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ON TROPICAL QUADRIC SURFACES

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Abstract. After introducing Tropical Algebraic Varieties, we give a polyhedral description of tropical hypersurfaces. Using TOPCOM and GAP, we show that there exist 59 types of two dimensional tropical quadric surfaces. We also show a criterion for a quadric hypersurface to be non-degenerate in terms of a tropical rank.

1. Introduction

Tropical geometry is the geometry over the *tropical semiring*, one of idempotent semirings. Idempotent semirings appear in various areas of applied mathematics, including control theory, optimization and mathematical physics [6], [7], [14]. Our work is over the tropical semiring, which is the *min-plus algebra* $(\mathbb{R}, \oplus, \odot)$. The underlying set is the set of real numbers $\mathbb{R}$, sometimes augmented by $+\infty$ as an additive neutral element. In this paper we take the set of real numbers $\mathbb{R}$ as the underlying set. The arithmetic operations of *tropical addition* $\oplus$ and *tropical multiplication* $\odot$ are

$$x \oplus y \ := \ \min\{x, y\} \qquad \text{and} \qquad x \odot y \ := \ x + y\,.$$

Max-plus algebra is occasionally taken as a tropical semiring, where $x \oplus y := \max\{x, y\}$ and $x \odot y := x + y$. Tropical geometries over either choices are symmetrical with respect to the origin.

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While linear algebra and matrix theory over idempotent semirings are well-developed and have had numerous successes in applications [4], [6], [7], the notion of tropical geometry was introduced only in the past decade, since then this theory has developed rapidly and led to many applications. A brief survey on tropical algebraic geometry can be found at [9], [12], [13] and [18]. Tropical algebraic geometry is a new area in the field of algebraic geometry in which complicated geometric or algebraic problems can be transformed into purely combinatorial or polyhedral problems. The theory of tropical algebraic geometry is very much in its beginning. The basics of tropical algebraic geometric formalisms including tropical spectrum and tropical scheme have been presented just recently by Mikhalkin [12].

A *tropical monomial* is an expression of the form

$$c \odot x_1^{a_1} \odot \cdots \odot x_n^{a_n}, \tag{1}$$

where the powers of the variables are computed tropically as well, that is,

$$x_i^{a_i} = x_1 \odot x_1 \odot \cdots \odot x_1 = x_1 + x_1 + \cdots + x_1 = a_i x_1.$$

Hence the tropical monomial (1) represents the classical linear function

$$\mathbb{R}^n \to \mathbb{R}\ ,\ \ (x_1, \ldots, x_n) \mapsto c + a_1x_1 + \cdots + a_nx_n.$$

We note that in (1), the coefficient c could be 0. A *tropical polynomial* is a finite tropical sum of tropical monomials,

$$F \ = \ \bigoplus_{a \in \mathcal{A}} c_a \odot x_1^{a_1} \odot \cdots \odot x_n^{a_n},\ \ \mathcal{A} \subset \mathbb{N}_0^n\ ,\ \ c_a \in \mathbb{R}, \tag{2}$$

where $\mathbb{N}_0$ is a set of non-negative integers. It is easy to check that the tropical polynomial F is a piecewise linear concave function, given as the minimum of linear functions $(x_1, \ldots, x_n) \mapsto c_a + a_1x_1 + \cdots + a_nx_n$, $\forall\ a \in \mathcal{A}$. The set $\mathcal{A} \subset \mathbb{N}_0^n$ is called the *support* of the tropical polynomial F, and the *degree* of the tropical polynomial F is d if its support $\mathcal{A}$ is equal to the set $\{(a_1, a_2, \cdots, a_n) \in \mathbb{N}_0^n \mid a_1 + a_2 + \cdots + a_n = d\}$. The coefficients c_a are any real numbers, including 0.

For a given tropical polynomial F like (2), we define the *tropical hypersurface* $\mathcal{T}(F)$ as the set of all points $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ with the property that F is not linear at x. Equivalently $\mathcal{T}(F)$ is the set of points x at which the minimum is attained by two or more of the linear functions. This is also described as a "corner locus" of a piecewise linear function F by Gathmann [9]. Thus a tropical hypersurface is an $n-1$ dimensional polyhedral complex in $\mathbb{R}^n$. If the degree of the tropical polynomial F is d, then we say that the corresponding tropical hypersurface $\mathcal{T}(F)$ is of degree d. Especially when $d = 2$, $\mathcal{T}(F)$ is called a tropical *quadric* hypersurface.

Every tropical variety is a finite intersection of tropical hypersurfaces [3]. But not every intersection of tropical hypersurfaces is a tropical variety. Tropical varieties are also known as *logarithmic limit sets* [1], or *non-archimedean amoebas* [5]. Tropical curves are the key ingredient in Mikhalkin's formula for planar Gromov-Witten invariants [11].

The n-dimensional real vector space $\mathbb{R}^n$ is a module over the tropical semiring $(\mathbb{R}, \oplus, \odot)$, with the operations of coordinatewise tropical addition

$$(a_1, \ldots, a_n) \oplus (b_1, \ldots, b_n) = \big(\min\{a_1, b_1\}, \ldots, \min\{a_n, b_n\}\big).$$

and tropical scalar multiplication

$$\lambda \odot (a_1, a_2, \ldots, a_n) = \big(\lambda + a_1, \lambda + a_2, \ldots, \lambda + a_n\big).$$

Similarly to the real projective n-space $\mathbb{P}^n$, we introduce the *tropical projective n-space* as follows:

$$\mathbb{TP}^n = \mathbb{R}^{n+1} / \sim,$$

where $\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \lambda \odot \mathbf{y}$ for some $\lambda \in \mathbb{R}$. Since tropical scalar multiplication over a vector is a coordinatewise ordinary addition, the tropical projective n-space is also described by $\mathbb{TP}^n = \mathbb{R}^{n+1}/\mathbb{R}(1, 1, \ldots, 1)$. After normalizing one of coordinates x_i by setting $x_i = 0$, we often identify the tropical projective n-space $\mathbb{TP}^n$ with a tropical affine space $\mathbb{R}^n$.

The main aim of this paper is to show the classification of tropical quadric surfaces in $\mathbb{TP}^3$. The motivation is from the work on tropical linear spaces by D. Speyer [17]. However we give just computational results due to the lack of proper mathematical methods for this problem. For the notations and the terminologies in this paper we follow [16] and [19].

2. Tropical algebraic variety

Before describing tropical curves and surfaces, we give an algebraic description of tropical algebraic varieties over an algebraically closed field K with a valuation into the reals, denoted by $\deg : K^* \to \mathbb{R}$. For simplicity, let K be the algebraic closure of $\mathbb{C}(t)$ of rational polynomials, that is,

$$K = \overline{\mathbb{C}(t)}.$$

The degree of a rational polynomial in a variable t is the smallest exponent in the numerator polynomial minus the smallest exponent in the denominator polynomial. This definition of degree extends uniquely to the algebraic closure K of the field $\mathbb{C}(t)$. An algebraic function $p(t) \in K$ can be locally expressed as a *Puiseux series*

$$p(t) = c_1 t^{q_1} + c_2 t^{q_2} + c_3 t^{q_3} + \cdots .$$

Here $c_1, c_2, \ldots$ are non-zero complex numbers and $q_1 < q_2 < \cdots$ are rational numbers with bounded denominators. Then $\deg p(t)$ is the smallest exponent q_1. The degree of an n-tuple of algebraic functions is the n-tuple of their degrees. This gives a map

$$\deg : (K^*)^n \to \mathbb{Q}^n \subset \mathbb{R}^n, \quad K^* = K \backslash \{0\}. \tag{3}$$

Let I be an ideal in the polynomial ring $K[x_1, \ldots, x_n]$ and consider its affine variety

$$V(I) = \{(a_1, \cdots, a_n) \in (K^*)^n \mid f(a_1, \cdots, a_n) = 0, \ \forall\, f \in I\}.$$

The image of $V(I)$ under the map (3) is a subset of $\mathbb{Q}^n$. We take its topological closure. The resulting subset of $\mathbb{R}^n$ is the tropical variety $\mathcal{T}(I)$. Notice that all affine varieties we consider are subsets of $(K^*)^n$. To avoid this restriction, some authors take ideals in the Laurent polynomial ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Definition 1. A *tropical algebraic variety* is a subset of $\mathbb{R}^n$ of the form

$$\mathcal{T}(I) \quad = \quad \overline{\deg(V(I))}\,,$$

where I is an ideal of $K[x_1, \ldots, x_n]$ and $K = \overline{\mathbb{C}(t)}$.

