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A PROPERTY ON G-INVARIANT MINIMAL HYPERSURFACES WITH CONSTANT SCALAR CURVATURES IN S^5

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Abstract. Let $G = O(2) \times O(2) \times O(2)$ and let M^4 be a closed G-invariant minimal hypersurface with constant scalar curvature in S^5. In this paper, we prove a property on M^4.

0. Introduction

Let M^n be a closed minimally immersed hypersurface in the unit sphere S^{n+1}, and h its second fundamental form. Denote by R and S its scalar curvature and the square norm of h, respectively. It is well known that $S = n(n-1) - R$ from the structure equations of both M^n and S^{n+1}. In particular, S is constant if and only if M has constant scalar curvature. In 1968, J. Simons [8] observed that if $S \leq n$ everywhere and S is constant, then $S \in \{0, n\}$. Clearly, M^n is an equatorial sphere if $S = 0$. And when $S = n$, M^n is indeed a product of spheres, due to the works of Chern, do Carmo, and Kobayashi [3] and Lawson [5].

We are concerned about the following conjecture posed by Chern [9].

Chern Conjecture *For n-dimensional closed minimal hypersurfaces in the unit sphere S^{n+1} with constant scalar curvature, the values S of*

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the squared norm of the second fundamental forms should be discrete.

C. K. Peng and C. L. Terng [7] proved

Theorem [Peng and Terng, 1983] *Let M^n be a closed minimally immersed hypersurface with constant scalar curvature in S^{n+1}. If $S > n$, then $S > n + 1/(12n)$.*

S. Chang [2] proved the following theorem by showing that $S = 3$ if $S \geq 3$ and M^3 has multiple principal curvatures at some point.

Theorem [Chang, 1993]*A closed minimally immersed hypersurface with constant scalar curvature in S^4 is either an equatorial 3-sphere, a product of spheres, or a Cartan's minimal hypersurface. In particular, $S = 0, 3$ or 6.*

H. Yang and Q. M. Cheng [10] proved

Theorem [Yang and Cheng, 1998] *Let M^n be a closed minimally immersed hypersurface with constant scalar curvature in S^{n+1}. If $S > n$, then $S \geq n + n/3$.*

Let $G \simeq O(k) \times O(k) \times O(q) \subset O(2k+q)$ and set $2k + q = n + 2$. Then W. Y. Hsiang [4] investigated G-*invariant, minimal hypersurfaces,* M^n in S^{n+1}, by studying their generating curves, M^n/G, in the orbit space S^{n+1}/G. He showed that there exit infinitely many closed minimal hypersurfaces in S^{n+1} for all $n \geq 2$, by proving the following theorem

Theorem [Hsiang, 1987] *For each dimension $n \geq 2$, there exist infinitely many, mutually noncongruent closed G-invariant minimal hypersurfaces in S^{n+1}, where $G \simeq O(k) \times O(k) \times O(q)$ and $k = 2$ or 3.*

We studied G-invariant minimal hypersurfaces , in stead of minimal ones, with constant scalar curvature in S^5. In this paper, we shall prove the following theorem:

Theorem. *Let $G \simeq O(2) \times O(2) \times O(2)$ and let M^4 be a closed G-invariant minimal hypersurface with constant scalar curvature in S^5. If $S > 4$, then M^4 does not have 3 distinct principal curvatures anywhere.*

1. Preliminaries

Let M^n be a manifold of dimension n immersed in a Riemannian manifold N^{n+1} of dimension $n+1$. Let $\overline{\nabla}$ and $\langle\, ,\, \rangle$ be the connection and metric tensor respectively of N^{n+1} and let $\bar{\mathcal{R}}$ be the curvature tensor with respect to the connection $\overline{\nabla}$ on N^{n+1}. Choose a local orthonormal frame field $e_1, \ldots, e_{n+1}$ in N^{n+1} such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n. Denote the dual coframe by $\{\omega_A\}$. Here we will always use $i, j, k, \ldots,$ for indices running over $\{1, 2, \ldots, n\}$ and $A, B, C, \ldots,$ over $\{1, 2, \ldots, n+1\}$.

As usual, the *second fundamental form* h and the *mean curvature* H of M^n in N^{n+1} are respectively defined by

$$h(v, w) = \langle \overline{\nabla}_v w,\, e_{n+1} \rangle \quad \text{and} \quad H = \sum_i h(e_i,\, e_i).$$

M^n is said to be minimal if H vanishes identically. And the *scalar curvature* $\bar{R}$ of N^{n+1} is defined by

$$\bar{R} = \sum_{A,B} \langle \bar{\mathcal{R}}(e_A,\, e_B) e_B,\, e_A \rangle.$$

Then the structure equations of N^{n+1} are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

where $K_{ABCD} = \langle \bar{\mathcal{R}}(e_A,\, e_B) e_D,\, e_C \rangle$. When N^{n+1} is the unit sphere S^{n+1}, we have

$$K_{ABCD} = \delta_{AC}\, \delta_{BD} - \delta_{AD}\, \delta_{BC}.$$

Next, we restrict all tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n. Then

$$\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0.$$

By Cartan's lemma, we can write

$$\omega_{(n+1)i} = -\sum_j h_{ij}\,\omega_j .$$

Here,

$$h_{ij} = -\omega_{(n+1)i}(e_j) = -\langle \overline{\nabla}_{e_j} e_{n+1},\, e_i\rangle = \langle \overline{\nabla}_{e_j} e_i,\, e_{n+1}\rangle = h(e_j, e_i) = h(e_i, e_j).$$

Second, from

$$d\,\omega_i = \sum_j \omega_{ij}\wedge\omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\,\omega_{ij} = \sum_l \omega_{il}\wedge\omega_{lj} - \frac{1}{2}\sum_{l,m} R_{ijlm}\,\omega_l\wedge\omega_m,$$

we find the curvature tensor of M^n is

$$R_{ijlm} = K_{ijlm} + h_{il}\,h_{jm} - h_{im}\,h_{jl}. \tag{1.1}$$

If M^n is a piece of minimally immersed hypersurface in the unit sphere S^{n+1} and R is the scalar curvature of M^n, then we have

$$R = n(n-1) - S, \tag{1.2}$$

where $S = \sum_{i,j} h_{ij}^2$ is the *square norm* of h.

