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MULTIPLICATION MODULES OVER PULLBACK RINGS (I)

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Abstract. First, we give a complete description of the multiplication modules over local Dedekind domains. Second, if R is the pullback ring of two local Dedekind domains over a common factor field then we give a complete description of separated multiplication modules over R.

0. Introduction

Throughout this paper *all rings will be commutative rings with non-zero identities and all modules will be unitary.* Let R be a commutative ring and M an R-module. Then M is called a *multiplication module* if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take

$$I = (N :_R M) = \{r \in R : rM \subseteq N\}.$$

Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\bar{R}$. Denote the pullback

$$R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\} \tag{1}$$

by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i by P_i for $i = 1, 2$ and denote the kernel

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of the homomorphism $v : R \longrightarrow \bar{R}$ by P. Then $P = P_1 \oplus P_2$ and

$$\begin{array}{ccccc} & & & & R_1/P_1 \\ & & & \cong & \\ R/P & \cong & \bar{R} & & \\ & & & \cong & \\ & & & & R_2/P_2 \end{array}$$

Since $P_1P_2 = P_2P_1 = 0$, R is not an integral domain. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$). Furthermore, for $i \neq j$, the sequence $0 \longrightarrow P_i \longrightarrow R \longrightarrow P_j \longrightarrow 0$ is an exact sequence of R-modules (see [6].)

An R-module S is called to be *separated* if there exists an R_i-module S_i, $i = 1, 2$, such that S is an R-submodule of $S_1 \oplus S_2$. Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for those of rings,

$$S = (S/P_2S \rightarrow S/PS \leftarrow S/P_1S)$$

[6, Corollary 3.3] and $S \leq (S/P_2S) \oplus (S/P_1S)$. Also, S is separated if and only if $P_1S \cap P_2S = 0$ [6, Lemma 2.9].

A *separated representation* of an R-module M is an R-module epimorphism $\varphi : S \longrightarrow M$ such that S is separated and such that, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \twoheadrightarrow M$ with S' separated, then f is one-to-one. Assume that $\varphi : S \longrightarrow M$ is a separated representation. If M is finitely generated, so is S [6, Corollary 2.10]. An exact sequence $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [6, Proposition 2.3]. Every module has a separated representation, which is unique up to isomorphism [6, Theorem 2.8].

1. Multiplication modules over Dedekind domains

The purpose of this section is to give a complete description of local multiplication modules over Dedekind domains.

Lemma 1.1. *If M is a non-zero multiplication module over a quasi-local ring R, then the R-module M is indecomposable.*

Proof. Let R be a quasi-local ring with unique maximal ideal Q. Assume that $M = A \oplus B$, where A and B are submodules of the R-module M. Since M is a multiplication module, there exist ideals I and J of R such that $A = IM$ and $B = JM$.

Suppose that $I \neq R$ and $J \neq R$. Then $I \subseteq Q$ and $J \subseteq Q$. This implies that $M = A + B = IM + JM \subseteq QM$, so $M = QM$. By [1, Proposition 1], $M = 0$, a contradiction. Hence, either $I = R$ or $J = R$.

If $I = R$, then $B = M \cap B = IM \cap B = A \cap B = 0$. Or, if $J = R$, then $A = A \cap M = A \cap JM = A \cap B = 0$.

Therefore, M is indecomposable. □

Compare Proposition 1.2 with [7, Theorem 2.8].

Proposition 1.2. *If M is a non-zero finitely generated multiplication module over a commutative ring R, then the R_P-module M_P is an indecomposable multiplication module for all prime/maximal ideals P of R.*

Proof. Let P be any prime/maximal ideal of R. Then by [1, Lemma 2], the R_P-module M_P is a multiplication module. Since R_P is a local ring with unique maximal ideal PR_P, it follows from Lemma 1.1 that the R_P-module M_P is indecomposable. □

Lemma 1.3. *If M is a multiplication module over a commutative ring R, then $Ann_M I = (Ann_R M :_R I)M$ for any ideal I of R.*

Proof. Assume that M is a multiplication module over a commutative ring R. Let I be any ideal of R. Then $\mathrm{Ann}_M I = (\mathrm{Ann}_M I :_R M)M$. Notice that

$$a \in \mathrm{Ann}_M I :_R M \Longleftrightarrow aM \subseteq \mathrm{Ann}_M I \Longleftrightarrow aIM = 0$$
$$\Longleftrightarrow aI \subseteq \mathrm{Ann}_R M \Longleftrightarrow a \in \mathrm{Ann}_R M :_R I.$$

Then $\mathrm{Ann}_M I :_R M = \mathrm{Ann}_R M :_R I$. Hence, $\mathrm{Ann}_M I = (\mathrm{Ann}_R M :_R I)M$, as required. □

Let M be a multiplication module over a commutative domain R. Then M is not necessarily faithful over R. The example of this is given below.