An ideal $I \subset K[x_1, \ldots, x_n]$ is *homogeneous* if all monomials $x_1^{i_1} \cdots x_n^{i_n}$ appearing in a given generator of I have the same total degree $i_1 + \cdots + i_n$. Such a homogeneous ideal I defines a projective variety $V(I)$ in projective space $\mathbb{P}_K^{n-1}$ minus the coordinate hyperplanes $x_i = 0$ (due to our restriction). Its image under the degree map (3) becomes a subset of tropical projective space $\mathbb{TP}^{n-1} = \mathbb{R}^n/\mathbb{R}(1, 1, \ldots, 1)$.

Definition 2. A *tropical projective variety* is a subset of $\mathbb{TP}^n$ of the form

$$\mathcal{T}(I) \quad = \quad \overline{\deg(V(I))}/\mathbb{R}(1, 1, \ldots, 1)\,,$$

where I is a homogeneous ideal of $K[x_1, \ldots, x_n]$. To draw a tropical projective variety in a tropical affine space $\mathbb{R}^n$, we normalize one of coordinates x_i by setting $x_i = 0$, similar to an ordinary projective variety.

We set local ring and its maximal ideal of K as follows:

$$R_K \;=\; \{\, p(t) \in K \;:\; \deg p(t) \geq 0 \,\} \quad \text{and}$$

$$M_K \;=\; \{\, p(t) \in K \;:\; \deg p(t) > 0 \,\}.$$

In our case, the residue field $k = R_K/M_K$ is a complex number field $\mathbb{C}$.

Every polynomial $f \in K[x]$ maps to a tropical polynomial $\mathrm{Trop}(f)$ in the following way. If

$$f(x_1, \ldots, x_n) = \sum_{a \in \mathcal{A}} p_a(t)\, x_1^{a_1} \cdots x_n^{a_n}, \; p_a(t) \in K^* \tag{4}$$

and $c_a = \deg p_a(t)$, then we define $\mathrm{Trop}(f)$ to be the tropical polynomial F in (2).

Let $w \in \mathbb{R}^n$ be a fixed vector. The *w-weight* of a term $p_a(t)\, x_1^{a_1} \cdots x_n^{a_n}$ in (4) is $\deg p_a(t) + a_1 w_1 + \cdots + a_n w_n$. Next we will define the *initial form* $\mathrm{in}_w(f)$ of a polynomial f. Set $\tilde{f}(x_1, \ldots, x_n) = f(t^{w_1} x_1, \ldots, t^{w_n} x_n)$. Let ν be the smallest weight of any term of $\tilde{f}$, so that $t^{-\nu} \tilde{f}$ is a non-zero element in $R_K[x]$. Define $\mathrm{in}_w(f)$ as the image of $t^{-\nu}\tilde{f}$ in $\mathbb{C}[x]$ under the map

$$\iota : R_K[x] \longrightarrow (R_K/M_K)[x] = \mathbb{C}[x], \tag{5}$$

that is, $\mathrm{in}_w(f) = \iota(t^{-\nu}\tilde{f})$. We set $\mathrm{in}_w(0) = 0$. The initial form $\mathrm{in}_w(f)$ is a polynomial in $\mathbb{C}[x]$.

Given any ideal $I \subset K[x]$, then its *initial ideal* is defined to be

$$\mathrm{in}_w(I) = \langle\, \mathrm{in}_w(f) \; : \; f \in I \,\rangle \subset \mathbb{C}[x].$$

We give the proof of the following theorem due to Speyer and Sturmfels because first it is fundamental in tropical algebraic geometry, and secondly we want to give a little more details following the line of their proof.

Theorem 3. (Speyer and Sturmfels [19])
For an ideal $I \subset K[x]$, the following subsets of $\mathbb{R}^n$ coincide:

(*a*) *Tropical algebraic variety $\mathcal{T}(I)$.*

(*b*) $\bigcap_{f \in I} \mathcal{T}(\mathrm{Trop}(f))$.

(*c*) $\{\, w \in \mathbb{R}^n \mid \mathrm{in}_w(I)$ contains no monomial $\}$.

Proof. First consider any point $w = (\deg(u_1), \ldots, \deg(u_n)) \in \mathcal{T}(I)$ for some $(u_1, u_2, \cdots, u_n) \in V(I)$. For any $f \in I$ we have $f(u_1, \ldots, u_n) = 0$ and this implies that the minimum in the definition of $F = \mathrm{Trop}(f)$ is attained at least twice at w, from which we conclude the inclusion $\mathcal{T}(I) \subseteq \bigcap_{f\in I} \mathcal{T}(\mathrm{Trop}(f))$. It is equivalent to the fact that $\mathrm{in}_w(f)$ is not a monomial. Thus we may say that $\bigcap_{f\in I} \mathcal{T}(\mathrm{Trop}(f)) \subseteq \{w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ contains no monomial}\}$.

The only thing we need to prove is that $\{w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ contains no monomial}\} \subseteq \mathcal{T}(I)$. Consider any vector $w \in \mathbb{R}^n$ such that $w = (\deg(v_1), \ldots, \deg(v_n))$ for some $v \in (K^*)^n$ and $\mathrm{in}_w(I)$ does not contain any monomial. Since the image of the map deg is dense in $\mathbb{R}$ and the set defined in (a) is closed, it suffices to prove that $w = (\deg(u_1), \ldots, \deg(u_n))$ for some $u \in V(I)$. By making the change of coordinates $x_i = x_i \cdot v_i^{-1}$, we may assume that $w = (0, 0, \ldots, 0)$.

Since $\mathbb{C}$ is algebraically closed, by applying the Nullstellensatz, we can get a point $\bar{u} \in V(\mathrm{in}_w(I)) \subset (\mathbb{C}^*)^n$. Let $\bar{m}$ denote the maximal ideal in $\mathbb{C}[x]$ corresponding to $\bar{u}$. Set $S = \{f \in R_K[x] \mid \iota(f) \notin \bar{m}\}$, where ι is the map (5). Let $f_1, f_2 \in S$. It is obvious that $f_1 f_2 \in R_K[x]$ and $\iota(f_1 f_2) = \iota(f_1)\iota(f_2) \notin \bar{m}$ since $\bar{m}$ is maximal, especially prime. This tells us that if $f_1, f_2 \in S$ then $f_1 f_2 \in S$, from which we conclude that S is a multiplicative set of $R_K[x]$. We next claim that $S \bigcap I = \emptyset$. Contrarily let $f \in S \bigcap I$. Then $\mathrm{in}_w(f)(\bar{u}) = 0$ since $\bar{u} \in V(\mathrm{in}_w(I))$ and $\mathrm{in}_w(f) \in \mathrm{in}_w(I)$. From this fact, it follows that $\iota(f) = \mathrm{in}_w(f) \in \bar{m}$, which is a contradiction to the fact that $\iota(f) \notin S$.

