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ACZÉL-CHUNG FUNCTIONAL EQUATION IN ALMOST EVERYWHERE SENSE

JAEYOUNG CHUNG

Abstract. We consider an n-dimensional version of the functional equation of Aczél and Chung in almost everywhere sense.

1. Introduction

In this article we consider an n-dimensional version of the functional equation of Aczél and Chung[2]

$$\sum_{j=1}^{l} f_j(x+\beta_j y) = \sum_{k=1}^{m} g_k(x)h_k(y), \tag{1.1}$$

in *almost everywhere sense*, where f_j, g_k, $h_k : \mathbb{R}^n \to \mathbb{C}$ and $\beta_j \in \mathbb{R}^n$ for $j = 1,\ldots,l$, $k = 1,\ldots,m$. For $\beta_j = (\beta_{j,1},\ldots,\beta_{j,n})$, $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$ we denote by $\beta_j\, y = (\beta_{j,1}y_1,\ldots,\beta_{j,n}y_n)$ and $\beta_j^{-1} = (\beta_{j,1}^{-1},\ldots,\beta_{j,n}^{-1})$, $j = 1,\ldots,l$. Similarly as in [2] we assume that $\beta_{j,p} \neq 0$ and $\beta_{i,p} \neq \beta_{j,p}$ for all $p = 1,\ldots,n$, $i \neq j$, $i,j = 1,\ldots,l$.

As a result, following the approach as in [5] we find the locally integrable solutions of the equation (1.1).

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2. Convolutional approach to the equation of Aczél and Chung

In this section we consider the equation (1.1). Here we impose the natural condition that $\{g_1, \ldots, g_m\}$ and $\{h_1, \ldots, h_m\}$ are linearly independent without which the number of the functions in the righthand side of the equation (1.1) can be reduced to a smaller one. We denote by $L^1_{loc}(\mathbb{R}^n)$ the space of all locally integrable functions on $\mathbb{R}^n$.

Theorem 2.1. *Let* f_j, g_k, $h_k \in L^1_{loc}(\mathbb{R}^n)$, $j = 1, \ldots, l$, $k = 1, \ldots, m$, *satisfy the equation* (1.1) *a.e.* $(x, y) \in \mathbb{R}^{2n}$. *Then* $f_j(x) = \widetilde{f}_j(x)$, $g_k(x) = \widetilde{g}_k(x)$, $h_k(x) = \widetilde{h}_k(x)$, $j = 1, \ldots, l$, $k = 1, \ldots, m$, *a.e.* $x \in \mathbb{R}^n$, *where* $\widetilde{f}_j$, $\widetilde{g}_k$, $\widetilde{h}_k$, $j = 1, \ldots, l$, $k = 1, \ldots, m$, *is a smooth solution of the equation*

$$\sum_{j=1}^{l} \widetilde{f}_j(x + \beta_j y) = \sum_{k=1}^{m} \widetilde{g}_k(x)\widetilde{h}_k(y), \quad x, y \in \mathbb{R}^n.$$

Proof. Consider a nonnegative function $\psi \in C^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$ and $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^n : |x| \le 1\}$. We denote the function $\psi_t(x) := t^{-n}\psi(x/t)$, $t > 0$. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Note that for each $t > 0$, $(f * \psi_t)(x) := \int f(y)\psi_t(x - y) dy$ is a smooth function in $\mathbb{R}^n$ and $(f * \psi_t)(x) \to f(x)$, *a.e.* as $t \to 0^+$. Convolving the tensor product $\psi_t(x)\psi_s(y)$ in each side of (1.1) we have, for $j = 1, \ldots, l$,

$$\begin{aligned}
&[f_j(x + \beta_j y) * (\psi_t(x)\psi_s(y))](\xi\,, \eta) \\
&= \int\!\!\int f_j(x + \beta_j y)\, \psi_t(\xi - x)\psi_s(\eta - y) dx\, dy \\
&= \int\!\!\int f_j(x)\psi_t\,(\xi - x + \beta_j\, y)\, \psi_s(\eta - y)\, dy\, dx \\
&= \int\!\!\int f_j(x)\psi_t\,(\xi - x + y)\, \psi_{s,\beta_j}\,(\beta_j\, \eta - y)\ dy\, dx \\
&= \int\!\!\int f_j(x)(\psi_t * \psi_{s,\beta_j})(\xi + \beta_j\, \eta - x)\, dx \\
&= (f_j * \psi_t * \psi_{s,\beta_j})(\xi + \beta_j\, \eta)
\end{aligned}$$

where $\psi_{s,\beta_j}(x) = (\beta_{j,1}\cdots\beta_{j,n})^{-1}\psi_s(\beta_j^{-1}x)$. Similarly we have, for $k = 1,\ldots,m$,

$$[(g_k(x)h_k(y)) * (\psi_t(x)\psi_s(y))](\xi,\eta) = (g_k * \psi_t)(\xi)(h_k * \psi_s)(\eta).$$