Example 1.4. Let R be a local domain with unique maximal ideal $P \neq 0$ and let $M = R/P^2$. Then M is a cyclic R-module and so it is a multiplication module over R. However, $\mathrm{Ann}_R M = P^2 \neq 0$. □

Compare the following result with [5, Lemma 4.1].

Proposition 1.5. *Let M be a faithful multiplication module over a commutative domain R. Then the following are true:*

(1) *$\mathrm{Ann}_M I = 0$ for any non-zero ideal I of R.*
(2) *M is a torsion-free R-module.*
(3) *M can be regarded as a submodule of the localization $M_{(0)}$ at the zero ideal (0), which is prime, of R.*

Proof. (1) Assume that M is a faithful multiplication module over a commutative domain R. Then for any non-zero ideal I of R,

$$(\mathrm{Ann}_R M :_R I) = 0 :_R I = 0.$$

By Lemma 1.3, $\mathrm{Ann}_M I = (\mathrm{Ann}_R M :_R I)M = 0M = 0$.

(2) Assume $rm = 0$, where $0 \neq r \in R$ and $m \in M$. Then by (1), $m \in \mathrm{Ann}_M r = 0$. Hence, M is torsion-free.

(3) Define a map $\varphi : M \longrightarrow M_{(0)}$ by $\varphi(m) = m/1$, where $m \in M$. Then it is clear that φ is an R-homomorphism. Assume that $m/1 = 0$, where $m \in M$. Then there exists an element $s \in R\backslash(0)$ such that $sm = 0$. By (2), $m = 0$. Hence, φ is a monomorphism. □

A *Dedekind domain* is a commutative domain with the property that every non-zero fractional ideal is invertible. Every integral ideal of a commutative domain is a fractional ideal. Let R be an integral domain. Then it is well-known that the following are equivalent:

(1) R is a Dedekind domain.
(2) R is integrally closed and Noetherian, and every proper prime ideal of R is maximal.
(3) Every proper ideal of R is uniquely a product of maximal ideals.

Lemma 1.6. *Let M be a non-zero multiplication over a local Dedekind domain with unique maximal ideal Q. Assume that M is faithful over R. Then the following are true:*

(1) There is an element x in M uniquely determined by units of R such that $M = Rx$. Further, $M \cong R$.
(2) Every non-zero submodule of M is of the form $Q^n x$, where n is a non-negative integer.

Proof. (1) ((Existence)) By [1, Proposition 4], M is cyclic. There exists an element $x \in M$ such that $M = Rx$. Further, x is non-zero.

((Uniqueness)) Assume that there exist elements $x, y \in M$ such that $M = Rx$ and $M = Ry$. Then $Rx = Ry$. $x \in Ry$, so there exists an element $a \in R$ such that $x = ay$. Similarly, there exists an element $b \in R$ such that $y = bx$. $(1 - ab)x = 0$. By Proposition 1.5 (2), $1 - ab = 0$. Hence, a and b are units of R. Further, we have $M = Rx \cong R/\mathrm{Ann}_R(x) = R/\mathrm{Ann}_R(M) = R/0 \cong R$.

(2) Let N be a non-zero submodule of M. Then there exists a non-zero ideal I of R such that $N = IM$. Since R is local Dedekind with maximal ideal Q, there is a non-negative integer n such that $I = Q^n$.

Hence

$$N = IM = Q^n Rx = Q^n x.$$

Conversely, suppose that there is a non-negative integer n such that $Q^n x = 0$. Then $x \in \mathrm{Ann}_M(Q^n) = 0$ by Proposition 1.5 (1). This contradiction shows that for every non-negative integer n, $Q^n x$ is a non-zero submodule of M. □

Theorem 1.7. *Let M be a non-zero multiplication module over a local Dedekind domain with unique maximal ideal Q. Then the only one of the following two statements holds:*

(1) *There is a positive integer n such that $M \cong R/Q^n$.*

(2) *$M \cong R$.*

Proof. Assume that M is a non-zero multiplication module over a local Dedekind domain with unique maximal ideal Q. Then M is a non-zero multiplication module over the local ring R, so by [1, Proposition 4], there exists an element $x \in M$ such that $M = Rx$.

