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ANOTHER METHOD FOR A KUMMER-TYPE TRANSFORMATION FOR THE GENERALIZED HYPERGEOMETRIC FUNTION $_2F_2$ DUE TO PARIS

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Abstract. A Kummer-type transformation formula for the generalized hypergeometric function $_2F_2$ deduced by Exton, rederived in two simple and transparent ways by Miller and generalized by Paris, is again derived by another method.

In 1997, Exton [1, Eq. (12)] deduced four new reduction formulas for the Kampé de Fériet function and, as a special case of one of his four main results, he obtained the following interesting identity:

$$e^{-y}{}_2F_2\left[\begin{matrix} a, & 1+\frac{1}{2}a \\ b, & \frac{1}{2}a \end{matrix};y\right] = {}_2F_2\left[\begin{matrix} b-a-1, & 2+a-b \\ b, & 1+a-b \end{matrix};-y\right]. \tag{1}$$

This result is an analog of the so-called Kummer's first type transformation formula [5] for the confluent hypergeometric function:

$$e^{-y}{}_1F_1\left[\begin{matrix} a \\ b \end{matrix};y\right] = {}_1F_1\left[\begin{matrix} b-a \\ b \end{matrix};-y\right]. \tag{2}$$

Recently Miller [3] derived the result (1) in two simple and transparent ways.

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Very recently Paris [4] gave a general result of (1) in the form:

$$e^{-y}{}_2F_2\left[\begin{matrix} a, & c+1 \\ b, & c \end{matrix};y\right] = {}_2F_2\left[\begin{matrix} b-a-1, & \alpha+1 \\ b, & \alpha \end{matrix};-y\right]. \tag{3}$$

where α is given by $\alpha \equiv \frac{c(1+a-b)}{a-c}$.

Evidently (3) reduces to (1) by taking $c = \frac{1}{2}a$. The aim of this research note is to derive (3) by another method.

For this, in fact, first we shall prove the following (presumably new) general result:

$$\frac{e^{-y}\Gamma(\rho+\gamma-\alpha-\beta)}{\Gamma(\rho+\gamma-\alpha)\Gamma(\rho+\gamma-\beta)}{}_2F_2\left[\begin{matrix} \rho, & \rho+\gamma-\alpha-\beta \\ \rho+\gamma-\alpha, & \rho+\gamma-\beta \end{matrix};y\right]$$

$$= \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{\Gamma(\rho+\gamma+r)r!}{}_1F_1\left[\begin{matrix} \gamma+r \\ \rho+\gamma+r \end{matrix};-y\right]. \tag{4}$$

In order to prove (4), consider the integral

$$I = \int_0^1 x^{\gamma-1}(1-x)^{\rho-1}e^{-xy}{}_2F_1\left[\begin{matrix} \alpha,\beta \\ \gamma \end{matrix};x\right]dx. \tag{5}$$

Writing $e^{-xy} = e^{-y}\cdot e^{(1-x)y}$ in (5) and expressing $e^{(1-x)y}$ as a series.

Changing the order of integration and summation, evaluating the integral with the help of a known result [2, p.849, Eq.(4)]:

$$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1}{}_2F_1\left[\begin{matrix} \alpha,\beta \\ \gamma \end{matrix};x\right]dx = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}, \tag{6}$$

and summing up the series, we get (7)

$$I = e^{-y}\frac{\Gamma(\rho+\gamma-\alpha-\beta)\Gamma(\rho)\Gamma(\gamma)}{\Gamma(\rho+\gamma-\alpha)\Gamma(\rho+\gamma-\beta)}{}_2F_2\left[\begin{matrix} \rho, & \rho+\gamma-\alpha-\beta \\ \rho+\gamma-\alpha, & \rho+\gamma-\beta \end{matrix};y\right].$$

On the other hand, in (5), expressing ${}_2F_1$ as a series, changing the order of integration and summation, evaluating the integral with the help of a known result [2, p.318, 3.383(1)]:

$$\int_0^u x^{n-1}(u-x)^{m-1}e^{\beta x}dx = B(m,n)u^{m+n-1}{}_1F_1\left[\begin{matrix} n \\ m+n \end{matrix};\beta u\right], \tag{8}$$

we get

$$I = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r\Gamma(\gamma+r)\Gamma(\rho)}{(\gamma)_r\Gamma(\rho+\gamma+r)r!}{}_1F_1\left[\begin{matrix} \gamma+r \\ \rho+\gamma+r \end{matrix};-y\right]. \tag{9}$$

From (7) and (9), we get (4). This completes the proof of (4).

In (4), if we take $\alpha = -1, \beta = b-c-1, \gamma = b-a-1$ and $\rho = a$, we get, after a little simplification, Paris' result (3). Consequently we obtain Exton's result for $c = \frac{1}{2}a$.

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