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THE STABILITY OF A DERIVATION ON A BANACH ALGEBRA

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Abstract. In this article, we show that for an approximate derivation on a Banach $*$-algebra, there exist a unique derivation near the an approximate derivation and for an approximate derivation on a C^*-algebra, there exist a unique derivation near the approximate derivation.

1. Introduction

In 1940, S. M. Ulam [23] posed the following problem concerning the stability of homomorphism. We are given a group G_1 and a metric group (G_2, d). Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x,\ y \in G_1$, then a homomorphism $g : G_1 \to G_2$ exists with $d(f(x), g(x)) \leq \varepsilon$ for all $x \in G_1$

As an answer to the problem of Ulam, D. H. Hyers [8] in 1941 has proved the stability of linear functional equation. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [17]:

Let $f : X \to Y$ be a mapping with X, Y Banach spaces, such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for all fixed $x \in X$. Suppose that there exist

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constant $\theta \geq 0$ *and* $p \in [0,1)$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$*, then there exists a unique* $\mathbb{R}$ *-linear mapping* $T : X \to Y$ *such that*

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x, y \in X$.

Since then, the stability problems of several functional equation have been extensively investigated by a number of authors (for instance, [1, 2, 4, 5, 6, 7, 9, 12, 11, 15, 18, 19, 20, 21, 22]). In Particular, P. Găvruta [7] in 1994 generalized the result of Rassias as the following:

Let G, X *be an abelian group and a Banach spaces, respectively. Denote by* $\varphi : G \times G \to [0, \infty)$ *a mapping such that*

$$\widetilde{\varphi}(x,y) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y)$$

or all x, $y \in X$. *Suppose that* $f : G \to X$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y)$$

for all x, $y \in G$. *Then there exists a unique additive mapping* $T : G \to Y$ *such that*

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x,y)$$

for all $x \in G$.

In 1988, B. E. Johnson [10] investigated almost algebra $*$-homomorphisms between Banach $*$-algebras. Recently, C. Park [16] also proved the generalized Hyers-Ulam-Rassias stability of algebra homomorphisms on Banach $*$-algebras and automorphisms on C^*-algebra.

Now we deal with a derivation d in algebra A, i.e., a linear mapping of A into A such that

$$d(xy) = xd(y) + d(x)y.$$

for all $x, y \in A$.

The main purpose of this paper is to prove that for an approximate derivation on a Banach $*$-algebra, there exist a unigue derivation near the approximate derivation and for an approximate derivation on a C^*-algebra, there exist a unigue derivation near the an approximate derivation.

2. Stability of derivations on a Banach $*$-algebra

In this section, let A be a Banach $*$-algebra and A_{sa} the set of self-adjoint elements in A. We also denote by

$$\mathbf{S} := \{\lambda \in \mathbb{C} |\ |\lambda| = 1\}.$$

As a matter of convenience, for given mapping $f : A \to A$, we use the following abbreviation:

$$D_\mu f(x, y) := f(\mu x + \mu y) - \mu f(x) - \mu f(y)$$

for all $\mu \in \mathbb{C}$ and for all $x, y \in A$.

Let $\varphi : A \times A \to [0, \infty)$ be a function satisfying

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty \tag{2.1}$$

for all $x, y \in A$.

Theorem 2.1. *Let $\varphi : A \times A \to [0, \infty)$ be a mapping satisfying (2.1). Suppose that $f : A \to A$ is a function such that*

$$\|D_\mu f(x, y)\| \le \varphi(x, y), \tag{2.2}$$

$$\|f(zw) - zf(w) - f(z)w\| \le \varphi(z, w) \tag{2.3}$$

for all $x, y \in A$, for all $\mu = 1, i$, and for all $z, w \in A_{sa}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for all fixed $x \in A$, then there exists a unique derivation $d : A \to A$ such that

$$\|f(x) - d(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, y) \tag{2.4}$$

for all $x \in A$.

Proof. Let $\mu = 1$ in (2.2). Then, by Găvruta's theorem, there exists a unique additive mapping $d : A \to A$ satisfying the inequality (2.4), where

$$d(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$.

If we use the same argument as in the proof of [17, Theorem], then we see that d is $\mathbb{R}$-linear mapping. By replacing y by $2^{n-1}x$ in (2.2), then we find that

$$\|f(2^n \mu x) - 2\mu f(2^{n-1}x)\| \leq \varphi(2^{n-1}x, 2^{n-1}x)$$

for $\mu = 1, i$ and for all $x \in A$.

Then, from this, we can show that

$$\|\mu f(2^n x) - 2\mu f(2^{n-1}x)\| = \|f(2^n x) - 2f(2^{n-1}x)\| \leq \varphi(2^{n-1}x, 2^{n-1}x)$$

for $\mu = 1, i$ and for all $x \in A$.

Thus we obtain that

$$\begin{aligned} \|f(2^n \mu x) - 2\mu f(2^{n-1}x)\| &\leq \|f(2^n \mu x) - 2\mu f(2^{n-1}x)\| \\ &\quad + \|\mu f(2^n x) - 2\mu f(2^{n-1}x)\| \\ &\leq \varphi(2^{n-1}x, 2^{n-1}x) + \varphi(2^{n-1}x, 2^{n-1}x) \end{aligned}$$

for $\mu = 1, i$ and for all $x \in A$.