(1) Assume that $\mathrm{Ann}_R(M) \neq 0$. Then there exists a non-zero element $r \in R$ such that $rM = 0$. So, $rx = 0$. This implies $\mathrm{Ann}_R x \neq 0$. Since R is local Dedekind with maximal Q, there is a positive integer n such that $\mathrm{Ann}_R x = Q^n$. Hence, $M = Rx \cong R/\mathrm{Ann}_R x = R/Q^n$.

(2) Or, assume that $\mathrm{Ann}_R(M) = 0$. By Lemma 1.6 (1), $M \cong R$. □

2. The Separated Case

The aim of this section is to give a complete description of the separated multiplication R-modules where R is the pullback ring as described in (1)

Lemma 2.1. *Let R and R' be any commutative rings, $f : R \to R'$ a ring homomorphism, and M an R'-module. If f is surjective and M is a multiplication R'-module, then M is a multiplication R-module.*

Proof. Since f is a surjective homomorphism, we can give M an R-module structure. Let N be any R-submodule of M. Then N is an R'-submodule of M. So, $N = I'M$ for some ideal I' of R'. Set $I = f^{-1}(I')$. Then I is an ideal of R and

$$f(I) = f(f^{-1}(I')) = I' \cap f(R) = I' \cap R' = I'.$$

Hence, $IM = f(I)M = I'M = N$, as required. □

Let R be the pullback ring as described in (1) and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module. Suppose that π_i is the projection map of R onto R_i. If for each $i \in \{1, 2\}$, S_i is a multiplication R_i-module, then it follows from Lemma 2.1 that each S_i is a multiplication R-module.

Lemma 2.2. *Let R be a commutative ring and I an ideal of R. Let M be a multiplication R-module and let N be an R-submodule of M such that $I \subseteq (N :_R M)$. Then M/N is a multiplication R/I-module.*

Proof. Let L be any submodule of M such that $N \subseteq L$. Then $(L :_R M)M = L$ since M is a multiplication R-module.

Clearly, $(L/N :_{R/I} M/N)M/N \subseteq L/N$. Conversely, let l be any element of L. Then there exist elements $a_1, \cdots, a_n \in L :_R M$ and elements $x_1, \cdots, x_n \in M$ such that $l = a_1x_1 + \cdots a_nx_n$. So, $a_1 + I, \cdots, a_n + I \in R/I$ and $x_1 + N, \cdots, x_n + N \in M/N$. Further, for each $i \in \{1, \cdots, n\}$, $(a_i + I)M/N = (a_iM + N)/N \subseteq (L + N)/N = L/N$, and so $a_i + I \in L/N :_{R/I} M/N$. This implies

$$\begin{aligned} l + N &= a_1x_1 + \cdots + a_nx_n + N \\ &= (a_1 + I)(x_1 + N) + \cdots + (a_n + I)(x_n + N) \\ &\in (L/N :_{R/I} M/N)M/N \end{aligned}$$

Hence, $L/N \subseteq (L/N :_{R/I} M/N)M/N$. Thus, $L/N = (L/N :_{R/I} M/N)M/N$.

Therefore, M/N is a multiplication R/I-module. □

Corollary 2.3. *If M is a multiplication module over a commutative ring R, then for every submodule N of M, the R-module M/N is a multiplication module.*

Proof. Take $I = 0$ in Lemma 2.2. □

Let N be an R-submodule of M. Then N is said to be *pure* in M if any finite system of equations over N which is solvable in M is also solvable in N. It is well-known that every direct summand of a module over a commutative ring is pure.

We can use Corollary 2.3 to see that every direct summand of a multiplication module is also a multiplication module. This can be proved alternatively by making use of the notion of a pure submodule of a module as follows.

Lemma 2.4. *Let M be a multiplication module over a commutative ring R. Then the following are true.*

(1) *If N is a pure submodule of M, then N is a multiplication module.*
(2) *Every direct summand of a multiplication module over a commutative ring is also a multiplication module.*

Proof. (1) Let K be any submodule of N. Then K is a submodule of M, so there exists an ideal I of R such that $K = IM$. Clearly, $IN \subseteq N \cap IM$.