Given a symmetric 2-tensor $T = \sum_{i,j} T_{ij}\,\omega_i\,\omega_j$ on M^n, we also define its covariant derivatives, denoted by $\nabla T, \nabla^2 T$ and $\nabla^3 T$, etc. with components $T_{ij,k}$, $T_{ij,kl}$ and $T_{ij,klp}$, respectively, as follows:

$$(1.3)\sum_k T_{ij,k}\,\omega_k = d\,T_{ij} + \sum_s T_{sj}\,\omega_{si} + \sum_s T_{is}\,\omega_{sj},$$
$$\sum_l T_{ij,kl}\,\omega_l = d\,T_{ij,k} + \sum_s T_{sj,k}\,\omega_{si} + \sum_s T_{is,k}\,\omega_{sj} + \sum_s T_{ij,s}\,\omega_{sk},$$
$$\sum_p T_{ij,klp}\,\omega_p = d\,T_{ij,kl} + \sum_s T_{sj,kl}\,\omega_{si} + \sum_s T_{is,kl}\,\omega_{sj} + \sum_s T_{ij,sl}\,\omega_{sk} + \sum_s T_{ij,ks}\,\omega_{sl}.$$

In general, the resulting tensors are no longer symmetric, and the rule to switch sub-index obeys the Ricci formula as follows:

$$(1.4)\, T_{ij,kl} - T_{ij,lk} = \sum_s T_{sj}\, R_{sikl} + \sum_s T_{is}\, R_{sjkl},$$

$$T_{ij,klp} - T_{ij,kpl} = \sum_s T_{sj,k}\, R_{silp} + \sum_s T_{is,k}\, R_{sjlp} + \sum_s T_{ij,s}\, R_{sklp},$$

$$T_{ij,klpm} - T_{ij,klmp} = \sum_s T_{sj,kl}\, R_{sipm} + \sum_s T_{is,kl}\, R_{sjpm} + \sum_s T_{ij,sl}\, R_{skpm} + \sum_s T_{ij,ks}\, R_{slpm}.$$

For the sake of simplicity, we always omit the comma (,) between indices in the special case $T = \sum_{i,j} h_{ij}\, \omega_i\, \omega_j$ with $N^{n+1} = S^{n+1}$.

Since $\sum_{C,D} K_{(n+1)iCD}\, \omega_C \wedge \omega_D = 0$ on M^n when $N^{n+1} = S^{n+1}$, we find

$$d\left(\sum_j h_{ij}\, \omega_j\right) = \sum_{j,l} h_{jl}\, \omega_l \wedge \omega_{ji}.$$

Therefore,

$$\sum_{j,l} h_{ijl}\, \omega_l \wedge \omega_j = \sum_j \left(dh_{ij} + \sum_l h_{lj}\, \omega_{li} + \sum_l h_{il}\, \omega_{lj}\right) \wedge \omega_j = 0;$$

i.e., h_{ijl} is symmetric in all indices.

Moreover, in the case that M^n is minimal, we have

$$(1.5)\quad \sum_l h_{ijll} = \sum_l h_{lijl}$$
$$= \sum_l \left\{ h_{lilj} + \sum_m (h_{mi} R_{mljl} + h_{lm} R_{mijl}) \right\} = (n-1) h_{ij}$$
$$+ \sum_{l,m} \{-h_{mi} h_{ml} h_{lj} + h_{lm}(\delta_{mj}\delta_{il} - \delta_{ml}\delta_{ij} + h_{mj} h_{il} - h_{ml} h_{ij})\}$$
$$= n h_{ij} - \sum_{l,m} h_{lm} h_{ml} h_{ij} = (n-S) h_{ij}.$$

It follows that

$$(1.6)\qquad \frac{1}{2}\, \Delta S = (n-S) S + \sum_{i,j,l} h_{ijl}^2.$$

2. G-invariant Hypersurface in S^5

For $G \simeq O(2)\times O(2)\times O(2)$, $\mathbb{R}^6$ splits into the orthogonal direct sum of irreducible invariant subspaces, namely

$$\mathbb{R}^6 \simeq \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 = \{(X, Y, Z)\}$$

where X, Y, Z are two generic 2-vectors. Here if we set $x = |X|$, $y = |Y|$ and $z = |Z|$, then the orbit space $\mathbb{R}^6/G$ can be parametrized by (x, y, z); $x, y, z \in \mathbb{R}_+$ and the orbital distance metric is given by $ds^2 = dx^2 + dy^2 + dz^2$. By restricting the above G-action to the unit sphere $S^5 \subset \mathbb{R}^6$, it is easy to see that

$$S^5/G \simeq \{(x, y, z) : x^2 + y^2 + z^2 = 1;\ x,\ y,\ z \geq 0\}$$

which is isometric to a spherical triangle of $S^2(1)$ with $\pi/2$ as its three angles. The orbit labeled by (x, y, z) is exactly $S^1(x) \times S^1(y) \times S^1(z)$.

To investigate those *G-invariant minimal hypersurfaces*, M^4, in S^5 we study their generating curves, $\gamma(s) = M^4/G$, in the orbit space S^5/G.

To prove our theorem, we need two Lemmas which was proved in [8].

Lemma 2.1. *Let M^4 be a G-invariant hypersurface in S^5. Then there is a local orthonormal frame field $e_1, \ldots, e_5$ in S^5 such that after restriction to M^4, the $e_1, \ldots, e_4$ are tangent to M^4 and $h_{ij} = 0$ if $i \neq j$.*

Lemma 2.2. *Let M^4 be a G-invariant hypersurface in S^5 and let $\{e_A\}$ be a local orthonormal frame field in S^5 as in Lemma 2.1. Then,*

(a) all $h_{ijl} = 0$ except when $\{i, j, l\}$ is a permutation of either $\{i, i, n\}$,

(b) all $h_{ijkl} = 0$ except when $\{i, j, k, l\}$ is a permutation of either $\{i, i, j, j\}$.