Consider the induced map

$$\phi \,:\, R_K \longrightarrow S^{-1}R_K[x]/S^{-1}(I \cap R_K[x])\,, \text{ where } \phi(c) = [\,c/1\,]\,.$$

Let P be a minimal prime ideal of $S^{-1}R_K[x]/S^{-1}(I \cap R_K[x])$. We claim that $\phi^{-1}(P) = \{0\}$. Suppose not, and pick $c \in R_K \backslash \{0\}$ with $\phi(c) \in P$. Then $\phi(c)$ is a zero-divisor in $S^{-1}R_K[x]/S^{-1}(I \cap R_K[x])$, so we can find

$f \in S$ such that $cf \in I$. Since c^{-1} exists in K, this implies $f \in I$ which is a contradiction.

Now, $\phi^{-1}(P) = \{0\}$ implies that $P \otimes_{R_K} K$ is a proper ideal in $K[x]/I$. There exists a maximal ideal of $K[x]/I$ containing $P \otimes_{R_K} K$, and, since K is algebraically closed, this maximal ideal has the form $\langle x_1 - u_1, \ldots, x_n - u_n \rangle$ for some $u \in V(I) \subset (K^*)^n$. We claim that $u_i \in R_K$ and $u_i \cong \bar{u}_i \mod M_K$. Since $\bar{u}_i \in \mathbb{C}^*$ for each i, this implies $\deg(u_1) = \cdots = \deg(u_1) = 0$, and hence complete the proof.

To prove the last claim, consider any $x_i - u_i \in I$. By clearing denominators, we get $a_i x - b_i \in I \cap R_K[x]$ with $b_i/a_i = u_i$, and not both a_i and b_i lie in M_k. If $a_i \in M_K$, then $a_i x - b_i \cong -b_i \mod M_K$. Hence $\text{in}_w(I)$ contains $b_i \in K^*$ and hence equals the unit ideal, which is a contradiction. If $a_i \notin M_K$ and $-b_i/a_i \ncong \bar{u}_i \mod M_K$, then $\iota(a_i x - b_i) \notin \bar{m}$. This means that $a_i x - b_i \in S$ and is a unit of $S^{-1}R_K[x]$, so P is the unit ideal. But then P is not prime, also a contradiction. This completes the proof of the last claim. □

From this theorem, we see that a tropical variety is an intersection of tropical hypersurfaces. Furthermore it has been shown with an algorithmic implementation that every ideal I has, so called, a finite *tropical basis* B so that $\mathcal{T}(I) = \bigcap_{f \in B} \mathcal{T}(\text{Trop}(f))$ [3]. However there are examples showing that arbitrary finite intersection of tropical hypersurfaces may not be tropical varieties.

Notice that if an ideal I is a principal ideal generated by a polynomial f, then the above equivalent condition (c) gives the reason why we defined the tropical hypersurface $\mathcal{T}(F)$ as the set of points x at which the minimum is attained at least twice, where $F = \text{Trof}(f)$.

The following theorem says that the dimension of the polyhedral complex of a tropical variety $\mathcal{T}(I)$ coincides with the Krull dimension of the ring $K[x_1, \ldots, x_n]/I$.

Theorem 4. Bieri-Groves Theorem, [2] *If I is a prime ideal and $K[x]/I$ has Krull dimension r, then $\mathcal{T}(I)$ is a pure polyhedral complex of dimension r.*

3. Polyhedral construction of tropical hypersurfaces

We return to the tropical semiring $(\mathbb{R}, \oplus, \odot)$, and aim to derive an elementary polyhedral description of tropical hypersurfaces.

Let $\mathcal{A}$ be a finite subset of $\mathbb{N}_0^2$ and consider a tropical polynomial

$$F(x,y) \quad = \sum_{(i,j)\in\mathcal{A}} \omega_{ij} \odot x^i \odot y^j \,,\; \omega_{ij} \in \mathbb{R}, \tag{6}$$

Set P to be the convex hull of the points (i, j, ω_{ij}). This is a three-dimensional polytope. The lower faces of P project bijectively onto the convex hull of $\mathcal{A}$ by deleting the last coordinate. This is, by definition, a *regular subdivision* Δ of $\mathcal{A}$. The following proposition gives a way to draw the tropical curve $\mathcal{T}(F)$ in the plane $\mathbb{R}^2$. This is described at various papers [3],[4] without proof.

Proposition 5. *The tropical curve $\mathcal{T}(F)$ is an embedded graph in $\mathbb{R}^2$ which is the negative of the dual to the regular subdivision Δ of the support $\mathcal{A}$ of the tropical polynomial f. The segments of $\mathcal{T}(F)$ arise from the interior edges of Δ, and the rays of $\mathcal{T}(F)$ arise from the boundary edges of Δ. Corresponding edges of $\mathcal{T}(F)$ and Δ are perpendicular. Furthermore, this could be generalized to the higher dimensional tropical hypersurfaces.*

Proof. For any two points $(i_1, j_1), (i_2, j_2) \in \mathcal{A}$, we consider a system of linear inequalities:

$$\begin{aligned} \omega_{i1j1} + i_1\, x + j_1\, y \;&=\; \omega_{i2j2} + i_2\, x + j_2\, y \\ &\leq\; \omega_{ij} + i\, x + j\, y \;\;\forall\;\, (i,j) \in \mathcal{A}\,. \end{aligned} \tag{7}$$

It is obvious that a point $(x,y) \in \mathcal{T}(F)$ if and only if the point (x,y) satisfies the above inequalities (7) for some two points $(i_1,j_1),(i_2,j_2) \in \mathcal{A}$. Let $\widetilde{\pi}$ be one of the lower faces of the polytope P. Then there corresponds an outward normal vector $\mathbf{n}$ to $\widetilde{\pi}$. We normalize the normal vector $\mathbf{n}$ so that the last coordinate is -1. Set $\mathbf{n} = (u,v,-1)$. Suppose that $(i_1,j_1,\omega_{i_1j_1}),(i_2,j_2,\omega_{i_2j_2})$ are arbitrary two vertices of the polygone $\widetilde{\pi}$. Since $\mathbf{n}$ is an outward normal vector to the lower face $\widetilde{\pi}$ of P, we have

$$
\begin{aligned}
(8) \qquad i_1\,u + j_1\,v - \omega_{i_1j_1} &= i_2\,u + j_2\,v - \omega_{i_2j_2} \\
&\geq i\,u + j\,v - \omega_{ij} \;\; \forall \;\; (i,j) \in \mathcal{A}\,.
\end{aligned}
$$

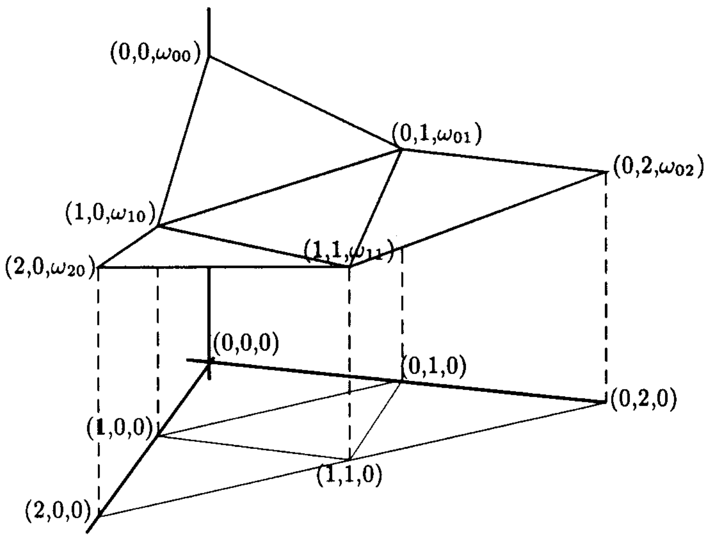
Let π be the projection of $\widetilde{\pi}$ by deleting the last coordinates. The inequalities (8) says that the point $(-u,-v)$, which is a negative of the dual point (u,v) corresponding to the polygone π, belongs to $\mathcal{T}(F)$.

