Honam Mathematical J. **28** (2006), No. 1, pp. 45–56

DEDUCTIVE SYSTEMS OF BL-ALGEBRAS

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Abstract. We give characterizations of a deductive system of a BL-algebra, and discuss how to generate a deductive system by a set.

1. Introduction

Fuzzy logic grows as a new discipline from the necessity to deal with vague data and imprecise information caused by the indistinguishability of objects in certain experimental environments. As mathematical tools fuzzy logic is only using $[0,1]$-valued maps and certain binary operations $*$ on the real unit interval $[0,1]$ known also as left-continuous t-norms. It took sometime to understand partially ordered monoids of the form $([0,1], \leq, *)$ as *algebras* for $[0,1]$-valued interpretations of a certain type of non-classical logic–the so-called monoidal logic. BL-algebras arise naturally in the analysis of the proof theory of propositional fuzzy logics. Indeed, in [3], Hájek introduced the system of basic logic (BL) axioms for propositional logic and defined the class of BL-algebras (see Definition 2.1). In [4], Ko and Kim investigated some properties of BL-algebras, and they [5] also studied relationships between closure operators and BL-algebras.

In this paper, we show that a deductive system of a BL-algebra can be represented as a union of special sets, and give a characterization

Received February 17, 2006. Revised March 15, 2006.

2000 Mathematics Subject Classification : 03B52, 94D05, 06D35.

Key words and phrases :(implicative, power) BL-algebra, deductive system.

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of a deductive system of a BL-algebra. We discuss how to generate a deductive system by a set, and investigates some related properties.

2. Preliminaries

A lattice $(L; \leq, \wedge, \vee, \odot, \rightsquigarrow, 0, 1)$ is called a *residuated lattice* if it satisfies the following conditions:

(R1) $(L; \odot, 1)$ is a commutative monoid,

(R2) $(\forall x, y, z \in L)\ (x \leq y \Rightarrow x \odot z \leq y \odot z)$,

(R3) $(\forall x, y, z \in L)\ (x \odot y \leq z \Leftrightarrow x \leq y \rightsquigarrow z)$.

Definition 2.1. [3] A *BL-algebra* is a residuated lattice $(L;\leq,\wedge,\vee,\odot,\rightsquigarrow,0,1)$ that satisfies the following conditions:

(B1) $(\forall x, y \in L)\ (x \wedge y = x \odot (x \rightsquigarrow y))$,

(B2) $(\forall x, y \in L)\ (x \vee y = ((x \rightsquigarrow y) \rightsquigarrow y) \wedge ((y \rightsquigarrow x) \rightsquigarrow x))$,

(B3) $(\forall x, y \in L)\ ((x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1)$.

Example 2.2. [4] Let X be a nonempty set and let $\mathcal{P}(X)$ be the family of all subsets of X. Define operations $\odot$ and $\rightsquigarrow$ by

$$A \odot B = A \cap B \text{ and } A \rightsquigarrow B = A^c \cup B$$

for all $A, B \in \mathcal{P}(X)$, respectively. Then $(\mathcal{P}(X), \subset, \cap, \cup, \odot, \rightsquigarrow, \emptyset, X)$ is a BL-algebra.

We call $\mathcal{P}(X)$ the *power BL-algebra* of X.

Proposition 2.3. [3, 6] *In a BL-algebra $(L; \leq, \wedge, \vee, \odot, \rightsquigarrow, 0, 1)$, we have the following properties:*

(p1) $(\forall x \in L)\ (x = 1 \rightsquigarrow x)$,

(p2) $(\forall x \in L)\ (1 = x \rightsquigarrow x)$,

(p3) $(\forall x, y \in L)\ (x \odot y \leq x, y)$,

(p4) $(\forall x, y \in L)\ (x \odot y \leq x \wedge y)$,

(p5) $(\forall x, y \in L)\ (y \leq x \rightsquigarrow y)$, (p6) $(\forall x, y \in L)\ (x \odot y \leq x \rightsquigarrow y)$,

(p7) $(\forall x, y \in L)\ (x \leq y \Leftrightarrow 1 = x \rightsquigarrow y)$,

(p8) $(\forall x, y \in L)\ (x = y \Leftrightarrow 1 = x \rightsquigarrow y = y \rightsquigarrow x)$,

(p9) $(\forall x, y \in L)\ (x \odot (x \rightsquigarrow y) \leq y)$,

(p10) $(\forall x, y, z \in L)\ (x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z))$.

(p11) $(\forall x, y, z \in L)\ ((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \leq x \rightsquigarrow (y \rightsquigarrow z))$,

(p12) $(\forall x, y, z \in L)\ (x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y))$,

(p13) $(\forall x, y, z \in L)\ (x \leq y \Rightarrow z \rightsquigarrow x \leq z \rightsquigarrow y,\ y \rightsquigarrow z \leq x \rightsquigarrow z)$.

3. Deductive systems of BL-algebras

We begin with the following inequality in a BL-algebra L:

(p14) $(\forall x, y, z \in L)\ (x \rightsquigarrow (y \rightsquigarrow z) \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z))$.

The following example shows that the inequality (p14) does not hold in a BL-algebra.

Example 3.1. In the Łukasiewicz structure $([0,1], \leq, \min, \max, \odot, \rightsquigarrow, 0, 1)$ which is a BL-algebra (see [4]) where

$$x \odot y = \max\{0, x + y - 1\} \text{ and } x \rightsquigarrow y = \min\{1, 1 - x + y\},$$

we have $0.5 \rightsquigarrow (0.4 \rightsquigarrow 0.3) \nleq (0.5 \rightsquigarrow 0.4) \rightsquigarrow (0.5 \rightsquigarrow 0.3)$.