Under such frame field as Lemma 2.1, we have

$$e_k(h_{ii}) = h_{iik} - \sum_s h_{si}\omega_{si}(e_k) - \sum_s h_{is}\omega_{si}(e_k) = h_{iik}. \tag{2.1}$$

Hence, in the case M^4 is minimal, by differentiating $\sum_k h_{kk} = 0$ we have

$$0 = (e_j\, e_i - \nabla_{e_j} e_i)\left(\sum_k h_{kk}\right) = \sum_k h_{kkij}. \tag{2.2}$$

Moreover, we have

$$e_k\left(\sum_{i,j} h_{ij}^2\right) = 2\sum_{i,j} h_{ij} e_k(h_{ij}) = 2\sum_{i,j} h_{ij}h_{ijk}. \tag{2.3}$$

Hence, in the case S is constant, by differentiating $\sum_{i,j} h_{ij}^2 = S$ twice, we have

$$0 = (e_l\, e_k - \nabla_{e_l} e_k)\left(\frac{1}{2}\sum_{i,j} h_{ij}^2\right) = \sum_{i,j}(h_{ij}h_{ijkl} + h_{ijk}h_{ijl}). \tag{2.4}$$

3. G-invariant Minimal Hypersurface in S^5

Throughout the following two sections , we assume that $G \simeq O(2) \times O(2) \times O(2)$ and M^4 is a closed G-invariant minimal hypersurface with constant scalar curvature in S^5. Let $\{e_A\}$ be a local orthonormal frame field in S^5 as in Lemma 2.1. Then by differentiating $\sum_i h_{ii} = 0$ and $\sum_i h_{ii}^2 = S$ with respect to e_4 respectively, we have

$$h_{114} + h_{224} + h_{334} + h_{444} = 0, \tag{3.1}$$

$$h_{11}h_{114} + h_{22}h_{224} + h_{33}h_{334} + h_{44}h_{444} = 0. \tag{3.2}$$

By differentiating (3.1) and (3.2) with respect to e_4 respectively, we have

$$h_{1144} + h_{2244} + h_{3344} + h_{4444} = 0, \tag{3.3}$$

$$\sum_i h_{ii}h_{ii44} + \sum_i h_{ii4}^2 = 0, \tag{3.4}$$

since

$$e_4(h_{ii4}) = h_{ii44} - \sum_s \{h_{si4}\,\omega_{si}(e_4) + h_{is4}\,\omega_{si}(e_4) + h_{iis}\,\omega_{s4}(e_4)\} = h_{ii44}.$$

Since $e_4(h_{ii44}) = h_{ii444}$ in the same way, by differentiating (3.3) and (3.4) with respect to e_4 respectively, we also have

$$h_{11444} + h_{22444} + h_{33444} + h_{44444} = 0, \tag{3.5}$$

$$\sum_i h_{ii} h_{ii444} + 3 \sum_i h_{ii4} h_{ii44} = 0. \tag{3.6}$$

¿From (1.5), we also have

$$h_{ii11} + h_{ii22} + h_{ii33} + h_{ii44} = (4 - S) h_{ii}. \tag{3.7}$$

And, (1.6) and Lemma 2.2 imply

$$\frac{1}{2}\,\Delta S = (4 - S)S + \sum_{i,j,l} h_{ijl}^2 = (4 - S)S + 3 \sum_{i \neq 4} h_{ii4}^2 + h_{444}^2.$$

Since S is constant, we have

$$3 \sum_{i \neq 4} h_{ii4}^2 + h_{444}^2 = S(S - 4). \tag{3.8}$$

Now, by differentiating it once and twice with respect to e_4 respectively, we have

$$3 \sum_{i \neq 4} h_{ii4}\, h_{ii44} + h_{444}\, h_{4444} = 0, \tag{3.9}$$

$$3 \sum_{i \neq 4} h_{ii4}\, h_{ii444} + h_{444}\, h_{44444} + 3 \sum_{i \neq 4} h_{ii44}^2 + h_{4444}^2 = 0. \tag{3.10}$$

Here, if $i \neq 4$, we know

(3.11)

$$h_{ii4} = h_{i4i} = e_i(h_{i4}) + \sum_s h_{s4}\omega_{si}(e_i) + h_{is}\omega_{s4}(e_i) = (h_{44} - h_{ii})\,\omega_{4i}(e_i)$$

and

$$(3.12)\quad h_{iiii} = e_i(h_{iii}) + \sum_s \{h_{sii}\omega_{si}(e_i) + h_{isi}\omega_{si}(e_i) + h_{iis}\omega_{si}(e_i)$$
$$= 3h_{ii4}\,\omega_{4i}(e_i).$$

Moreover, if i, $j \neq 4$ and $i \neq j$, then

$$(3.13)\quad h_{iijj} = e_j(h_{iij}) + \sum_s \{h_{sij}\omega_{si}(e_j) + h_{isj}\omega_{si}(e_j) + h_{iis}\omega_{sj}(e_j)$$
$$= h_{ii4}\,\omega_{4j}(e_j).$$

And, if $i \neq 4$, then $h_{444i} = 0$ by Lemma 2.2. Hence, we have

$$(3.14)\qquad h_{444ii} = e_i(h_{444i}) + \sum_s \{h_{s44i}\omega_{s4}(e_i) + h_{4s4i}\omega_{s4}(e_i) + h_{44si}\omega_{s4}(e_i) + h_{444s}\omega_{si}(e_i)\}$$
$$= (h_{4444} - 3h_{44ii})\omega_{4i}(e_i).$$

And, since

$$e_4(h_{4i4i}) = e_4(h_{44ii}) = h_{44ii4} - \sum_s \{h_{s4ii}\omega_{s4}(e_4) + h_{4sii}\omega_{s4}(e_4) + h_{44si}\omega_{si}(e_4) + h_{44is}\omega_{si}(e_4)\}$$
$$= h_{44ii4}$$

and $e_4(h_{44i4}) = 0 = e_4(h_{444i})$, we have

$$\begin{aligned}
h_{ii444} &= e_4(h_{4ii4}) = e_4\{h_{4i4i} + (h_{ii} - h_{44})(1 + h_{ii}h_{44})\} \\
&= h_{44ii4} + (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\
&= h_{44i4i} + h_{i4i}R_{i4i4} + h_{4ii}R_{i4i4} + h_{444}R_{4ii4} \\
&\quad + (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\
&= e_i(h_{44i4}) + h_{i4i4}\,\omega_{i4}(e_i) + h_{4ii4}\,\omega_{i4}(e_i) + h_{4444}\,\omega_{4i}(e_i) + h_{44ii}\,\omega_{i4}(e_i) \\
&\quad + h_{i4i}R_{i4i4} + h_{4ii}R_{i4i4} + h_{444}R_{4ii4} \\
&\quad + (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444})
\end{aligned}$$

$$= h_{i4i4}\,\omega_{i4}(e_i) + h_{4ii4}\,\omega_{i4}(e_i) + h_{4444}\,\omega_{4i}(e_i) + h_{44ii}\,\omega_{i4}(e_i)$$
$$+ h_{i4i}R_{i4i4} + h_{4ii}R_{i4i4} + h_{444}R_{4ii4}$$
$$+ (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444})$$
$$= (h_{4444} - h_{44ii} - 2h_{ii44})\,\omega_{4i}(e_i) + 2h_{ii4}R_{i4i4} + h_{444}R_{4ii4}$$
$$+ (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}).$$