Now suppose that the line segment connecting (i_1,j_1) and (i_2,j_2) is a boundary edge of the regular subdivision Δ. Consider a pencil of planes $\widetilde{\pi}_t$ containing the line through two vertices $(i_1,j_1,\omega_{i_1j_1})$, $(i_2,j_2,\omega_{i_2j_2})$ and staying below the polytope P. Let $\mathbf{n}_t$ be the normalized normal vector to $\widetilde{\pi}_t$ so that $\mathbf{n}_t = (u_t,v_t,-1)$. As the plane $\widetilde{\pi}_t$ approaches to the vertical plane to the xy plane, normal vector $\mathbf{n}_t$ approaches to the infinity. All normal vectors $\mathbf{n}_t$ lie on the plane which is perpendicular to the line segment $(i_1,j_1,\omega_{i_1j_1})$, $(i_2,j_2,\omega_{i_2j_2})$. Then the negative of the projection of the those normal vectors $\mathbf{n}_t$ by deleting the third coordinate forms a ray of the tropical curve $\mathcal{T}(F)$ starting from the point $(-u,-v)$, which is also perpendicular to the line segment between (i_1,j_1) and (i_2,j_2). If the line segment connecting (i_1,j_1) and (i_2,j_2) is an interior edge of the regular subdivision Δ, then similarly we can show that it corresponds to a finite line segment of $\mathcal{T}(F)$. All these descriptions naturally extend to the higher dimensional cases too. $\square$

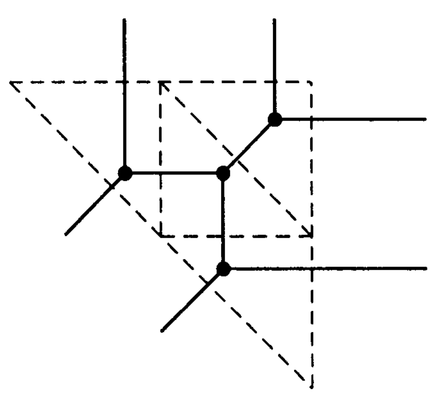
We give a concrete example. After normalizing with $z = 0$, consider a quadric tropical polynomial

$$F(x,y) = \omega_{20}\odot x^2 \oplus \omega_{11}\odot x\odot y \oplus \omega_{02}\odot y^2 \oplus \omega_{10}\odot x \oplus \omega_{01}\odot y \oplus \omega_{00}\,, \quad \omega_{ij}\in\mathbb{R},$$

where $2\,\omega_{10} < \omega_{00}+\omega_{20}$, $2\,\omega_{11} < \omega_{20}+\omega_{02}$, $2\,\omega_{01} < \omega_{00}+\omega_{02}$ with a few more inequalities.



The above figure shows the lower faces of the polytope P and the corresponding regular subdivision, and the corresponding tropical curve is shown below.



4. The Rank of a Tropical Matrix

For more details on the ranks of tropical matrices, we refer to [8]. The set $\mathbb{R}^d$ of real d-vectors and the set $\mathbb{R}^{d\times n}$ of real $d \times n$-matrices are semimodules over the semiring $(\mathbb{R}, \oplus, \odot)$. The operations of matrix addition and matrix multiplication are well defined. Here we give three different ranks for a tropical matrix, which are identical for an ordinary matrix.

Definition 6. *The* Barvinok rank *of a matrix $M \in \mathbb{R}^{d\times n}$ is the smallest integer r for which M can be written as the tropical sum of r matrices, each of which is the tropical product of a $d \times 1$-matrix and a $1 \times n$-matrix.*

Definition 7. *The* Kapranov rank *of a matrix $M \in \mathbb{R}^{d\times n}$ is the smallest dimension of any tropical linear space containing the columns of M.*

Definition 8. *A square matrix $M = (m_{ij}) \in \mathbb{R}^{r\times r}$ is* tropically singular *if the minimum in*

$$\begin{aligned}\det(M): &= \bigoplus_{\sigma\in\mathcal{S}_r} m_{1\sigma_1} \odot m_{2\sigma_2} \odot \cdots \odot m_{r\sigma_r}.\\ &= \min\{\, m_{1\sigma_1} + m_{2\sigma_2} + \cdots + m_{r\sigma_r} \,:\, \sigma \in \mathcal{S}_r \,\}.\end{aligned}$$

is attained at least twice. Here $\mathcal{S}_r$ denotes the symmetric group on $\{1, 2, \ldots, r\}$. The tropical rank *of a matrix $M \in \mathbb{R}^{d\times n}$ is the largest integer r such that M has a non-singular $r \times r$ minor.*

These three definitions are easily seen to agree for $r = 1$, but in general they are not equivalent:

Theorem 9. ([8]) *For every matrix M with entries in the tropical semiring $(\mathbb{R}, \oplus, \odot)$, we have*

$$\text{tropical rank}\,(M) \quad \le \quad \text{Kapranov rank}\,(M) \quad \le \quad \text{Barvinok rank}\,(M). \tag{9}$$

Both of these inequalities can be strict.

Next we discuss the relationship between the rank of a symmetric tropical square matrix and the degeneracy of the corresponding tropical quadric hypersurface. This is well known for an ordinary quadric hypersurface. We say a tropical quadric hypersurface to be *degenerate* if it contains an unbounded half hyperplane with multiplicity greater than or equal to 2, and *non-degenerate* otherwise. First, the following Lemma gives a sufficient condition for a given quadric hypersurface to be degenerate.

Lemma 10. *Let $\mathbf{a}_1, \cdots, \mathbf{a}_n$ be row vectors of an $n \times n$ tropical symmetric matrix A. If $\mathbf{a}_j = \lambda \odot \mathbf{a}_i$ for some $\lambda \in \mathbb{R}$ and $i \neq j$, then the tropical quadric hypersurface $\mathcal{T}(F)$ in $\mathbb{TP}^{n-1}$ determined by the tropical symmetric matrix A is degenerate.*

Proof. Let $\mathbf{a}_i = (a_{i1}, a_{i2}, \cdots, a_{in})$. Since $\mathbf{a}_i$ are row vectors of a tropical symmetric matrix A, $a_{ij} = a_{ji}$, $1 \le i, j \le n$. For simplicity, we may assume that $\mathbf{a}_2 = \lambda \odot \mathbf{a}_1$. Then

$$a_{12} = a_{21} = \lambda + a_{11},\ a_{22} = \lambda + a_{12} = \lambda + \lambda + a_{11} = 2\lambda + a_{11},\ \cdots$$

If we normalize the given quadric hypersurface with $x = 0$, then the defining equation is given by

$$a_{11} + a_{12}y + a_{22}y^2 + \cdots = a_{11} + (\lambda + a_{11})y + (2\lambda + a_{11})y^2 + \cdots$$

