \Rightarrow f(A) \in IVF\alpha OS(Y)).$$

Example 6.2. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. Define IVF sets A_1 and A_2 on X, and B_1, B_2, B_3, B_4 and B_5 on Y as follows:

$$\begin{aligned}
&A_1(a) = [0.1, 0.3],\ A_1(b) = [0, 0.3],\ A_1(c) = [0.2, 0.4],\\
&A_2(a) = [0.2, 0.3],\ A_2(b) = [0.1, 0.4],\ A_2(c) = [0.1, 0.3],\\
&B_1(1) = [0.2, 0.4],\ B_1(2) = [0, 0.3],\ B_1(3) = \mathbf{0},\\
&B_2(1) = [0.2, 0.3],\ B_2(2) = [0.1, 0.4],\ B_2(3) = \mathbf{0},\\
&B_3(1) = [0.1, 0.3],\ B_3(2) = [0, 0.3],\ B_3(3) = \mathbf{0},\\
&B_4(1) = [0.2, 0.4],\ B_4(2) = [0.1, 0.4],\ B_4(3) = \mathbf{0},\\
&B_5(1) = \mathbf{1},\ B_5(2) = \mathbf{1},\ B_5(3) = \mathbf{0}.
\end{aligned}$$

Then $\tau = \{\tilde{0}, \tilde{1}, A_1, A_2, A_1 \cap A_2, A_1 \cup A_2\}$ and $\kappa = \{\tilde{0}, \tilde{1}, B_1, B_2, B_1 \cap B_2, B_3, B_4, B_5\}$ are IVF topologies on X and Y, respectively. The mapping $f : X \to Y$ defined by $f(a) = f(c) = 1$ and $f(b) = 2$ is an IVF open mapping.

Using Diagram 5.1, we have the following relations.

IVF open mapping

$\Downarrow$

IVF α-open mapping

$\Downarrow$ $\searrow$

IVF semiopen mapping IVF preopen mapping

Diagram 6.1

The reverse implications are not true in the above diagram in general, as seen in the following example.

Example 6.3. (1) Let A_1 and A_2 be IVF sets in $X = \{x, y, z\}$ given by $A_1(x) = [0.1, 0.3]$, $A_1(y) = [0.1, 0.3]$, $A_1(z) = \mathbf{0}$, $A_2(x) = [0.1, 0.3]$, $A_2(y) = [0.1, 0.3]$, $A_2(z) = [0.7, 0.9]$, and let B_1, B_2, B_3, B_4 be IVF sets in $Y = \{a, b, c, d\}$ defined by $B_1(a) = [0.1, 0.3]$, $B_1(b) = [0.1, 0.3]$, $B_1(c) = \mathbf{0}$, $B_1(d) = \mathbf{0}$, $B_2(a) = [0.7, 0.9]$, $B_2(b) = [0.1, 0.3]$, $B_2(c) = \mathbf{0}$, $B_2(d) = \mathbf{0}$, $B_3(a) = [0.1, 0.3]$, $B_3(b) = \mathbf{1}$, $B_3(c) = \mathbf{0}$, $B_3(d) = \mathbf{0}$, $B_4(a) = \mathbf{1}$, $B_4(b) = [0.1, 0.3]$, $B_4(c) = \mathbf{0}$, $B_4(d) = \mathbf{0}$. Consider an IVF topology $\tau = \{\tilde{0}, \tilde{1}, A_1, A_2\}$ on X and IVF topologies $\kappa_1 = \{\tilde{0}, \tilde{1}, B_1, B_2, B_4\}$, $\kappa_2 = \{\tilde{0}, \tilde{1}, B_1, B_3\}$, and $\kappa_3 = \{\tilde{0}, \tilde{1}, B_1, B_1^c\}$ on Y. A mapping $f : (X, \tau) \to (Y, \kappa_1)$ defined by $f(x) = b$ and $f(y) = f(z) = a$ is an IVF α-open mapping, which is not an IVF open mapping since $f(\tilde{1}) \notin \kappa_1$. A mapping $f : (X, \tau) \to (Y, \kappa_2)$ defined by $f(x) = b$ and $f(y) = f(z) = a$ is an IVF semiopen mapping, which is not an IVF preopen mapping since $f(A_2) \nsubseteq \text{int}(\text{cl}(f(A_2)))$, and hence not an IVF α-open mapping. A mapping $f : (X, \tau) \to (Y, \kappa_3)$ defined by $f(x) = b$ and $f(y) = f(z) = a$ is an IVF preopen mapping, which is not an IVF semiopen mapping since $f(\tilde{1}) \nsubseteq \text{cl}(\text{int}(f(\tilde{1})))$, and so not an IVF α-open mapping.