Here, from (1.1) if $i \neq 4$ then

$$R_{i4i4} = K_{i4i4} + h_{ii}h_{44} = 1 + h_{ii}h_{44} \quad \text{and} \quad R_{4ii4} = -1 - h_{44}h_{ii}.$$

Hence, if $i \neq 4$ then

$$h_{ii444} = (3 + 4h_{ii}h_{44} - h_{44}^2)h_{ii4} - (2 + 3h_{ii}h_{44} - h_{ii}^2)h_{444} + (h_{4444} - h_{44ii} - 2h_{ii44})\omega_{4i}(e_i). \tag{3.15}$$

Now, to prove our Theorem we need the following two lemmas which are from [8].

Lemma 3.1. *Suppose $h_{ii} = h_{44} = \lambda$ at some point p for $i = 1, 2$ or 3. Then,*

$$S = \frac{12\lambda^4 + 4\lambda^2}{5\lambda^2 - 1}. \tag{3.16}$$

Lemma 3.2. *If $S > 4$ and $i = 1, 2, 3$, then for all i, $h_{44} \neq h_{ii}$ anywhere.*

4. Proof of the Theorem

Throughout this section, $\{e_A\}$ is such a local frame field in S^5 as in Lemma 2.1.

Theorem 4.1. *If $S > 4$, then M^4 does not have 3 distinct principal curvatures anywhere.*

Proof. Suppose that M^4 has 3 distinct principal curvatures at some point, say, p. Let $h_{ii} = \lambda_i$. Then by Lemma 3.2, without loss of generality we may assume that $\lambda_1 = \lambda_2 = \lambda$ and $\lambda, \lambda_3, \lambda_4$ are distinct at p. From now on, we evaluate all calculations at p. Then (3.1) and (3.2) imply

$$(4.1) \qquad \begin{cases} h_{114} + h_{224} + h_{334} + h_{444} = 0, \\ \lambda\, h_{114} + \lambda\, h_{224} + \lambda_3\, h_{334} + \lambda_4\, h_{444} = 0. \end{cases}$$

Since $S > 4$, by using (3.8) and (4.1) we see that $h_{114} \neq 0$ or $h_{224} \neq 0$. Hence, without loss of generality we can put $h_{114} = b\, h_{224}$ for some b. Thus, (4.1) becomes

$$(4.2) \qquad \begin{cases} (1+b)\, h_{224} + h_{334} + h_{444} = 0, \\ (1+b)\lambda\, h_{224} + \lambda_3\, h_{334} + \lambda_4\, h_{444} = 0. \end{cases}$$

Hence, from (3.11) and (4.2) we have

$$(4.3) \qquad \begin{aligned} h_{114} &= (\lambda_4 - \lambda)\,\omega_{41}(e_1) = (\lambda_4 - \lambda_3)\, a\, b, \\ h_{224} &= (\lambda_4 - \lambda)\,\omega_{42}(e_2) = (\lambda_4 - \lambda_3)\, a, \\ h_{334} &= (\lambda_4 - \lambda_3)\,\omega_{43}(e_3) = (\lambda - \lambda_4)\, a\,(1+b), \end{aligned}$$

for some nonzero real number a, since $S > 4$.

Since h_{ijl} is symmetric in all indices, (1.4) implies

$$(4.4) \qquad h_{3311} - h_{1133} = (\lambda_3 - \lambda)(1 + \lambda_3\,\lambda) = h_{3322} - h_{2233}.$$

By the way, (3.13) and (4.3) imply

$$(4.5) \qquad \begin{cases} h_{3311} - h_{1133} &= h_{334}\,\omega_{41}(e_1) - h_{114}\,\omega_{43}(e_3) \\ &= (\lambda - \lambda_4)\, a\,(1+b)\frac{\lambda_4 - \lambda_3}{\lambda_4 - \lambda}\, a\, b - (\lambda_4 - \lambda_3)\, a\, b \frac{\lambda - \lambda_4}{\lambda_4 - \lambda_3}\, a\,(1+b) \\ &= (\lambda_3 - \lambda) a^2 b (1+b), \\ h_{3322} - h_{2233} &= h_{334}\,\omega_{42}(e_2) - h_{224}\,\omega_{43}(e_3) = (\lambda_3 - \lambda) a^2 (1+b). \end{cases}$$

Hence, from (4.4) and (4.5) we get

$$(4.6) \qquad (\lambda_3 - \lambda) a^2 b (1+b) = (\lambda_3 - \lambda)(1 + \lambda_3\,\lambda) = (\lambda_3 - \lambda) a^2 (1+b)$$

and so,

$$b = -1 \quad \text{or} \quad b = 1. \tag{4.7}$$

Therefore, to prove Lemma 4.2 we only need to show that $b \neq -1$ and $b \neq 1$.

Case 1. In the case $b = -1$: We compute $6h_{114}h_{11444}$ in *Step* 1 and *Step* 2 respectively by using different ways, and show that in *Step* 3 they are not equal mutually. Now, (4.6) implies $(\lambda_3 - \lambda)(1 + \lambda_3 \lambda) = 0$, i.e.,

$$\lambda \neq 0, \quad \lambda_3 = \frac{-1}{\lambda} \quad \text{and} \quad \lambda_4 = \frac{1}{\lambda} - 2\lambda. \tag{4.8}$$

Hence,

$$S = 2\lambda^2 + \lambda_3^2 + \lambda_4^2 = 6\lambda^2 + \frac{2}{\lambda^2} - 4. \tag{4.9}$$

¿From (4.3) and (4.2), we have

(4.10)

$$h_{114} = -h_{224}, \; h_{334} = h_{444} = 0, \; \omega_{41}(e_1) = -\omega_{42}(e_2) \text{ and } \omega_{43}(e_3) = 0.$$

Hence, from (3.8) and (4.10) we have

$$6h_{114}^2 = S(S-4). \tag{4.11}$$

Let $h_{114}\omega_{41}(e_1) = c$. Then, by using (4.3) and (4.8) we have

$$c(\lambda_4 - \lambda) = h_{114}^2 \quad \text{and so} \quad c = \frac{h_{114}^2}{\lambda_4 - \lambda} = \frac{h_{114}^2\,\lambda}{1 - 3\lambda^2}. \tag{4.12}$$