From this equation, if we apply Proposition 5, then we conclude that along y-axis, the height is proportional. So the lower face of the convex hull along y-axis has the length 2. So the corresponding half hyperplane has the multiplicity 2. Hence the given quadric hypersurface $\mathcal{T}(F)$ is degenerate. □

Proposition 11. *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

be an $n \times n$ symmetric tropical matrix, which is a matrix corresponding to a tropical quadric hypersurface $\mathcal{T}(F)$ in $\mathbb{TP}^{n-1}$. Then the tropical quadric hypersurface $\mathcal{T}(F)$ is non-degenerate if and only if each 2×2 minors of the matrix A along the diagonal is tropically non-singular and diagonally dominant, that is,

$$2\,a_{ij} < a_{ii} + a_{jj}, \quad \text{for each} \quad i < j\,.$$

Proof. (i) $\Rightarrow$ Suppose that the tropical quadric surface $\mathcal{T}(F)$ is non-degenerate and a 2×2 minor of the matrix A along the diagonal,

$$\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{pmatrix}$$

is tropically singular or $2\,a_{ij} > a_{ii} + a_{jj}$, for some $i < j$. Similar to the argument of the previous lemma, it follows that the edge of the regular subdivision of the support set Δ of F between vertices corresponding to x_i^2 and x_j^2 has the length 2. Hence $\mathcal{T}(F)$ is degenerate, which is contradict to the assumption.

(ii) $\Leftarrow$ Now suppose that each 2×2 minors of the matrix A along the diagonal is tropically non-singular and diagonally dominant. Then it is easy to check that this is equivalent to saying that each boundary edge of the regular subdivision of the support set Δ of F has length 1, which tells us that $\mathcal{T}(F)$ is non-degenerate. □

The relationship between the geometry of the tropical quadric hypersurfaces T(F) and the various ranks of its defining tropical matrix A remains to be further clarified.

Consider a non-degenerate tropical quadric in the tropical plane, which was described at Example 9.3 [18]:

$$f_1(x,y) = 0x^2 + 1xy + 0y^2 + 1x + 1y + 0.$$

The corresponding tropical symmetric matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

It is interesting to notice that A is tropically singular and the tropical rank of A is 2.

5. The Classification of Tropical Quadratic Plane Curves.

A tropical quadric curve in $\mathbb{TP}^2$ is defined by a tropical quadric polynomial

$$\begin{aligned} F &= a_1 \odot x \odot x \ \oplus \ a_2 \odot y \odot y \ \oplus \ a_3 \odot z \odot z \\ &\oplus a_4 \odot x \odot y \ \oplus \ a_5 \odot x \odot z \ \oplus \ a_6 \odot y \odot z, \ a_i \in \mathbb{R}, \end{aligned}$$

which is also represented by a 3×3 symmetric tropical matrix

$$\begin{pmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{pmatrix} \tag{10}$$

In this section we classify tropical quadric plane curves $\mathcal{T}(F)$. This is already done by examining the various cases coming from the definition of tropical hypersurface $\mathcal{T}(F)$ [16]. Here instead we use two programs, TOPCOM and GAP, and get the same result. The reason for doing the same classification is to make sure that this gives us the correct answer. And then we want to extend our computational method to the two dimensional case. That is, we would like to see how many different types of tropical quadric surfaces in $\mathbb{TP}^3$ exist, which will be shown at the next section.

TOPCOM is a package for computing triangulation of point configurations and oriented matroids [15], and GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory [10].

Let $\mathcal{A}$ be the support set of the tropical quadric polynomial F. We first use TOPCOM to compute all possible regular subdivisions of the support $\mathcal{A}$, the support set of a general tropical plane quadric curve. We set a file "con.dat" consisting of the lattice points of the support set $\mathcal{A}$.

```
[[2,0,0],[0,2,0],[0,0,2],[1,1,0],[1,0,1],[0,1,1]]
```

In the following, all computations were done under the Linux operating system. We use TOPCOM command to execute the data file "con.dat" and produce the result in a file "con.res" as follows:

```
points2triangs --regular < con.dat > con.res
```

The file "con.res" contains a list of all regular subdivisions of the support set $\mathcal{A}$ as follows:

```
T[1]:={{0,1,2}};
T[2]:={{0,2,5},{0,1,5}};
T[3]:={{1,2,4},{0,1,4}};
T[4]:={{1,2,3},{0,2,3}};
T[5]:={{1,2,3},{2,3,4},{0,3,4}};
T[6]:={{0,2,3},{2,3,5},{1,3,5}};
T[7]:={{1,2,4},{0,3,4},{1,3,4}};
T[8]:={{0,1,4},{2,4,5},{1,4,5}};
T[9]:={{0,2,5},{1,3,5},{0,3,5}};
T[10]:={{0,1,5},{2,4,5},{0,4,5}};
```

```
T[11]:={{1,3,5},{2,4,5},{0,3,5},{0,4,5}};
T[12]:={{2,3,4},{0,3,4},{2,3,5},{1,3,5}};
T[13]:={{0,3,4},{1,3,4},{2,4,5},{1,4,5}};
T[14]:={{0,3,4},{1,3,5},{2,4,5},{3,4,5}};
```

Here the integers $0, 1, 2, \cdots, 5$ are one of six lattice points of the support set $\mathcal{A}$:

$$0 = [2,0,0],\ 1 = [0,2,0],\ 2 = [0,0,2],\ 3 = [1,1,0], 4 = [1,0,1],\ 5 = [0,1,1].$$

In the above list of regular subdivisions of the support set $\mathcal{A}$ of a general quadric plane curve, there are several identical regular subdivisions due to symmetries. To find out a representative from a group of identical regular subdivisions, we use the program GAP. Since permutation groups in GAP do not act on 0, we increase the numbering by 1. The symmetry group is generated by the permutation of three vertices $1, 2, 3$ and the reflection of vertices 1 and 2. The symmetry group G is generated by two elements $(1,2,3)(4,6,5)$ and $(1,2)(5,6)$, that is,

$$G = \langle (1,2,3)(4,6,5), (1,2)(5,6) \rangle$$

Next we convert the above lists into GAP-objects as follows, and name it "con":

```
objects:=[
[[ 1,2,3 ]],
[[ 1,3,6 ],[ 1,2,6 ]],
[[ 2,3,5 ],[ 1,2,5 ]],
[[ 2,3,4 ],[ 1,3,4 ]],
[[ 2,3,4 ],[ 3,4,5 ],[ 1,4,5 ]],
[[ 1,3,4 ],[ 3,4,6 ],[ 2,4,6 ]],
[[ 2,3,5 ],[ 1,4,5 ],[ 2,4,5 ]],
```

```
[[ 1,2,5 ],[ 3,5,6 ],[ 2,5,6 ]],
[[ 1,3,6 ],[ 2,4,6 ],[ 1,4,6 ]],
[[ 1,2,6 ],[ 3,5,6 ],[ 1,5,6 ]],
[[ 2,4,6 ],[ 3,5,6 ],[ 1,4,6 ],[ 1,5,6 ]],
[[ 3,4,5 ],[ 1,4,5 ],[ 3,4,6 ],[ 2,4,6 ]],
[[ 1,4,5 ],[ 2,4,5 ],[ 3,5,6 ],[ 2,5,6 ]],
[[ 1,4,5 ],[ 2,4,6 ],[ 3,5,6 ],[ 4,5,6 ]]
];
```