Thus the equation (1.1) is converted to the following functional equation

$$F(x,y,t,s) = \sum_{k=1}^{m} G_k(x,\, t)H_k(y,\, s) \tag{2.1}$$

where

$$F(x,y,t,s) = \sum_{j=1}^{l}(f_j * \psi_t * \psi_{s,\beta_j})(x + \beta_j y), \tag{2.2}$$

$$G_k(x,t) = (g_k * \psi_t)(x),\ \ H_k(y,s) = (h_k * \psi_s)(y), \tag{2.3}$$

for $k = 1,\ldots,m$. We first prove that for all $k = 1,\ldots,m$, $\lim_{t\to 0^+} G_k(x,t)$ are smooth functions and equal to $g_k(x)$, a. e. $x \in \mathbb{R}^n$. We use mathematical induction on m. Note that $\lim_{t\to 0^+} F(x,y,t,s)$ is a smooth function of x for each $y \in \mathbb{R}^n$, $s > 0$ and $\{H_1,\ldots,H_m\}$ is linearly independent. For $m = 1$, we can choose $y_1 \in \mathbb{R}^n$, $s_1 > 0$ such that $H_1(y_1,s_1) := a_1 \neq 0$. Then we can write

$$\begin{aligned}\lim_{t\to 0^+} G_1(x,t) &= \lim_{t\to 0^+} a_1^{-1}F(x,y_1,t,s_1)\\ &= a_1^{-1}\sum_{j=1}^{l}(f_j * \psi_{s_1,\beta_j})(x + \beta_j\, y_1).\end{aligned}$$

Thus $\widetilde{g}_1(x) := \lim_{t\to 0^+} G_1(x,t)$ is a smooth function. Now choose $y_m \in \mathbb{R}^n$, $s_m > 0$ such that $H_m(y_m,s_m) := a_m \neq 0$. Then it follows from (2.1) that

$$G_m(x,t) = a_m^{-1}\left(F(x,y_m,t,s_m) - \sum_{k=1}^{m-1} b_k G_k(x,\, t)\right) \tag{2.4}$$

where $b_k = H_k(y_m,s_m)$, $k = 1,\ldots,m-1$. Putting (2.4) in (2.1) we have

$$F^*(x,y,t,s) = \sum_{k=1}^{m-1} G_k(x,\, t)H_k^*(y,\, s) \tag{2.5}$$

where

(2.6) $$F^*(x,y,t,s) = F(x,y,t,s) - a_m^{-1}F(x,y_m,t,s_m)H_m(y,s),$$

(2.7) $$H_k^*(y,s) = H_k(y,s) - a_m^{-1}b_kH_m(y,s),\ k = 1,\dots,m-1.$$

Since $\lim_{t\to 0^+} F(x,y,t,s)$ is a smooth function of x for each $y \in \mathbb{R}^n$, $s > 0$ and $\{H_1,\dots,H_m\}$ is linearly independent, it follows from (2.6) and (2.7) that

(2.8) $$\lim_{t\to 0^+} F^*(x,y,t,s)$$

is a smooth function of x for each $y \in \mathbb{R}^n$, $s > 0$ and

(2.9) $$\{H_1^*,\dots,H_{m-1}^*\}$$

is linearly independent. Assume that the conditions (2.8) and (2.9) imply that $G_k(x,t)$ converges locally uniformly to a smooth function $\widetilde{g}_k(x)$ for $1 \le k \le m-1$. Then by induction and the equation (2.4), $G_m(x,t)$ converges locally uniformly to a smooth function $\widetilde{g}_m(x)$. This proves the assertion. Changing the roles of G_k and H_k, we obtain that for all $k = 1,\dots,m$, $\widetilde{h}_k(y) := \lim_{s\to 0^+} H_k(y,s)$ are smooth functions and $h_k = \widetilde{h}_k$. a. e. $x \in \mathbb{R}^n$.