Moreover, by using (3.12), (3.13), (3.3), (3.7) and (4.10) we also have

$$\begin{aligned}
(4.13)\quad & h_{1111} = 3c, & h_{1122} &= -c, & h_{1133} &= 0, & h_{1144} &= (4-S)\lambda - 2c,\\
& h_{2211} = -c, & h_{2222} &= 3c, & h_{2233} &= 0, & h_{2244} &= (4-S)\lambda - 2c,\\
& h_{3311} = 0, & h_{3322} &= 0, & h_{3333} &= 0, & h_{3344} &= (4-S)\lambda_3,\\
& h_{4411} = -2c, & h_{4422} &= -2c, & h_{4433} &= 0, & h_{4444} &= (4-S)\lambda_4 + 4c.
\end{aligned}$$

Step 1. First we compute $6h_{114}h_{11444}$ by using one way. ¿From (3.14), (3.15) and (4.10), we have

$$(4.14) \qquad h_{44433} = 0, \quad h_{33444} - h_{44433} = 0 \quad \text{and so,} \quad h_{33444} = 0.$$

Now, $h_{1144} = h_{2244}$ from (4.13). Hence, from (3.5), (3.6), (4.10) and (4.14)

$$(4.15) \qquad \begin{cases} h_{11444} + h_{22444} + h_{44444} = 0, \\ \lambda h_{11444} + \lambda h_{22444} + \lambda_4 h_{44444} = 0. \end{cases}$$

Hence, we have

$$(4.16) \qquad h_{11444} = -h_{22444} \quad \text{and} \quad h_{44444} = 0.$$

Hence, from (3.10) and (4.16) we have

$$(4.17) \qquad 6h_{114}h_{11444} = -6h_{1144}^2 - 3h_{3344}^2 - h_{4444}^2.$$

Step 2. Second we compute $6h_{114}h_{11444}$ by using another way. By using (3.14), (3.15) and (4.10) we also have

(4.18)

$$6h_{114}h_{11444} = 6(3 + 4h_{11}h_{44} - h_{44}^2)h_{114}^2 - 6(2 + 3h_{11}h_{44} - h_{11}^2)h_{114}h_{444}$$
$$+6(h_{4444} - h_{4411} - 2h_{1144})h_{114}\,\omega_{41}(e_1)$$
$$= 6(h_{4444} - h_{4411} - 2h_{1144})c + 6(3 + 4\lambda\,\lambda_4 - \lambda_4^2)h_{114}^2.$$

Step 3. Now we show that they computed in *Step* 1 and *Step* 2 respectively, are not equal mutually. Suppose (4.18) = (4.17). Then the equality gives

(4.19)

$$6(h_{4444}-h_{4411}-2h_{1144})c+6(3+4\lambda\,\lambda_4-\lambda_4^2)h_{114}^2 = -6h_{1144}^2-3h_{3344}^2-h_{4444}^2.$$

By using (4.11) and (4.13), (4.19) becomes

$$\begin{aligned}&6[(4-S)\lambda_4+4c-(-2c)-2\{(4-S)\lambda-2c\}]c\\&\quad+(3+4\lambda\,\lambda_4-\lambda_4^2)S(S-4)\\&=-6\{(4-S)\lambda-2c\}^2-3\{(4-S)\lambda_3\}^2-\{(4-S)\lambda_4+4c\}^2..\end{aligned}$$

By using the fact that $S-4\neq 0$,

$$(4.20)\qquad \begin{aligned}&(S-4)(6\lambda^2+3\lambda_3^2+\lambda_4^2)+(36\lambda-14\lambda_4)c\\&\qquad+S(3+4\lambda\,\lambda_4-\lambda_4^2)+\frac{100c^2}{S-4}=0.\end{aligned}$$

Let $\lambda^2=t$. Then, from (4.8) and (4.9) we have

$$(4.21)\qquad \begin{cases}6\lambda^2+3\lambda_3^2+\lambda_4^2 &=6\lambda^2+3\frac{1}{\lambda^2}+\left(\frac{1}{\lambda}-2\lambda\right)^2=2S-2t+4\\ \lambda_4^2-4\lambda\,\lambda_4 &=\left(\frac{1}{\lambda}-2\lambda\right)^2-4\lambda\left(\frac{1}{\lambda}-2\lambda\right)=\frac{S}{2}+9t-6.\end{cases}$$

And, from (4.9) and (4.12) we also have

$$(4.22)\qquad \begin{cases}S &=6t+\frac{2}{t}-4,\qquad (S-4)t=2(3t-1)(t-1)\\ c &=\frac{h_{114}^2}{\lambda_4-\lambda}=\frac{S(S-4)\,\lambda}{6(1-3t)},\qquad c\lambda=\frac{S(S-4)t}{6(1-3t)}.\end{cases}$$

By using (4.21) and (4.22), the above (4.20) becomes

$$(4.23)\qquad (-110S^2+912S-720)t=85S^2+60S-144.$$

Therefore, from (4.9) and (4.23) we have a system of equations:

$$(4.24)\qquad \begin{cases}S=6t+\frac{2}{t}-4,\\ (-110S^2+912S-720)t=85S^2+60S-144.\end{cases}$$

To find such pairs of numbers S, t that satisfy the above system (4.24) of equations, let us eliminate S from a system of equations. Then we obtain a equation

$$(4.25) \qquad f(t) = 990t^5 - 1923t^4 + 1262t^3 - 142t^2 - 200t + 85 = 0.$$

Since $S > 4$, we have

$$6t + \frac{2}{t} - 4 > 4$$

and so, $0 < t < 1/3$ or $t > 1$.

