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DIFFERENTIAL EQUATIONS ON WARPED PRODUCTS

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Abstract. In this paper, we consider the problem of achieving a prescribed scalar curvature on warped product manifolds according to fiber manifolds with constant scalar curvature.

1. Introduction

One of the well-known problems in differential geometry is that of whether a given smooth function on a compact Riemannian manifold is necessarily the scalar curvature of some metric. In order to study these kinds of problems, we need some analytic methods in differential geometry, because they have the forms of differential equations.

In recent work, some authors have considered the problem of scalar curvature functions on a warped product manifold and obtained partial results about the existence and nonexistence of a warped metric with some prescribed scalar curvature function (cf. [5], [6],[7],[8], [9]).

In this paper, using the upper solution and lower solution methods, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M = B \times_f F$ in terms of its warping function f and the scalar curvatures of B and F. Using upper solution and lower solution

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methods, we treat the existence of a warping function f such that the resulting metric admits the prescribed scalar curvature function.

In this paper, we extend the results of [8]. That is, we show that if $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that

$$\frac{4n}{n+1} Bt^\beta \geq R(t) \geq \frac{4n}{n+1}\frac{C}{t^\alpha} \quad for \quad t \geq t_0,$$

where $t_0 > a$, $0 < \alpha < 2$, $0 < \beta < \frac{4}{n+1}$, C and B are positive constants, then equation (2.4) has a positive solution on $[a,\infty)$ and the resulting Lorentzian warped product metric is a future geodesically complete metric of positive scalar curvature outside a compact set.

2. Main Results

Let (N,g) be a Riemannian manifold of dimension n and let $f : [a,\infty) \to R^+$ be smooth function, where a is a positive number. The Lorentzian warped product of N and $[a,\infty)$ with warping function f is defined to be the product manifold $([a,\infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g. \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N,g). Then Theorem 2.9 implies that the scalar curvature $R(t,x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{2.2}$$

for $t \in [a,\infty)$ and $x \in N$ (for details, cf. [3] or [4]). If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u'' - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian n-manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [4], we have the following proposition.

Proposition 2.1 *If the scalar curvature of the fiber manifold N is arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $[a, \infty)$ such that the resulting Lorentzian warped product metric on $[a, \infty) \times_f N$ produces positive constant scalar curvature.*

However, the results of [4] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature.

Remark 2.2. By Remark 2.58 in [1] and Corollary 5.6 in [10], if (a, b) is a finite interval and $n = 3$, then all nonspacelike geodesics are incomplete. But on $(-\infty, +\infty)$ there exists a warping function so that all non-spacelike geodesics are complete. For Theorem 5.5 in [10] implies that all timelike geodesics are future (resp.past) complete on $(-\infty, +\infty) \times_{v(t)} N$ if and only if $\int_{t_0}^{+\infty} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$) and Remark 2.58 in [1] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty} v^{\frac{1}{2}} dt = +\infty$(resp. $\int_{-\infty}^{t_0} v^{\frac{1}{2}} dt = +\infty$) (cf. Theorem 4.1 and Remark 4.2 in [2]).□

We assume that the fiber manifold N of $M = [a, \infty) \times_f N$ has a positive scalar curvature, where a is a positive number. If we let $u(t) = t^{\alpha}$, where $\alpha \in (0, 1)$ is a constant, then we have

$$R(t, x) > -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} \geq -\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}, \quad t > a.$$

By the similar proof like as Theorem 2.4 in [7], we have the following:

Theorem 2.3 *If $R(g)$ is positive, then there is no positive solution to equation (2.3) with*

$$R(t) \leq \frac{-4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad for \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

If N is a positive scalar curvature, then any smooth function on N is the scalar curvature of some Riemannian metric. So we can take a

Riemannian metric g on N with scalar curvature $R(g) = \frac{4n}{n+1}k^2$, where k is a positive constant. Then equation (2.3) becomes

$$\frac{4n}{n+1}u^{''}(t) + \frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} - R(t,x)u(t) = 0. \tag{2.4}$$

If $R(t,x)$ is the function of only t-variable, then we have the following theorem.