Now we consider the regularity of the initial value of the right hand side of the equation (2.2). In (1.1), letting $s \to 0^+$, replacing x by $x - \beta_i y$, multiplying $\psi_s(y)$ and integrating with respect to y we have, for $i = 1,\dots,l$,

(2.10)
$$(f_i*\psi_t)(x) = -\sum_{j\neq i}(f_j*\psi_t*\psi_{s,\beta_j-\beta_i})(x)+\sum_{k=1}^{m}\int G_k(x-\beta_i y,\ t)\widetilde{h}_k(y)\psi_s(y)\,dy.$$

It follows from (2.10) and the locally uniform convergence of G_k that each $(f_i * \psi_t)(x)$, $i = 1,\dots,l$, converges locally uniformly to a smooth function $\widetilde{f}_i(x)$ as $t \to 0^+$. Finally, letting $s \to 0^+$ and $t \to 0^+$ in (2.1) and (2.2) we see that $\widetilde{f}_j$, $\widetilde{g}_k$, $\widetilde{h}_k$, $j = 1,\dots,l$, $k = 1,\dots,m$ is a smooth solution of classical version of the equation (1.1). This completes the proof. □

Combined with the result of Aczél and Chung[2] we have the following as a consequence of the above result.

Corollary 2.2. *Every solution* $f_j(x)$, $g_k(x)$, $h_k(x) \in L^1_{loc}(\mathbb{R})$, $j = 1, \ldots, l$, $k = 1, \ldots, m$, *of the equation* (1.1) *for the dimension* $n = 1$ *has the form of exponential polynomials*

$$E_p(x) = \sum_{k=1}^{N} p_k(x) e^{\beta_k x}$$

for almost everywhere $x \in \mathbb{R}$.

Examples. The well known Cauchy functional equations, Pexider equations, Jensen equations and quadratic functional equations are typical examples of the form (1.1). In addition to the equations, as consequences of our result the solutions of the equations

$$f(x+y) + f(x-y) - 2f(x) - 2g(y) = 0, \ a.e.\ (x,y) \in \mathbb{R}^{2n} \tag{2.11}$$

$$f(x+y) + f(x-y) - 2f(x)f(y) = 0, \ a.e.\ (x,y) \in \mathbb{R}^{2n} \tag{2.12}$$

$$f(x+y) + f(x-y) - 2f(x)g(y) = 0, \ a.e.\ (x,y) \in \mathbb{R}^{2n} \tag{2.13}$$

$$f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) - g(x)g(y) = 0, \ a.e.\ (x,y) \in \mathbb{R}^{2n} \tag{2.14}$$

$$f(x-y) - f(x)f(y) - g(x)g(y) = 0, \ a.e.\ (x,y) \in \mathbb{R}^{2n} \tag{2.15}$$

are equal almost everywhere x to smooth solutions of the corresponding classical functional equations, which can be found using induction for the dimension n. Thus the solutions of the above equations are given, respectively, by

(2.16)

$$g(x) = \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k, \quad a.e.\ \ x \in \mathbb{R}^n$$

$$f(x) = \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k + \sum_{j=1}^{n} b_j\, x_j + d, \quad a.e.\ \ x \in \mathbb{R}^n$$

(2.17)

$$f(x) = \cos(a_1x_1 + \cdots + a_nx_n), \quad a.e.\ \ x \in \mathbb{R}^n$$

(2.18)

$$g(x) = \cos(a_1x_1 + \cdots + a_nx_n), \quad a.e.\ \ x \in \mathbb{R}^n$$

$$f(x) = c_1 \cos(a_1x_1 + \cdots + a_nx_n) + c_2 \sin(a_1x_1 + \cdots + a_nx_n), \quad a.e.\ x \in \mathbb{R}^n$$

(2.19)

$$g(x) = c \sin(a_1x_1 + \cdots + a_nx_n), \quad a.e.\ \ x \in \mathbb{R}^n$$

$$f(x) = c^2 \sin^2(a_1x_1 + \cdots + a_nx_n) + d, \quad a.e.\ \ x \in \mathbb{R}^n$$

or

$$g(x) = a_1x_1 + \cdots + a_nx_n, \quad a.e.\ \ x \in \mathbb{R}^n$$

$$f(x) = (a_1x_1 + \cdots + a_nx_n)^2 + d, \quad a.e.\ \ x \in \mathbb{R}^n$$

(2.20)

$$g(x) = \pm \sin(a_1x_1 + \cdots + a_nx_n), \quad a.e.\ \ x \in \mathbb{R}^n$$

$$f(x) = \cos(a_1x_1 + \cdots + a_nx_n), \quad a.e.\ \ x \in \mathbb{R}^n$$

where $x = (x_1, \ldots, x_n)$ and all the coefficients are complex numbers.

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Jaeyoung Chung
Kunsan National University
Kunsan 573-701
South Korea
Email : jychung@kunsan.ac.kr