(2) Let A, B, C, D and E be IVF sets in $I = [0,1]$ defined by $A = \widetilde{\frac{3}{4}}$, $B = \widetilde{[\frac{1}{2}, \frac{5}{8}]}$, $C = \widetilde{\frac{3}{8}}$,

$$D(x) = \begin{cases} [\frac{1}{2}, \frac{5}{8}], & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} & \frac{1}{2} < x \leq 1, \end{cases}$$

and $E = \widetilde{\frac{1}{4}}$. Then a mapping $f : (I, \tau_1 = \{\tilde{0}, \tilde{1}, A\}) \to (I, \tau_2 = \{\tilde{0}, \tilde{1}, B\})$, $x \mapsto x$, is an IVF α-open mapping which is not an IVF open mapping. A mapping $f : (I, \tau_3 = \{\tilde{0}, \tilde{1}, C\}) \to (I, \tau_4 = \{\tilde{0}, \tilde{1}, B^c\})$, $x \mapsto x$, is an IVF preopen mapping which is not an IVF semiopen mapping, and hence not an IVF α-open mapping. A mapping $f : (I, \tau_5 = \{\tilde{0}, \tilde{1}, D\}) \to (I, \tau_6 = \{\tilde{0}, \tilde{1}, D, E\})$, $x \mapsto \min\{2x, 1\}$, is an IVF semiopen mapping which is not an IVF preopen mapping since $f(D) \nsubseteq \text{int}(\text{cl}(f(D)))$, and thus not an IVF α-open mapping.

Theorem 6.4. *Let $f : (X, \tau) \to (Y, \kappa)$ and $g : (Y, \kappa) \to (Z, \delta)$ be mappings of IVF topological spaces. If f is IVF open and g is IVF α-open* (resp. *IVF preopen*), *then $g \circ f$ is IVF α-open* (resp. *IVF preopen*).

Proof. Straightforward. □

Theorem 6.5. *A mapping $f : (X, \tau) \to (Y, \kappa)$ of IVF topological spaces is IVF α-open if and only if it is IVF preopen and IVF semiopen.*

Proof. Necessity follows from the diagram 5.1. Assume that f is IVF preopen and IVF semiopen and let A be an IVF open set in (X, τ). Then $f(A)$ is an IVF preopen set as well as IVF semiopen set in (Y, κ). It follows from Theorem 5.8 that $f(A)$ is an IVF α-open set in (Y, κ) so that f is an IVF α-open mapping. □

References

[1] K. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy Sets and Systems **20** (1986), 87–96.

[2] K. K. Azad, *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity,* J. Math. Anal App. **82** (1981), 14–32.

[3] R. Biawas, *Rosenfeld's fuzzy subgroups with interval-valued membership functions,* Fuzzy Sets and Systems **63** (1994), 87–90.

[4] M. B. Gorzałczany, *A method of inference in approximate reasoning based on interval-valued fuzzy sets,* Fuzzy Sets and Systems **21** (1987), 1–17.

[5] T. K. Mondal and S. K. Samanta, *Topology of interval-valued fuzzy sets,* Indian J. Pure Appl. Math. **30** (1999), no. 1, 23–38.

[6] T. K. Mondal and S. K. Samanta, *Connectedness in topology of interval-valued fuzzy sets,* Italian J. Pure Appl. Math. **18** (2005), 33–50.

[7] J. H. Park, J. S. Park and Y. C. Kwun, *On fuzzy inclusion in the interval-valued sense,* FSKG 2005, LANI 3613, Springer-Verlag (2005), 1–10.

[8] P. V. Ramakrishnan and V. Lakshmana Gomathi Nayagam, *Hausdorff interval valued fuzzy filters,* J. Korean Math. Soc. **39** (2002), no. 1, 137–148.

[9] M. K. Roy and R. Biswas, *I-v fuzzy relations and Sanchez's approach for medical diagnosis,* Fuzzy Sets and Systems **47** (1992), 35–38.

[10] W. Zeng and Y. Shi, *Note on interval-valued fuzzy set,* FSKG 2005, LANI 3613, Springer-Verlag (2005), 20–25.

[11] L. A. Zadeh, *Fuzzy sets,* Inform. Control **8** (1965), 338–353.

[12] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning I,* Inform. Sci. **8** (1975), 199-249.

Y. B. Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, South Korea
Email : ybjun@gnu.ac.kr

G. C. Kang
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, South Korea
Email : integral_kr@hanmail.net

M. A. Öztürk
Department of Mathematics
Faculty of Arts and Sciences
Cumhuriyet University
58140 Sivas, Turkey
Email : maozturk@cumhuriyet.edu.tr