Conversely, let $x \in N \cap IM$. Then there are elements $a_1, a_2, \cdots, a_r \in I$ and elements $x_1, x_2, \cdots, x_r \in M$ such that $x = a_1x_1 + a_2x_2 + \cdots + a_rx_r$. Since N is pure, we must have $x_1, x_2, \cdots, x_r \in N$. Hence,

$$x = a_1x_1 + a_2x_2 + \cdots + a_rx_r \in IN.$$

This shows that $N \cap IM \subseteq IN$. Therefore,

$$IN = N \cap IM = N \cap K = K.$$

Consequently, N is a multiplication module over R.

(2) Let N be any direct summand of M. Then as we have already known, N is pure in M. Therefore, by (1), N is a multiplication module. □

A module N is said to be *pure-injective* if any (infinite) system of equations (allowing infinitely many indeterminates) in N which is finitely solvable in N is solvable in N (see [7, Theorem 2.8, p.28]).

Theorem 2.5. *Let M be a non-zero multiplication module over a Dedekind domain R. If M is not faithful over R, then M is pure-injective.*

Proof. By [2, Proposition 2.10], M is Noetherian. Since $\text{Ann}_R(M)$ is a proper ideal of a Dedekind domain R, there are finitely many maximal ideals $Q_1, Q_2, \cdots, Q_n$ of R such that $\text{Ann}_R(M) = Q_1Q_2\cdots Q_n$. So,

$$Q_1Q_2\cdots Q_nM = \text{Ann}_R(M)M = 0.$$

By [8, Theorem 7.30], M is an Artinian module over R. Since M satisfies the a.c.c. and the d.c.c., M has a finite length. By [3, p.4064], M is pure-injective. □

Lemma 2.6. *Let R be the pullback ring as described in (1) and M a non-zero multiplication module over R. Then the following are true:*

(1) *If $(P_1 \oplus 0 + \text{Ann}_R(M)) \cap (0 \oplus P_2 + \text{Ann}_R(M)) = 0$, then M is separated.*

(2) *If either $\text{Ann}_R(M) \subseteq P_1 \oplus 0$ or $\text{Ann}_R(M) \subseteq 0 \oplus P_2$, then M is separated.*

(3) *If M is faithful over R, then it is separated.*

Proof. Let M be a non-zero multiplication module over the ring R. Then by [5, Corollary 1.7],

$$(P_1 \oplus 0)M \cap (0 \oplus P_2)M = ((P_1 \oplus 0 + \text{Ann}_R(M)) \cap (0 \oplus P_2 + \text{Ann}_R(M)))M.$$

(1) Assume that $(P_1 \oplus 0 + \mathrm{Ann}_R(M)) \cap (0 \oplus P_2 + \mathrm{Ann}_R(M)) = 0$. Then

$$(P_1 \oplus 0)M \cap (0 \oplus P_2)M = ((P_1 \oplus 0 + \mathrm{Ann}_R(M)) \cap (0 \oplus P_2 + \mathrm{Ann}_R(M)))M = 0.$$

Hence, by [6, Lemma 2.9], M is separated.

(2) We may assume that $\mathrm{Ann}_R(M) \subseteq P_1 \oplus 0$ since the proof of the other is similar. Then by the Modular Law,

$$\begin{aligned}(P_1 \oplus 0)M \cap (0 \oplus P_2)M &= ((P_1 \oplus 0 + \mathrm{Ann}_R(M)) \cap (0 \oplus P_2 + \mathrm{Ann}_R(M)))M \\ &= (((P_1 \oplus 0 + \mathrm{Ann}_R(M)) \cap (0 \oplus P_2)) + \mathrm{Ann}_R(M))M \\ &= ((P_1 \oplus 0 + \mathrm{Ann}_R(M)) \cap (0 \oplus P_2))M \\ &= ((P_1 \oplus 0) \cap (0 \oplus P_2))M \\ &= 0M \\ &= 0.\end{aligned}$$

Hence, by [6, Lemma 2.9] again, M is separated.

(3) If M is faithful, then it follows from (1) or (2) that M is separated. □

Let R be the pullback ring of two local Dedekind domains R_1, R_2. Assume that S is a separated R-module of an R_1-module S_1 and an R_2-module S_2. If for each $i \in \{1,2\}$, S_i is a non-zero faithful multiplication module over R_i, then by Lemma 2.6, for each $i \in \{1,2\}$, S_i is separated. Every non-zero faithful multiplication module over R is also separated since R is a local ring (see Section 0.)