For all t such that $0 < t < 1/3$,

$$\begin{aligned} f(t) &= 990t^5 - 1923t^4 + 1262t^3 - 142t^2 - 200t + 85 \\ &= 110(9t^2 - 6t + 1)t^3 - 1263t^4 + 1152t^3 - 142t^2 - 200t + 85 \\ &= 110(3t-1)^2t^3 + 421(1-3t)t^3 + 16(1-9t^2) + 67(1-3t) \\ &\quad + 731t^3 + 2t^2 + t + 2 > 0. \end{aligned}$$

Moreover, for all t such that $t > 1$

$$\begin{aligned} f(t) &= 990t^5 - 1923t^4 + 1262t^3 - 142t^2 - 200t + 85 \\ &= 962(t^2 - 2t + 1)t^3 + 100(t^2 - 2t + 1) + 28t^5 + t^4 + 300t^3 - 242t^2 - 15 \\ &= 962(t-1)^2t^3 + 100(t-1)^2 + 242(t^3 - t^2) + 15(t^3 - 1) \\ &\quad + 28t^5 + t^4 + 43t^3 > 0. \end{aligned}$$

Hence, $f(t) > 0$ for all t such that $0 < t < 1/3$ or $t > 1$. That is, there is no root of the equation (4.25). It follows that $b \neq -1$.

Case 2. In the case $b = 1$: We also compute h_{1144} in *Step* 1 and *Step* 2 respectively by using different ways, and show that in *Step* 3 they are not equal mutually. By using (4.6), we have

$$(4.26) \qquad 1 + \lambda_3 \lambda = 2a^2.$$

Case $2-1$. Suppose that $\lambda = 0$. Then, it follows from (4.26) that

$$(4.27) \qquad a^2 = \frac{1}{2}, \quad \lambda_4 = -\lambda_3 \neq 0 \quad \text{and} \quad S = 2\lambda_3^2.$$

¿From (4.3), we have

$$h_{114} = h_{224} = -h_{334} = -h_{444} = -2a\lambda_3 \neq 0, \tag{4.28}$$
$$\omega_{41}(e_1) = \omega_{42}(e_2) = 2a, \quad \omega_{43}(e_3) = -a.$$

Together with (3.12) and (3.13), (4.28) implies

$$h_{1111} = 6ah_{114}, \quad h_{1122} = 2ah_{114}, \quad h_{1133} = -ah_{114}, \tag{4.29}$$
$$h_{2211} = 2ah_{224}, \quad h_{2222} = 6ah_{224}, \quad h_{2233} = -ah_{224}.$$

Step 1. First we compute h_{1144} by using one way. ¿From (3.7) and (4.29) we have

(4.30)
$$h_{1144} = (4 - S)\lambda - h_{1111} - h_{1122} - h_{1133} = -7a\, h_{114} = 14a^2\lambda_3 = 7\lambda_3.$$

Step 2. Second we compute h_{1144} by using another way. Since $h_{1144} = h_{2244}$, (3.3), (3.4), (3.9) and (4.28) give a system of equations:

$$\begin{cases} 2h_{1144} + h_{3344} + h_{4444} = 0, \\ h_{3344} - h_{4444} = -8\lambda_3, \\ 6h_{1144} - 3h_{3344} - h_{4444} = 0. \end{cases} \tag{4.31}$$

Hence, we obtain

$$h_{1144} = -\frac{4}{5}\lambda_3. \tag{4.32}$$

Step 3. Now we show that they computed in *Step* 1 and *Step* 2 respectively, are not equal mutually. Suppose (4.30) = (4.32). Then the equality gives

$$7\lambda_3 = -\frac{4}{5}\lambda_3 \quad \text{and so,} \quad \lambda_3 = 0.$$

But, since $\lambda_3 \neq 0$, it follows that $\lambda \neq 0$.

Case 2 − 2. Suppose that $\lambda \neq 0$. Then, from (4.25) we have

$$(4.33) \qquad \lambda_3 = \frac{2a^2-1}{\lambda}, \quad \lambda_4 = \frac{1-2a^2}{\lambda} - 2\lambda$$

and

$$(4.34) \quad S = 2\lambda^2 + \lambda_3^2 + \lambda_4^2 = 6\lambda^2 + 2\left(\frac{2a^2-1}{\lambda}\right)^2 - 4(1-2a^2).$$

¿From (4.3), we have

(4.35)

$$\begin{cases} h_{114} = h_{224} = (\lambda_4 - \lambda_3)\,a, \;\; h_{334} = 2(\lambda - \lambda_4)\,a, \;\; h_{444} = 2(\lambda_3 - \lambda)\,a, \\ \omega_{41}(e_1) = \omega_{42}(e_2) = \frac{\lambda_4-\lambda_3}{\lambda_4-\lambda}\,a, \quad \omega_{43}(e_3) = 2\frac{\lambda-\lambda_4}{\lambda_4-\lambda_3}\,a. \end{cases}$$

Hence, (3.8), (4.34) and (4.35) imply

$$(4.36) \;\; S(S-4) = (16\lambda^2 + 10\lambda_3^2 + 18\lambda_4^2 - 12\lambda_3\,\lambda_4 - 24\lambda\,\lambda_4 - 8\lambda_3\,\lambda)\,a^2.$$

Let $\lambda^2 = t$ and $2a^2 - 1 = u$. Then by using (4.33) and (4.34) we have

$$(4.37) \qquad \lambda_3 = \frac{u}{\lambda}, \qquad \lambda_4 = \frac{-u}{\lambda} - 2\lambda, \qquad S = 6t + \frac{2u^2}{t} + 4u.$$

And, by using (4.36) and (4.37) we have

$$(4.38) \qquad S(S-4) = (68t + 20\frac{u^2}{t} + 56u)\,(u+1).$$

By eliminating S from (4.37) and (4.38), we have

$$(4.39) \; u^4 - tu^3 - (4t^2 + 7t)u^2 - (5t^3 + 18t^2)u + (9t^4 - 23t^3) = 0.$$

Step 1. First we compute h_{1144} by using one way. ¿From (3.12), (3.13) and (4.35),

$$(4.40) \qquad h_{1144} = (4-S)\lambda - \frac{4(\lambda_4-\lambda_3)^2}{\lambda_4-\lambda}a^2 + 2(\lambda_4-\lambda)a^2$$

and

$$(4.41)\quad h_{2244} = (4-S)\lambda - h_{224}\{\omega_{41}(e_1) + 3\omega_{42}(e_2) + \omega_{43}(e_3)\} = h_{1144}.$$

Step 2. Second we compute h_{1144} by using another way.