Theorem 2.4 *Suppose that $R(g) = \frac{4n}{n+1}k^2$ and $R(t,x) = R(t) \in C^\infty([a,\infty))$. Assume that for $t > t_0$, there exist an upper solution $u_+(t)$ and a lower solution $u_-(t)$ of equation (2.4) such that $0 < u_-(t) \leq u_+(t)$. Then there exists a solution $u(t)$ of equation (2.4) such that for $t > t_0$ $\;0 < u_-(t) \leq u(t) \leq u_+(t)$.*

Proof. We have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a,\infty)$ $\;\tilde{u}_-(t) \leq \tilde{u}_+(t)$. Since $R(t) \in C^\infty([a,\infty))$, there exists a positive constant d such that $|R(t)| \leq \frac{4n}{n+1}d^2$ for $t \in [a,t_0]$. We assume that $u_+(t) \geq 1$ for $t \in [a,t_0]$. Then we have

$$\begin{aligned}
&\frac{4n}{n+1}u_+^{''}(t) + \frac{4n}{n+1}k^2u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
&\leq \frac{4n}{n+1}u_+^{''}(t) + \frac{4n}{n+1}k^2u_+(t) + \frac{4n}{n+1}d^2u_+(t) \\
&= \frac{4n}{n+1}[u_+^{''}(t) + (k^2+d^2)u_+(t)].
\end{aligned}$$

And if we devide the given interval $[a,t_0]$ into small intervals $\{I_i\}_{i=1}^n$, then for each interval I_i we have an upper solution $u_+^i(t)$ by parallel transporting $c_1cos(\sqrt{k^2+d^2}t)$ such that $u_+^i(t) \geq 1$ for some constant c_1. That is to say, for each interval I_i, $\frac{4n}{n+1}u_+^i(t)^{''} + \frac{4n}{n+1}k^2u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+^i(t) \leq \frac{4n}{n+1}(u_+^i(t)^{''} + (k^2+d^2)u_+^i(t)) = 0$, which means that $u_+^i(t)$ is an upper solution for each interval I_i. Then put $\tilde{u}_+(t) = u_+^i(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution on $[a,b)$ such that $\tilde{u}_+(t) \geq 1$ for all $t \in [a,t_0]$.

Put $\tilde{u}_-(t) = e^{-\alpha t}$ for $t \in [a,t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a,t_0]$,

$\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} + R(t)u_-(t) \geq \frac{4n}{n+1}(u_-''(t) - d^2 u_-(t)) = \frac{4n}{n+1}e^{-\alpha t}(\alpha^2 - d^2) \geq 0$ for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a, \infty)$ $0 < \tilde{u}_-(t) \leq \tilde{u}_+(t)$. □

Theorem 2.5 *Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that*

$$\frac{4n}{n+1}Bt^\beta \geq R(t) \geq \frac{4n}{n+1}\frac{C}{t^\alpha} \quad for \quad t \geq t_0,$$

where $t_0 > a$, $0 < \alpha < 2$, $0 < \beta < \frac{4}{n+1}$, C and B are positive constants. Then equation (2.4) has a positive solution on $[a, \infty)$ and the resulting Lorentzian warped product metric is a future geodesically complete metric of positive scalar curvature outside a compact set.

Proof. We let $u_+ = t^m$ and $u_- = t^{-\delta}$, where m and δ are positive numbers. If we take m large enough so that $m\frac{4}{n+1} > 2$, then we have, $t \geq t_0$ for some large t_0,

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
&\leq \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C}{t^\alpha}u_+(t) \\
&= \frac{4n}{n+1}t^m\left[\frac{m(m-1)}{t^2} + \frac{k^2}{t^{m\frac{4}{n+1}}} - \frac{C}{t^\alpha}\right] \\
&\leq 0,
\end{aligned}$$

which is possible for large fixed m since $\alpha < 2$. And since the exponent $1 - \frac{4}{n+1}$ is less than 1 and $R(t) \leq \frac{4n}{n+1}Bt^\beta$, if we take $0 < \delta < 1$ so that $\delta\frac{4}{n+1} > \beta$, then