We now execute GAP command to compute the orbit representatives under the action of group G on the set of regular subdivisions of the support set $\mathcal{A}$.

```
gap> Read("/home/GAP/con");

gap> objects:=List(objects,i->Set(List(i,Set)));

[ [ [ 1,2,3 ] ],
[ [ 1,2,6 ], [ 1,3,6 ] ],
[ [ 1,2,5 ], [ 2,3,5 ] ],
[ [ 1,3,4 ], [ 2,3,4 ] ],
[ [ 1,4,5 ], [ 2,3,4 ], [ 3,4,5 ] ],
[ [ 1,3,4 ], [ 2,4,6 ], [ 3,4,6 ] ],
[ [ 1,4,5 ], [ 2,3,5 ], [ 2,4,5 ] ],
[ [ 1,2,5 ], [ 2,5,6 ], [ 3,5,6 ] ],
[ [ 1,3,6 ], [ 1,4,6 ], [ 2,4,6 ] ],
[ [ 1,2,6 ], [ 1,5,6 ], [ 3,5,6 ] ],
[ [ 1,4,6 ], [ 1,5,6 ], [ 2,4,6 ], [ 3,5,6 ] ],
[ [ 1,4,5 ], [ 2,4,6 ], [ 3,4,5 ], [ 3,4,6 ] ],
[ [ 1,4,5 ], [ 2,4,5 ], [ 2,5,6 ], [ 3,5,6 ] ],
[ [ 1,4,5 ], [ 2,4,6 ], [ 3,5,6 ], [ 4,5,6 ] ] ]

gap> G:=Group((1,2,3)(4,6,5),(1,2)(5,6));
```

```
  Group([ (1,2,3)(4,6,5),(1,2)(5,6) ])

gap> orb:=Orbits(G,objects,OnSetsSets);

  [ [ [ [ 1, 2, 3 ] ] ],
    [ [ [ 1,2,5 ], [ 2,3,5 ] ], [ [ 1,3,4 ], [ 2,3,4 ] ],
      [ [ 1,2,6 ], [ 1,3,6 ] ] ],
  [ [ [ 1,2,5 ], [ 2,5,6 ], [ 3,5,6 ] ],
      [ [ 1,4,5 ], [ 2,3,4 ], [ 3,4,5 ] ],
      [ [ 1,2,6 ], [ 1,5,6 ], [ 3,5,6 ] ],
      [ [ 1,3,6 ], [ 1,4,6 ], [ 2,4,6 ] ],
      [ [ 1,3,4 ], [ 2,4,6 ], [ 3,4,6 ] ],
      [ [ 1,4,5 ], [ 2,3,5 ], [ 2,4,5 ] ] ],
  [ [ [ 1,4,5 ], [ 2,4,5 ], [ 2,5,6 ], [ 3,5,6 ] ],
      [ [ 1,4,5 ], [ 2,4,6 ], [ 3,4,5 ], [ 3,4,6 ] ],
      [ [ 1,4,6 ], [ 1,5,6 ], [ 2,4,6 ], [ 3,5,6 ] ] ],
  [ [ [ 1,4,5 ], [ 2,4,6 ], [ 3,5,6 ], [ 4,5,6 ] ] ] ]

gap> L:=List(orb, i->i[1]);

  [ [ [ 1,2,3 ] ], [ [ 1,2,5 ], [ 2,3,5 ] ],
    [ [ 1,2,5 ], [ 2,5,6 ], [ 3,5,6 ] ],
    [ [ 1,4,5 ], [ 2,4,5 ], [ 2,5,6 ], [ 3,5,6 ] ],
    [ [ 1,4,5 ], [ 2,4,6 ], [ 3,5,6 ], [ 4,5,6 ] ] ]

gap> Length(L);

  5

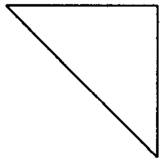
gap>
```

At last we get five non-isomorphic regular subdivisions of the support set $\mathcal{A}$:

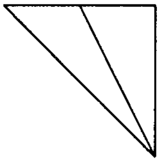
$\{[1,2,3]\}, \{[1,2,5],[2,3,5]\}, \{[1,2,5],[2,5,6],[3,5,6]\},$
$\{[1,4,5],[2,4,5],[2,5,6],[3,5,6]\}, \{[1,4,5],[2,4,6],[3,5,6],[4,5,6]\}$

Each of these regular subdivisions corresponds to the figures below, after a normalization with $z = 0$. These are exactly identical to the result described at [16]. We note that there are three different types of Euclidean plane conics (including degenerate cases), but there are five different tropical quadric plane curves (again including degenerate cases).

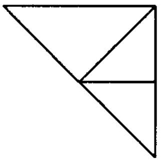
The bold lines represent rays with the multiplicity 2. For more detailed description on the multiplicity, we refer to [16]. We will call tropical curves containing half-rays without the multiplicity bigger than or equal to 2 to be *non-degenerate*, otherwise *degenerate*. Among the tropical quadric curves below, (d) and (e) are non-degenerate curves.



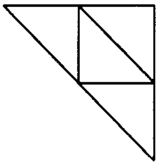
(a)



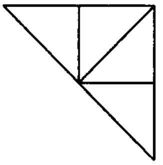
(b)



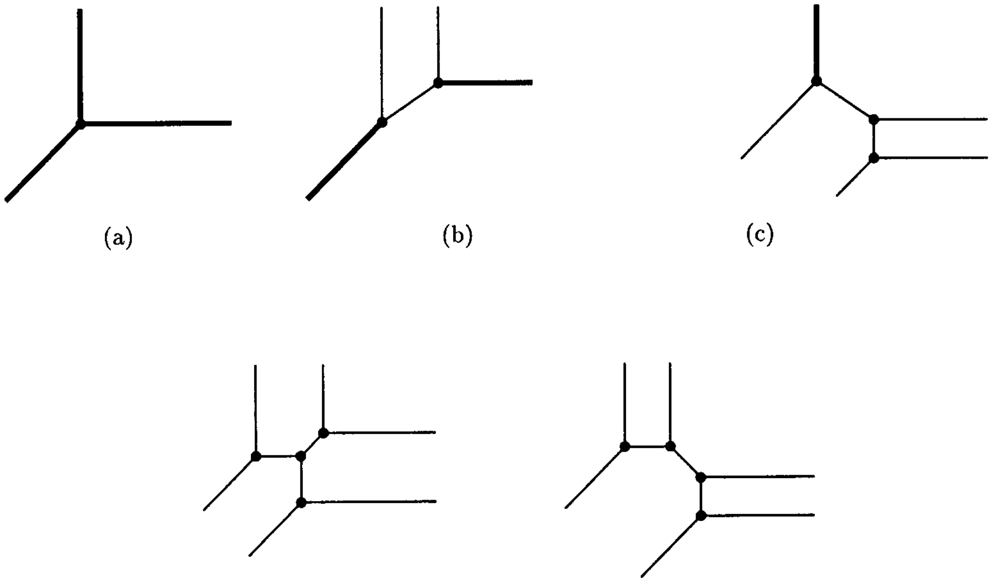
(c)



(d)



(e)



(a) (b) (c)

(d) (e)

6. The Classification of Tropical Quadric Surfaces.

Similarly to the previous classification of tropical quadric plane curves in $\mathbb{TP}^2$, we now intend to classify tropical quadric surfaces in $\mathbb{TP}^3$. It is not quite possible to classify tropical quadric surfaces without using computer, which will be clear later.