Since $h_{1144} = h_{2244}$, (3.3), (3.4) and (3.9) imply a system of equations:

(4.42)
$$\begin{cases} 2h_{1144} + h_{3344} + h_{4444} = 0, \\ 2\lambda\, h_{1144} + \lambda_3\, h_{3344} + \lambda_4\, h_{4444} = -(2h_{114}^2 + h_{334}^2 + h_{444}^2), \\ 6h_{114}\, h_{1144} + 3h_{334}\, h_{3344} + h_{444}\, h_{4444} = 0. \end{cases}$$

Here, since $\lambda_3 + \lambda_4 = -2\lambda$, (4.35) gives

$$2h_{114}^2 + h_{334}^2 + h_{444}^2 = 8S\, a^2.$$

Using (4.35) and (4.36), from the system (4.42) of equations we also compute

$$(4.43)\qquad h_{1144} = \frac{32(\lambda_4 - 3\lambda)}{S-4}\, a^4.$$

Step 3. Now, by using (4.39) we show that they computed in *Step* 1 and *Step* 2 respectively, are not equal mutually. Suppose (4.43) = (4.40). Then, we have

$$(4.44)\qquad \frac{32(\lambda_4 - 3\lambda)}{S-4}\, a^4 = (4-S)\lambda - \frac{4(\lambda_4-\lambda_3)^2}{\lambda_4 - \lambda} a^2 + 2(\lambda_4 - \lambda)a^2.$$

By using (4.37), from (4.44) we obtain

$$(4.45)\quad 5u^5 + (14t+7)u^4 + (28t^2+26t)u^3 + (4t^3+124t^2-10t)u^2 \\ - (93t^4 - 222t^3 - 4t^2)u - (54t^5 - 69t^4 - 38t^3) = 0.$$

Therefore, from (4.39) and (4.45) we obtain a system of equations:

(4.46)
$$\begin{cases} u^4 - tu^3 - (4t^2 + 7t)u^2 - (5t^3 + 18t^2)u + (9t^4 - 23t^3) = 0, \\ 5u^5 + (14t + 7)u^4 + (28t^2 + 26t)u^3 + (4t^3 + 124t^2 - 10t)u^2 \\ \quad -(93t^4 - 222t^3 - 4t^2)u - (54t^5 - 69t^4 - 38t^3) = 0. \end{cases}$$

To find such pairs of numbers t, u that satisfy the above system (4.46) of equations, let us eliminate u. Since $t > 0$, From (4.46), we have

$$\begin{aligned} &5u\{tu^3 + (4t^2 + 7t)u^2 + (5t^3 + 18t^2)u - (9t^4 - 23t^3)\} \\ &+ (14t + 7)\{tu^3 + (4t^2 + 7t)u^2 + (5t^3 + 18t^2)u - (9t^4 - 23t^3)\} \\ &+ (28t^2 + 26t)u^3 + (4t^3 + 124t^2 - 10t)u^2 \\ &\quad - (93t^4 - 222t^3 - 4t^2)u - (54t^5 - 69t^4 - 38t^3) = 0. \end{aligned}$$

(4.47)
$$\begin{aligned} &5t\{tu^3 + (4t^2 + 7t)u^2 + (5t^3 + 18t^2)u - (9t^4 - 23t^3)\} \\ &+ 5u\{(4t^2 + 7t)u^2 + (5t^3 + 18t^2)u - (9t^4 - 23t^3)\} \\ &+ (14t + 7)\{tu^3 + (4t^2 + 7t)u^2 + (5t^3 + 18t^2)u - (9t^4 - 23t^3)\} \\ &+ (28t^2 + 26t)u^3 + (4t^3 + 124t^2 - 10t)u^2 \\ &\quad - (93t^4 - 222t^3 - 4t^2)u - (54t^5 - 69t^4 - 38t^3) = 0. \end{aligned}$$

Since $t > 0$, (4.47) $\div\, t$ implies

(4.48)
$$\begin{aligned} &(67t + 68)u^3 + (105t^2 + 375t + 39)u^2 \\ &+ (-43t^3 + 714t^2 + 130t)u - (225t^4 - 443t^3 - 199t^2) = 0. \end{aligned}$$

$\{(4.48) \times u\}$ and (4.46) imply

$$\begin{aligned} &(67t + 68)\{tu^3 + (4t^2 + 7t)u^2 + (5t^3 + 18t^2)u - (9t^4 - 23t^3)\} \\ &+ (105t^2 + 375t + 39)u^3 + (-43t^3 + 714t^2 + 130t)u^2 \\ &\quad - (225t^4 - 443t^3 - 199t^2)u = 0. \end{aligned}$$

$$(4.49)\quad (172t^2+443t+39)u^3+(225t^3+1455t^2+606t)u^2 \\ +(110t^4+1989t^3+1423t^2)u-(603t^5-929t^4-1564t^3)=0.$$

$\{(4.48)\times(172t^2+443t+39)-(4.49)\times(67t+68)\}\div 3$ yields

$$(4.50)\quad (995t^4-590t^3+12462t^2-3102t+507)u^2 \\ =(4922t^5+12328t^4-35464t^3+3776t^2-1690t)u \\ -567t^6+14906t^5-17914t^4+306t^3-2587t^2.$$

For all $t>0$, we have

$$995t^4-590t^3+12462t^2-3102t+507 \\ =295(t-1)^2t^2+507(4t-1)^2+700t^4+4055t^2+954t>0.$$

Hence, multiplying (4.48) by $995t^4+\cdots+507$ we obtain

$$(67t+68)(995t^4-590t^3+12462t^2-3102t+507)u^3 \\ +(105t^2+375t+39)(995t^4-590t^3+12462t^2-3102t+507)u^2 \\ +(-43t^3+714t^2+130t)(995t^4-590t^3+12462t^2-3102t+507)u \\ -(225t^4-443t^3-199t^2)(995t^4-590t^3+12462t^2-3102t+507)=0.$$

And using (4.50) we obtain

$$(4.51)\quad (67t+68)u\{(4922t^5+12328t^4-35464t^3+3776t^2-1690t)u \\ -567t^6+14906t^5-17914t^4+306t^3-2587t^2\} \\ +(105t^2+375t+39)\{(4922t^5+12328t^4-35464t^3+3776t^2-1690t)u \\ -567t^6+14906t^5-17914t^4+306t^3-2587t^2\} \\ +(-43t^3+714t^2+130t)(995t^4-590t^3+12462t^2-3102t+507)u \\ -(225t^4-443t^3-199t^2)(995t^4-590t^3+12462t^2-3102t+507)=0.$$

And dividing (4.51) by $2t(67t+68)$ we obtain

$$
\begin{aligned}
(4.52)\qquad & (2461t^4+6164t^3-17732t^2+1888t-845)u^2\\
&+(3254t^5+32788t^4-32704t^3-1620t^2-5174t)u\\
&+(-2115t^6+16520t^5-10652t^4+10788t^3-9933t^2)=0.
\end{aligned}
$$