Lemma 2.7. *Let R be the pullback ring of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2. Assume that S is a separated R-module of an R_1-module S_1 and an R_2-module S_2. If S is a non-zero multiplication module over R, then for each $i \in \{1,2\}$, S_i is a non-zero multiplication module over R_i. The converse holds provided that either $P_1S_1 = 0$ or $P_2S_2 = 0$.*

Proof. Suppose that S is a non-zero multiplication module over R. $(0 \oplus P_2)S \subseteq S$ and $0 \oplus P_2 \subseteq ((0 \oplus P_2)S :_R S)$. By Lemma 2.2, $S/(0 \oplus P_2)S$ is a multiplication $R/0 \oplus P_2$-module. Further,

$$S_1 \cong S/(0 \oplus P_2)S \text{ and } R_1 \cong R/0 \oplus P_2 .$$

Hence, S_1 is a multiplication module over R_1. By a similar proof, we can show that S_2 is a multiplication module over R_2.

Conversely, we may assume that $P_2S_2 = 0$ since the proof of the other is similar. Assume that S_1 is a non-zero multiplication module over R_1 and that S_2 is a non-zero multiplication module over R_2. By [1, Proposition 4], there exists an element $s_1 \in S_1$ and an element $s_2 \in S_2$ such that $S_1 = R_1s_1$ and $S_2 = R_2s_2$. There exists an element $s_2' \in S_2$ such that $f_1(s_1) = f_2(s_2')$. Then $(s_1, s_2') \in S$. Hence, $R(s_1, s_2') \subseteq S$. Conversely, let $(u, v) \in S$. Then there exists an element $r_1 \in R_1$ and an element $r_2 \in R_2$ such that $u = r_1s_1$ and $v = r_2s_2$. There exists an element $r_2' \in R_2$ such that $v_1(r_1) = v_2(r_2')$. Then $(r_1, r_2') \in R$. Since $(u, v) \in S$, we have

$$f_2(r_2's_2') = v_2(r_2')f_2(s_2') = v_1(r_1)f_1(s_1) = f_1(r_1s_1) = f_1(u) = f_2(v).$$

This implies $v - r_2's_2' \in \mathrm{Ker}(f_2) = P_2S_2 = 0$. So, $v = r_2's_2'$. Thus,

$$(u, v) = (r_1s_1, r_2's_2') = (r_1, r_2')(s_1, s_2') \in R(s_1, s_2').$$

This shows that $S \subseteq R(s_1, s_2')$. Therefore, $S = R(s_1, s_2')$. By [1, Proposition 4] again, S is a multiplication module over R. □

Let R be the pullback ring as described in (1). Here is a list of indecomposable separated R-modules (see [3, Lemma 2.8]): for all positive integers n, m, $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$.

Proposition 2.8. *Let R be the pullback ring as described in (1). Then for all positive integers n, m such that either n or m is 1, $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ is a multiplication R-module.*

Proof. Assume that either n or m is 1. If $n = 1$, then $P_1(R_1/P_1^n) = 0$. Or, if $m = 1$, then $P_2(R_2/P_2^m) = 0$. Since R_1 is a multiplication module over R_1, it follows from Corollary 2.3 that the R_1-module R_1/P_1^n is a multiplication module. Similarly, the R_2-module R_2/P_2^m is a multiplication module. Hence, by Lemma 2.7, S is a multiplication module over R. □

Compare Theorem 2.9 with [4, Proposition 2.3].

Theorem 2.9. *Let R be the pullback ring as described in (1). Assume that S is a non-zero faithful multiplication module over R. Then the following are true:*

(1) *S is indecomposable.*

(2) *S is isomorphic to one of the following:*

(a) R.

(b) $(R_1 \to \bar{S} \leftarrow R_2/P_2^k)$.

(c) $(R_1/P_1^m \to \bar{S} \leftarrow R_2)$.

(d) $(R_1/P_1^m \to \bar{S} \leftarrow R_2/P_2^k)$.

Here, m and k are positive integers.

Proof. (1) Since S is a non-zero multiplication module over a local ring R, it follows from Lemma 1.1 that S is indecomposable.

(2) By Lemma 2.6, S is separated. There exists an R_1-module S_1 and R_2-module S_2 such that $S = (S_1 \to \bar{S} \leftarrow S_2)$. S is a non-zero multiplication module over R. By Lemma 2.7, S_i is a multiplication module over R_i for each $i \in \{1, 2\}$. By Theorem 1.7, $S_1 \cong R_1$ or $S_1 \cong R_1/P_1^m$ for some positive integer m, and $S_2 \cong R_2$ or $S_2 \cong R_2/P_2^k$ for some positive integer k. Hence, the results follows. □

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