A tropical quadric surface in $\mathbb{TP}^3$ is defined by a tropical quadric polynomial

$F = a_1 \odot x \odot x \oplus a_2 \odot y \odot y \oplus a_3 \odot z \odot z \oplus a_4 \odot w \odot w \oplus a_5 \odot x \odot y \oplus a_6 \odot x \odot z \oplus a_7 \odot x \odot w \oplus a_8 \odot y \odot z \oplus a_9 \odot y \odot w \oplus a_{10} \odot z \odot w, \; a_i \in \mathbb{R}.$

Let $\mathcal{A}$ be the support set of the tropical quadric polynomial F. As in the case of tropical conic case, with lattice points of $\mathcal{A}$, we set a file "quad.dat" as follows:

```
[[2,0,0,0],[0,2,0,0],[0,0,2,0],[0,0,0,2],
[1,1,0,0],[1,0,1,0],[0,1,1,0],[1,0,0,1],[0,1,0,1],[0,0,1,1]]
```

If we apply TOPCOM command,

```
points2triangs --regular < quad.dat > quad.res
```

then the result file "quad.res" gives the following lists of regular subdivisions, all of which is too long to print out.

```
T[1]:={{0,1,2,3}};
T[2]:={{1,2,3,4},{0,2,3,4}};
T[3]:={{0,2,3,6},{0,1,3,6}};
T[4]:={{0,2,3,7},{0,1,2,7}};
T[5]:={{1,2,3,5},{0,1,3,5}};
T[6]:={{0,1,3,8},{0,1,2,8}};
T[7]:={{1,2,3,9},{0,1,2,9}};
    ⋮
T[946]:={{0,4,5,9},{3,7,8,9},{1,4,6,7},{2,5,6,8},{4,5,8,9},
{4,5,6,8},{4,6,7,8},{4,7,8,9}};
T[947]:={{0,4,5,9},{3,7,8,9},{1,4,6,7},{2,5,6,8},{5,6,8,9},
{4,5,6,9},{4,6,7,9},{6,7,8,9}};
T[948]:={{0,4,5,9},{3,7,8,9},{1,4,6,7},{2,5,6,8},{4,5,7,9},
{4,5,6,7},{5,6,7,8},{5,7,8,9}};
```

From this list, we see that there are all together 948 subdivisions. Here the integers $0, 1, 2, \cdots, 9$ are one of 10 lattice points of the support set $\mathcal{A}$:

$0 = [2,0,0,0]$, $1 = [0,2,0,0]$, $2 = [0,0,2,0]$, $3 = [0,0,0,2]$,
$4 = [1,1,0,0]$, $5 = [1,0,1,0]$, $6 = [0,1,1,0]$, $7 = [0,1,0,1]$,
$8 = [0,0,1,1]$, $9 = [1,0,0,1]$.

In the above list of regular subdivisions of the support set $\mathcal{A}$ of a general quadric surface in $\mathbb{TP}^3$, there are again several identical regular subdivisions due to symmetries. So we give an action by the symmetry group over the set of lists of the above subdivisions. For this purpose we again use the computer algebra program GAP. Since permutation groups in GAP do not act on 0, we increase the numbering by 1. The symmetry group G is generated by two elements $(1,2,3,4)(5,7,9,10)(6,8)$ and $(1,2)(6,7)(8,10)$, that is,