To eliminating u^2 from (4.50) and (4.52), multiply (4.52) by $995t^4+\cdots+507$. Then, we obtain

$$
\begin{aligned}
&(2461t^4+6164t^3-17732t^2+1888t-845)(995t^4-590t^3\\
&\qquad\qquad+12462t^2-3102t+507)u^2\\
&+(3254t^5+32788t^4-32704t^3-1620t^2-5174t)(995t^4-590t^3\\
&\qquad\qquad+12462t^2-3102t+507)u\\
&+(-2115t^6+16520t^5-10652t^4+10788t^3-9933t^2)(995t^4-590t^3\\
&\qquad\qquad+12462t^2-3102t+507)=0.
\end{aligned}
$$

And using (4.50) we obtain

$$
\begin{aligned}
&(2461t^4+6164t^3-17732t^2+1888t-845)\{(4922t^5+12328t^4-35464t^3\\
&+3776t^2-1690t)u-567t^6+14906t^5-17914t^4+306t^3-2587t^2\}\\
&+(3254t^5+32788t^4-32704t^3-1620t^2-5174t)(995t^4-590t^3\\
&\qquad\qquad+12462t^2-3102t+507)u\\
&+(-2115t^6+16520t^5-10652t^4+10788t^3-9933t^2)(995t^4-590t^3\\
&\qquad\qquad+12462t^2-3102t+507)=0.
\end{aligned}
$$

And dividing the above equation by $4t(67t+68)$ we obtain

$$
\begin{aligned}
&(4.53)\\
&(57279t^7+282846t^6-697135t^5+698506t^4-129559t^3-69294t^2\\
&\qquad\qquad+36855t-4394)u\\
&\quad=(13059t^7-203082t^6+164525t^5+376306t^4-906107t^3\\
&\qquad\qquad+494522t^2-124805t+10478)t.
\end{aligned}
$$

In the same way as above, multiplying both sides of (4.50) by $57279t^7 + \cdots - 4394$ and using (4.53) we obtain an equation. And dividing both sides of the equality by $995t^4 + \cdots + 507$ we have

(4.54)
$$\begin{aligned}(13059t^7 - 203082t^6 + 164525t^5 + 376306t^4 - 906107t^3 \\ + 494522t^2 - 124805t + 10478)u \\ = (31959t^7 - 126930t^6 + 959993t^5 - 2470086t^4 + 2650385t^3 \\ - 1084542t^2 + 226831t - 12506)t.\end{aligned}$$

Last, using (4.53) and (4.54) we obtain an equation in which u is eliminated and dividing both sides of the equation by $32(995t^4 + \cdots + 507)$ we also obtain

(4.55)
$$\begin{aligned}52137t^{10} + 253062t^9 - 2033508t^8 + 5141910t^7 - 7134618t^6 \\ + 6230014t^5 - 3591608t^4 + 1378538t^3 - 343231t^2 + 50684t - 3380 \\ = (t-1)^2(3t-1)^2(5793t^6 + 43566t^5 - 123930t^4 + 139498t^3 \\ - 79719t^2 + 23644t - 3380) = 0.\end{aligned}$$

¿From (4.53), (4.54) and (4.36), we see that if $t = 1$ or $\frac{1}{3}$, then $u = -1$ and $S = 4$. But since $S > 4$, we know $t \neq 1$ and $t \neq \frac{1}{3}$. Hence, from (4.55) we have an equation

(4.56)
$$5793t^6 + 43566t^5 - 123930t^4 + 139498t^3 - 79719t^2 + 23644t - 3380 = 0.$$

Let $f(t) = 5793t^6 + 43566t^5 - 123930t^4 + 139498t^3 - 79719t^2 + 23644t - 3380$. Then, we have

$$\begin{aligned}f'(t) &= 34758t^5 + 217830t^4 - 495720t^3 + 418494t^2 - 159438t + 23644, \\ f''(t) &= 6(28965t^4 + 145220t^3 - 247860t^2 + 139498t - 26573), \\ f'''(t) &= 6(115860t^3 + 435660t^2 - 495720t + 139498) \\ &= 6(28965t + 131172)(2t-1)^2 + 6(26832t^2 + 3t + 8326) > 0.\end{aligned}$$

Since $f'''(t) > 0$ for all $t > 0$, f'' is increasing. And since $f''(0) < 0$, there is only one real number α $(5/12 < \alpha < 1/2)$ such that $f''(\alpha) = 0$. That is, f' has only one local minimum at α. For the α

$$\begin{aligned}
f'(\alpha) &= 34758\alpha^5 + 217830\alpha^4 - 495720\alpha^3 + 418494\alpha^2 - 159438\alpha + 23644 \\
&= \left(\frac{6\alpha}{5} - 1\right)(28965\alpha^4 + 145220\alpha^3 - 247860\alpha^2 + 139498\alpha - 26573) \\
&\quad + 72531\alpha^4 - 53068\alpha^3 + 3236\alpha^2 + 11947\alpha - 2929 + \frac{2}{5}\alpha^2 + \frac{3}{5}\alpha \\
&= 72531\alpha^4 - 53068\alpha^3 + 3236\alpha^2 + 11947\alpha - 2929 + \frac{2}{5}\alpha^2 + \frac{3}{5}\alpha \\
&> (8059\alpha^2 - 524\alpha - 886)(3\alpha - 1)^2 + (2\alpha + 11)(\alpha - 1)^2 \\
&\quad + 7175\alpha - 2054 \\
&> 0
\end{aligned}$$

since $8059\alpha^2 - 524\alpha - 886 > 0$ and $7175\alpha - 2054 > 0$. Hence $f'(t) > 0$ for all $t > 0$, and so f is increasing. It implies that the equation (4.56) has only one root β (≈ 0.654) between $\frac{3}{5}$ and $\frac{2}{3}$, since $f(\frac{3}{5}) < 0$ and $f(\frac{2}{3}) > 0$. For the root $t = \beta$, from (4.53) and (4.54) we compute that $u \approx -1.118$. But, since a is nonzero in (4.3), $u = 2a^2 - 1 > -1$. Therefore there is no pair t, u satisfying the system (4.46) of equations such that $t > 0$, $t \neq \frac{1}{3}$, $t \neq 1$ and $u > -1$. That is, it follows that $b \neq 1$, which completes the proof of our theorem. □

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