$$\begin{aligned}
&\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} - R(t)u_-(t) \\
&\geq \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}Bt^\beta u_-(t) \\
&= \frac{4n}{n+1}t^{-\delta}\left[\delta(\delta+1)t^{-2} + k^2 t^{\delta\frac{4}{n+1}} - Bt^\beta\right] \geq 0
\end{aligned}$$

for large t. Since $t > t_0 > a > 0$, we can take the lower solution $u_-(t) = t^{-\delta}$ so that $0 < u_-(t) < u_+(t)$. So by the upper and lower solution method, we obtain a positive solution $u(t) = f(t)^{\frac{n+1}{2}}$ such that $0 < u_-(t) \leq u(t) \leq u_+(t)$. Hence

$$\int_{t_0}^{+\infty} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} \left(\frac{u(t)^{\frac{2}{n+1}}}{1+u(t)^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt$$
$$\geq \int_{t_0}^{+\infty} \left(\frac{t^{-\delta\frac{2}{n+1}}}{1+t^{-\delta\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \geq \frac{1}{2}\int_{t_0}^{\infty} t^{-\frac{\delta}{n+1}} dt = \infty$$

and

$$\int_{t_0}^{+\infty} f(t)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} u(t)^{\frac{1}{n+1}} dt \geq \int_{t_0}^{+\infty} t^{-\frac{\delta}{n+1}} dt = +\infty$$

which, by Remark 2.2, implies that the resulting warped product metric is a future geodesically complete one. □

Theorem 2.6 *Assume that $R(t,x) = R(t) \in C^{\infty}([a,\infty))$ is a positive function such that*

$$\frac{4n}{n+1} B t^{\beta} \geq R(t) \geq \frac{C}{t^2} \quad for \quad t \geq t_0,$$

where $t_0 > a$, $0 < \beta < \frac{4}{n+1}$, B and C are positive constants. If $C > n(n-1)$, then equation (2.4) has a positive solution on $[a,\infty)$ and the resulting Lorentzian warped metric is a future nonspacelike geodesically complete metric of positive scalar curvature outside a compact set.

Proof. In case $C > n(n-1)$, we may take $u_+ = C_+ t^{\frac{n+1}{2}}$, where C_+ is a positive constant. Then

$$\frac{4n}{n+1} u_+''(t) + \frac{4n}{n+1} k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t) u_+(t)$$
$$\leq C_+ \frac{4n}{n+1} t^{\frac{n-3}{2}} \left[\frac{n^2-1}{4} + k^2 C_+^{-\frac{4}{n+1}} - \frac{n+1}{4n} C\right] \leq 0,$$

which is possible if we take C_+ to be large enough since $\frac{(n+1)(n-1)}{4} - \frac{n+1}{4n}C_+ < 0$. And we take $u_-(t)$ as in the proof of Theorem 2.5. In this case, we also obtain a positive solution as in Theorem 3.5. Hence

$$\int_{t_0}^{+\infty} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} \left(\frac{u(t)^{\frac{2}{n+1}}}{1+u(t)^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt$$
$$\geq \int_{t_0}^{+\infty} \left(\frac{t^{-\delta\frac{2}{n+1}}}{1+t^{-\delta\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \geq \frac{1}{2}\int_{t_0}^{\infty} t^{-\frac{\delta}{n+1}} dt = \infty$$

and

$$\int_{t_0}^{+\infty} f(t)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} u(t)^{\frac{1}{n+1}} dt \geq \int_{t_0}^{+\infty} t^{-\frac{\delta}{n+1}} dt = +\infty$$

which, by Remark 2.2, implies that the resulting warped product metric is a future nonspacelike geodesically complete one. □

Remark 2.7. By Theorem 3.5 and Corollary 3.6 in [5], the result in Theorem 2.5 is almost sharp as we can get as close to $\frac{n(n-1)}{t^2}$ as possible. For example, let $R(g) = \frac{4n}{n+1}k^2$ and $f(t) = tlnt$ for $t > a$. Then we have

$$R = \frac{1}{t^2}\left[\frac{4n}{n+1}\frac{k^2}{(lnt)^2} + \frac{2n}{lnt} + n(n-1)(1+\frac{1}{lnt})^2\right],$$

which converges to $\frac{n(n-1)}{t^2}$ as t goes to ∞. □

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