$$G = \langle (1,2,3,4)(5,7,9,10)(6,8), (1,2)(6,7)(8,10) \rangle$$

Similarly to the tropical conic case, we convert the above lists into GAP-objects, and name it "quad". Then we follow the same line of GAP command lines as before:

```
gap> Read("/home/GAP/con");

gap> objects:=List(objects,i->Set(List(i,Set)));
  [ [ [ 1,2,3,4 ] ],
  [ [ 1,3,4,5 ], [ 2,3,4,5 ] ],
          ⋮
  [ [ 1,5,6,10 ], [ 2,5,7,8 ], [ 3,6,7,9 ], [ 4,8,9,10 ], [
5,6,7,10 ], [ 5,7,8,10 ], [ 6,7,9,10 ],
        [ 7,8,9,10 ] ], [ [ 1,5,6,10 ], [ 2,5,7,8 ], [ 3,6,7,9
], [ 4,8,9,10 ], [ 5,6,7,8 ],
        [ 5,6,8,10 ], [ 6,7,8,9 ], [ 6,8,9,10 ] ] ]

gap> G:=Group((1,2,3,4)(5,7,9,10)(6,8),(1,2)(6,7)(8,10));

  Group([ ((1,2,3,4)(5,7,9,10)(6,8),(1,2)(6,7)(8,10) ])

gap> orb:=Orbits(G,objects,OnSetsSets);

gap> L:=List(orb, i->i[1]);
```

```
[ [ 1,2,3,4 ] ], [ [ 1,2,3,8 ], [ 1,3,4,8 ] ],
  [ [ 1,2,3,8 ], [ 1,3,8,9 ], [ 1,4,8,9 ] ],
  [ [ 1,2,3,8 ], [ 1,3,8,9 ], [ 1,8,9,10 ], [ 4,8,9,10 ] ],
  [ [ 1,2,6,8 ], [ 1,4,6,8 ], [ 2,3,6,8 ], [ 3,4,6,8 ] ],
  [ [ 1,2,6,8 ], [ 1,4,6,8 ], [ 2,3,6,8 ], [ 3,6,8,9 ],
    [ 4,6,8,9 ] ],
  [ [ 1,2,6,8 ], [ 1,4,6,8 ], [ 2,3,6,9 ], [ 2,6,8,9 ],
    [ 4,6,8,9 ] ],
  [ [ 1,2,6,8 ], [ 1,4,6,8 ], [ 2,6,7,8 ], [ 3,6,7,8 ],
    [ 3,6,8,9 ], [ 4,6,8,9 ] ],
  [ [ 1,2,6,8 ], [ 1,4,6,8 ], [ 2,6,7,8 ], [ 3,6,7,9 ],
    [ 4,6,7,8 ], [ 4,6,7,9 ] ],
  [ [ 1,2,6,8 ], [ 1,4,6,8 ], [ 2,6,7,8 ], [ 3,6,7,9 ],
    [ 4,6,8,9 ], [ 6,7,8,9 ] ],
  [ [ 1,2,6,8 ], [ 1,4,8,9 ], [ 1,6,8,9 ], [ 2,3,6,9 ],
    [ 2,6,8,9 ] ],
  [ [ 1,2,6,8 ], [ 1,4,8,9 ], [ 1,6,8,9 ], [ 2,6,7,8 ],
    [ 3,6,7,8 ], [ 3,6,8,9 ] ],
  [ [ 1,2,6,8 ], [ 1,4,8,9 ], [ 1,6,8,9 ], [ 2,6,7,8 ],
    [ 3,6,7,9 ], [ 6,7,8,9 ] ],
  [ [ 1,2,6,8 ], [ 1,4,8,9 ], [ 1,6,8,9 ], [ 2,6,7,9 ],
    [ 2,6,8,9 ], [ 3,6,7,9 ] ],
  [ [ 1,2,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,6,7,8 ],
    [ 3,6,7,8 ], [ 3,6,8,9 ], [ 4,8,9,10 ] ],
  [ [ 1,2,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,6,7,8 ],
    [ 3,6,7,9 ], [ 4,8,9,10 ], [ 6,7,8,9 ] ],
  [ [ 1,2,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,6,7,9 ],
    [ 2,6,8,9 ], [ 3,6,7,9 ], [ 4,8,9,10 ] ],
  [ [ 1,2,6,8 ], [ 1,6,8,10 ], [ 2,6,7,8 ], [ 3,4,6,7 ],
    [ 4,6,7,8 ], [ 4,6,8,10 ] ],
  [ [ 1,2,6,8 ], [ 1,6,8,10 ], [2,6,7,8 ], [ 3,4,6,8 ],
```

```
    [ 3,6,7,8 ], [ 4,6,8,10 ] ],
[ [ 1,2,6,8 ], [ 1,6,8,10 ], [ 2,6,7,8 ], [ 3,6,7,8 ],
    [ 3,6,8,9 ], [ 4,6,8,9 ], [ 4,6,8,10 ] ],
[ [ 1,2,6,8 ], [ 1,6,8,10 ], [ 2,6,7,8 ], [ 3,6,7,8 ],
    [ 3,6,8,9 ], [ 4,8,9,10 ], [ 6,8,9,10 ] ],
[ [ 1,2,6,8 ], [ 1,6,8,10 ], [ 2,6,7,8 ], [ 3,6,7,8 ],
    [ 3,6,8,10 ], [ 3,8,9,10 ], [ 4,8,9,10 ] ],
[ [ 1,2,6,8 ], [ 1,6,8,10 ], [ 2,6,7,8 ], [ 3,6,7,9 ],
    [ 4,8,9,10 ], [ 6,7,8,9 ], [ 6,8,9,10 ] ],
[ [ 1,2,6,9 ], [ 1,2,8,9 ], [ 1,4,8,9 ], [ 2,3,6,9 ] ],
[ [ 1,2,6,9 ], [ 1,2,8,9 ], [ 1,4,8,9 ], [ 2,6,7,9 ],
    [ 3,6,7,9 ] ],
[ [ 1,2,6,9 ], [ 1,2,8,9 ], [ 1,8,9,10 ], [ 2,6,7,9 ],
    [ 3,6,7,9 ], [ 4,8,9,10 ] ],
[ [ 1,2,6,9 ], [ 1,2,9,10 ], [ 2,3,6,9 ], [ 2,4,9,10 ] ],
[ [ 1,2,6,9 ], [ 1,2,9,10 ], [ 2,3,6,9 ], [ 2,8,9,10 ],
    [ 4,8,9,10 ] ],
[ [ 1,2,6,9 ], [ 1,2,9,10 ], [ 2,6,7,9 ], [ 2,8,9,10 ],
    [ 3,6,7,9 ], [ 4,8,9,10 ] ],
[ [ 1,2,6,10 ], [ 2,3,6,9 ], [ 2,4,9,10 ], [ 2,6,9,10 ] ],
[ [ 1,2,6,10 ], [ 2,3,6,9 ], [ 2,6,8,9 ], [ 2,6,8,10 ],
    [ 4,6,8,9 ], [ 4,6,8,10 ] ],
[ [ 1,2,6,10 ], [ 2,3,6,9 ], [ 2,6,8,9 ], [ 2,6,8,10 ],
    [ 4,8,9,10 ], [ 6,8,9,10 ] ],
[ [ 1,2,6,10 ], [ 2,3,6,9 ], [ 2,6,9,10 ], [ 2,8,9,10 ],
    [ 4,8,9,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,8 ], [ 2,6,8,10 ], [ 3,4,6,7 ],
    [ 4,6,7,8 ], [ 4,6,8,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,8 ], [ 2,6,8,10 ], [ 3,4,8,10 ],
    [ 3,6,7,8 ], [ 3,6,8,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,8 ], [ 2,6,8,10 ], [ 3,6,7,8 ],
```

```
    [ 3,6,8,9 ], [ 4,6,8,9 ], [ 4,6,8,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,8 ], [ 2,6,8,10 ], [ 3,6,7,8 ],
    [ 3,6,8,9 ], [ 4,8,9,10 ], [ 6,8,9,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,8 ], [ 2,6,8,10 ], [ 3,6,7,8 ],
    [ 3,6,8,10 ], [ 3,8,9,10 ], [ 4,8,9,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,8 ], [ 2,6,8,10 ], [ 3,6,7,9 ],
    [ 4,6,7,8 ], [ 4,6,7,9 ], [ 4,6,8,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,8 ], [ 2,6,8,10 ], [ 3,6,7,9 ],
    [ 4,6,8,9 ], [ 4,6,8,10 ], [ 6,7,8,9 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,8 ], [ 2,6,8,10 ], [ 3,6,7,9 ],
    [ 4,8,9,10 ], [ 6,7,8,9 ], [ 6,8,9,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,9 ], [ 2,6,8,9 ], [ 2,6,8,10 ],
    [ 3,6,7,9 ], [ 4,6,8,9 ], [ 4,6,8,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,9 ], [ 2,6,8,9 ], [ 2,6,8,10 ],
    [ 3,6,7,9 ], [ 4,8,9,10 ], [ 6,8,9,10 ] ],
[ [ 1,2,6,10 ], [ 2,6,7,9 ], [ 2,6,9,10 ], [ 2,8,9,10 ],
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[ [ 1,5,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,5,6,8 ],
    [ 2,6,7,8 ], [ 3,6,7,8 ], [ 3,6,8,9 ], [ 4,8,9,10 ] ],
[ [ 1,5,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,5,6,8 ],
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[ [ 1,5,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,5,6,8 ],
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[ [ 1,5,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,5,7,8 ],
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[ [ 1,5,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,5,7,8 ],
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[ [ 1,5,6,8 ], [ 1,6,8,9 ], [ 1,8,9,10 ], [ 2,5,7,8 ],
    [ 3,6,7,9 ], [ 4,8,9,10 ], [ 5,6,7,8 ], [ 6,7,8,9 ] ],
[ [ 1,5,6,8 ], [ 1,6,8,10 ], [ 2,5,6,8 ], [ 2,6,7,8 ],
    [ 3,6,7,8 ], [ 3,6,8,9 ], [ 4,6,8,9 ], [ 4,6,8,10 ] ],
```

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[[ 1,5,6,8 ], [ 1,6,8,10 ], [ 2,5,6,8 ], [ 2,6,7,8 ],
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[[ 1,5,6,8 ], [ 1,6,8,10 ], [ 2,5,6,8 ], [ 2,6,7,8 ],
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[[ 1,5,6,8 ], [ 1,6,8,10 ], [ 2,5,7,8 ], [ 3,5,6,8 ],
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[[ 1,5,6,8 ], [ 1,6,8,10 ], [ 2,5,7,8 ], [ 3,6,7,8 ],
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[[ 1,5,6,8 ], [ 1,6,8,10 ], [ 2,5,7,8 ], [ 3,6,7,9 ],
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[[ 1,5,6,10 ], [ 2,5,6,9 ], [ 2,5,9,10 ], [ 2,6,7,9 ],
   [ 2,8,9,10 ], [ 3,6,7,9 ], [ 4,8,9,10 ], [ 5,6,9,10 ]],
[[ 1,5,6,10 ], [ 2,5,6,10 ], [ 2,6,7,9 ], [ 2,6,9,10 ],
   [ 2,8,9,10 ], [ 3,6,7,9 ], [ 4,8,9,10 ] ],
[[ 1,5,6,10 ], [ 2,5,7,8 ], [ 3,6,7,9 ], [ 4,8,9,10 ],
   [ 5,6,7,8 ], [ 5,6,8,10 ], [ 6,7,8,9 ], [ 6,8,9,10 ]]]

gap> Length(L);

  59

gap>
```

From this computation, we conclude that there are 59 types of tropical quadric surfaces in $\mathbb{TP}^3$. Our next aim is to get this same number through a mathematical proof, and further extend to the higher dimensional case. We also tried to do the same computation for the tropical quadric 3-folds in $\mathbb{TP}^4$. However TOPCOM was not able to produce the lists of the regular subdivisions in a reasonable time. Lastly we summarize our computational results in the following claim.

Claim 12. *The number of different types of tropical quadric surfaces in $\mathbb{TP}^3$ is 59. The number of non-degenerate tropical quadrics is 14.*

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