By dividing both sides by 2^n in preceding inequality, we have

$$\left\| \frac{f(2^n \mu x)}{2^n} - \mu \frac{f(2^{n-1}x)}{2^{n-1}} \right\| \leq \frac{1}{2^{n-1}} \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in A$.

Hence it follows that

$$d(\mu x) = \lim_{n\to\infty} \frac{f(2^n \mu x)}{2^n} = \lim_{n\to\infty} \mu \frac{f(2^n x)}{2^n} = \mu d(x)$$

for $\mu = 1, i$ and for all $x \in A$.

For all element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So we note that

$$d(\lambda x) = d(sx+itx) = sd(x)+td(ix) = sd(x)+itd(x) = (s+it)d(x) = \lambda d(x)$$

for all $\lambda \in \mathbb{C}$ and for all $x \in A$.

Therefore we conclude that

$$d(\zeta x + \eta y) = \zeta d(x) + \eta d(y)$$

for all $\zeta, \eta \in \mathbb{C}$ and for all $x, y \in A$, i.e., d is $\mathbb{C}$-linear mapping.

If we substitute $z := 2^n z$ in(2.3) and multiply $1/2^n$, then we deduce that

$$\left\| \frac{f(2^n zw)}{2^n} - zf(w) - \frac{f(2^n z)}{2^n} w \right\| \le \frac{1}{2^n} \varphi(2^n z, w)$$

for all $z, w \in A_{sa}$ and thence we get

$$d(zw) = \lim_{n\to\infty} \frac{f(2^n zw)}{2^n} = \lim_{n\to\infty} \left[zf(w) + \frac{f(2^n z)}{2^n} w \right] = zf(w) + d(z)w$$

for all $z, w \in A_{sa}$.

Replace w by $2^n w$ in the above relation and divide both sides by 2^n, then we have by additivity of d

$$d(zw) = z \frac{f(2^n w)}{2^n} + d(z)w,$$

for all $z, w \in A_{sa}$.

Taking the limit, we arrive at

$$d(zw) = zd(w) + d(z)w,$$

for all $z, w \in A_{sa}$.

For elements $x, y \in A$, $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$ and $y = \frac{y+y^*}{2} + i\frac{y-y^*}{2i}$, where $x_1 := \frac{x+x^*}{2}$, $x_2 := \frac{x-x^*}{2i}$ and $y_1 := \frac{y+y^*}{2}$, $y_2 := \frac{y-y^*}{2i}$ are selfadjoint.

Since d is $\mathbb{C}$-linear mapping,

$$\begin{aligned} d(xy) &= d(x_1y_1 - x_2y_2 + ix_1y_2 + ix_2y_1) \\ &= (x_1 + ix_2)(d(y_1) + id(y_2)) + (d(x_1) + id(x_2))(y_1 + iy_2) \\ &= xd(y) + d(x)y \end{aligned}$$

for all $x, y \in A$, which the proof is now complete. □

Theorem 2.2. *Let $\varphi : A \times A \to [0, \infty)$ be a mapping satisfying (2.1). Suppose that $f : A \to A$ is a function such that the inequalities (2.2) and (2.3) for all $\mu \in \mathbf{S}$, for all $x, y \in A$, and for all $z, w \in A_{sa}$. Then there exists a unique derivation $d : A \to A$ such that the inequality (2.4) for all $x \in A$.*

Proof. Set $\mu = 1 \in \mathbf{S}$ in (2.2). As in the proof of Theorem 2.1, there exists a unique additive mapping $d : A \to A$ satisfying the inequality (2.4), where

$$d(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$.

Using the same method as in the proof of Theorem 2.1, we find that

$$d(\mu x) = \lim_{n\to\infty} \frac{f(2^n \mu x)}{2^n} = \lim_{n\to\infty} \mu \frac{f(2^n x)}{2^n} = \mu d(x)$$

for $\mu \in \mathbf{S}$ and for all $x \in A$.

Here and now, let $\lambda \in \mathbb{C}$ be nonzero and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3}$. According to the theorem in [13], we see that there exist elements $\mu_1, \mu_2, \mu_3 \in \mathbf{S}$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. Observe that $d(x) = d(3 \cdot \frac{1}{3}x) = 3d(\cdot\frac{1}{3}x)$ for all $x \in A$. Then we have $d(\frac{1}{3}x) = \frac{1}{3}d(x)$ for all $x \in A$.

Thus it follows that for all $x \in A$,

$$\begin{aligned} d(\lambda x) &= d\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x\right) = Md\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x\right) = \frac{M}{3}d\left(3\frac{\lambda}{M}x\right) \\ &= \frac{M}{3}d(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(d(\mu_1 x) + d(\mu_2 x) + d(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)d(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}d(x) = \lambda d(x) \end{aligned}$$

Hence, by additivity of d, we obtain that

$$d(\zeta x + \eta y) = \zeta d(x) + \eta d(y)$$

for all $x \in A$ and for all $\zeta, \eta \in \mathbb{C}$ with $\zeta \neq 0$, $\eta \neq 0$. In particular, set $\zeta = 1, \eta = -1$, and $y = x$ in the above relation. Then $d(0) = d(x - x) = d(x) - d(x) = 0$. So $d(0x) = 0 = 0d(x)$ for all $x \in A$. Therefore d is a $\mathbb{C}$-linear mapping. The rest of the proof goes through by the same way as that of Theorem 2.1. The proof of Theorem is complete. □

Corollary 2.3. *Let $\varphi : A \times A \to [0, \infty)$ be a mapping. Suppose that there exists $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|D_\mu f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p),$$

$$\|f(zw) - zf(w) - f(z)w\| \leq \theta(\|z\|^p + \|w\|^p)$$

for all $\mu \in \mathbf{S}$, for all $x, y \in A$, and for all $z, w \in A_{sa}$. Then there exists a unique derivation $d : A \to A$ such that

$$\|f(x) - d(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. In Theorem 2.2, defining $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, we obtain the desired result. □

3. Stability of derivations on a unital C^*-algebra

From now on, let A be a unital C^*-algebra and $U(A)$ the set of unitary elements in A. Futhermore, e is the identity element of A.

Theorem 3.1. *Let $\varphi : A \times A \to [0, \infty)$ be a mapping satisfying (2.1). Suppose that $f : A \to A$ is a function such that (2.2) and*

$$\|f(uy) - uf(y) - f(u)y\| \le \varphi(u, y) \tag{3.1}$$

for all $\mu = 1, i$, for all $u \in U(A)$, and for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique derivation $d : A \to A$ such that the inequality (2.4) for all $x \in A$.

Proof. If we utilize the same way as in the proof of Theorem 2.1, then there exists a unique $\mathbb{C}$-linear mapping $d : A \to A$ such that

$$\|f(x) - d(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in A$, where

$$d(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in A$.

Replace $2^n y$ instead of y in (3.1) and divide both sides by 2^n, then we have

$$\left\|\frac{f(2^n uy)}{2^n} - u\frac{f(2^n y)}{2^n} - f(u)y\right\| \le \frac{1}{2^n}\varphi(u, 2^n y)$$

for all $u \in U(A)$ and for all $x \in A$.

Now taking the limit, we get

$$d(uy) = \lim_{n\to\infty} \frac{f(2^n uy)}{2^n} = \lim_{n\to\infty}\left[u\frac{f(2^n y)}{2^n} + f(u)y\right] = ud(y) + f(u)y$$

for all $u \in U(A)$ and for all $y \in A$.

In the above relation, setting $y = e$ yields $d(u) = ud(e) + f(u)$ for all $u \in U(A)$. Considering $u = e$ in this relation, we obtain $f(e) = 0$. In particular, we note that

$$2^{-n}d(e) = d(2^{-n}e) = \lim_{n\to\infty} \frac{f(2^n \cdot 2^{-n}e)}{2^n} = \lim_{n\to\infty} \frac{f(e)}{2^n} = 0.$$

Hence we conculde that $d(e) = 0$. So that $d(u) = f(u)$ for all $u \in U(A)$.

Since d is $\mathbb{C}$-linear mapping and all $x \in A$ is a finite combination of unitary elements, i.e., $x = \sum_{j=1}^{m} \lambda_j v_j$,

$$d(xy) = d(\sum_{j=1}^{m} \lambda_j u_j y) = \sum_{j=1}^{m} \lambda_j u_j d(y) + \sum_{j=1}^{m} \lambda_j f(u_j) y = xd(y) + d(x)y$$

for all $x, y \in A$. This completes the proof of Theorem. □

Theorem 3.2. *Let $\varphi : A \times A \to [0, \infty)$ be a mapping satisfying (2.1). Suppose that $f : A \to A$ is a function such that (2.2) and (3.1) for all $\mu \in \mathbf{S}$, for all $u \in U(A)$, and for all $x, y \in A$. Then there exists a unique derivation $d : A \to A$ such that the inequality (2.4) for all $x \in A$.*

Proof. By the same reasoning as in the proof of Theorem 2.2, there exists a unique $\mathbb{C}$-linear mapping $d : A \to A$ such that

$$\|f(x) - d(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in A$. The rest of proof is same to the proof of Theorem 3.1. □

Corollary 3.3. *Let $\varphi : A \times A \to [0, \infty)$ be a mapping. Suppose that there exists $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\|D_\mu f(x, y)\| \le \theta(\|x\|^p + \|y\|^p),$$

$$\|f(uy) - uf(y) - f(u)y\| \le \theta(\|u\|^p + \|y\|^p)$$

for all $\mu \in \mathbf{S}$, for all $x, y \in A$, and for all $u \in U(A)$. Then there exists a unique derivation $d : A \to A$ such that

$$\|f(x) - d(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. Considering $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ in the theorem 3.2, we arrive at the conclusion of the corollary. □

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