Definition 3.2. A BL-algebra L satisfying the inequality (p14) is said to be *implicative.*

Example 3.3. The power BL-algebra $\mathcal{P}(X)$ of a set X is an implicative BL-algebra.

In what follows, let L denote a BL-algebra unless otherwise specified. For every $a_1, a_2, \cdots, a_n \in L$, we define

$$P(a_1, a_2, \cdots, a_{n-1} \backslash a_n) := \begin{cases} a_n & \text{if } n = 1, \\ a_1 \rightsquigarrow P(a_2, a_3, \cdots, a_{n-1} \backslash a_n) & \text{if } n > 1. \end{cases}$$

Definition 3.4. [2, 6] A subset D of L is called a *deductive system* of L if it satisfies the following conditions:

(ds1) $1 \in D$,

(ds2) $(\forall x, y \in L)$ $(x \in D,\ P(x\backslash y) \in D \Rightarrow y \in D)$.

Lemma 3.5. [4] *Let D be a nonempty subset of L. Then D is a deductive system of L if and only if it satisfies:*

(ds3) $(\forall a, b \in D)$ $(a \odot b \in D)$,

(ds4) $(\forall a \in D)(\forall b \in L)$ $(a \leq b \Rightarrow b \in D)$.

Theorem 3.6. *If D is a deductive system of L, then*

(i) $(\forall x \in L)(\forall y \in D)$ $(P(x\backslash y) \in D)$,

(ii) $(\forall x \in L)(\forall y_1, y_2 \in D)$ $(P(P(y_1, y_2\backslash x)\backslash x) \in D)$.

Proof. (i) Let $x \in L$ and $y \in D$. Then

$$P(y, x\backslash y) = P(x, y\backslash y) = P(x\backslash 1) = 1 \in D,$$

and so $P(x\backslash y) \in D$.

(ii) Let $x \in L$ and $y_1, y_2 \in D$. Then

$$\begin{aligned} y_1 &\leq P(P(y_1\backslash x)\backslash x) && \text{by (p9)} \\ &\leq P(P(y_2, y_1\backslash x)\backslash P(y_2\backslash x)) && \text{by (p12)} \\ &= P(y_2\backslash P(P(y_2, y_1\backslash x)\backslash x)) && \text{by (p10).} \end{aligned}$$

It follows from (p10), Lemma 3.5 and (ds2) that

$$P(P(y_1, y_2\backslash x)\backslash x) = P(P(y_2, y_1\backslash x)\backslash x) \in D.$$

This completes the proof. □

Definition 3.7. For any $a, b \in L$, we define

$$\mathfrak{L}(a, b) := \{x \in L \mid P(a, b\backslash x) = 1\}.$$

Obviously $a, b, 1 \in \mathfrak{L}(a, b)$ for all $a, b \in L$.

Theorem 3.8. *Let X be a nonempty set and let $\mathcal{P}(X)$ be the power BL-algebra of X. Then for every subsets A and B of X, we have*

$$\mathfrak{L}(A,B) = \begin{cases} \mathcal{P}(X) & \text{if } A \cap B = \emptyset, \\ \{C \subset X \mid A \subset C\} & \text{if } A \subset B, \\ \{U \subset X \mid C \subset U\} & \text{if } A \cap B = C. \end{cases}$$

Proof. For every subsets A and B of X, we have

$$\begin{aligned} \mathfrak{L}(A,B) &= \{C \in \mathcal{P}(X) \mid P(A, B\backslash C) = X\} \\ &= \{C \in \mathcal{P}(X) \mid A \rightsquigarrow (B \rightsquigarrow C) = X\} \\ &= \{C \in \mathcal{P}(X) \mid A \subset B \rightsquigarrow C\} \\ &= \{C \in \mathcal{P}(X) \mid A \odot B \subset C\} \\ &= \{C \in \mathcal{P}(X) \mid A \cap B \subset C\}. \end{aligned}$$

This completes the proof. □

Example 3.9. For a set $X = \{a, b, c\}$, consider the power BL-algebra $\mathcal{P}(X)$. Then

- $\mathfrak{L}(\{a\},\{a\}) = \mathfrak{L}(\{a\},\{a,b\}) = \mathfrak{L}(\{a\},\{a,c\}) = \mathfrak{L}(\{a\},X)$
  $= \mathfrak{L}(\{a,b\},\{a,c\}) = \{\{a\},\{a,b\},\{a,c\},X\}$,
- $\mathfrak{L}(\{b\},\{b\}) = \mathfrak{L}(\{b\},\{a,b\}) = \mathfrak{L}(\{b\},\{b,c\})$
  $= \mathfrak{L}(\{b\},X) = \mathfrak{L}(\{a,b\},\{b,c\}) = \{\{b\},\{a,b\},\{b,c\},X\}$,
- $\mathfrak{L}(\{c\},\{c\}) = \mathfrak{L}(\{c\},\{a,c\}) = \mathfrak{L}(\{c\},\{b,c\})$
  $= \mathfrak{L}(\{c\},X) = \mathfrak{L}(\{a,c\},\{b,c\}) = \{\{c\},\{a,c\},\{b,c\},X\}$,
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- $\mathfrak{L}(\{b,c\},X) = \{\{b,c\},X\}$,
- $\mathfrak{L}(\{a\},\{b\}) = \mathfrak{L}(\{a\},\{c\}) = \mathfrak{L}(\{b\},\{c\}) = \mathfrak{L}(\{a\},\{b,c\})$
  $= \mathfrak{L}(\{b\},\{a,c\}) = \mathfrak{L}(\{c\},\{a,b\}) = \mathcal{P}(X)$.

Theorem 3.10. *Assume that L is implicative and let $a, b \in L$. Then $\mathfrak{L}(a,b)$ is a deductive system of L.*

Proof. Let $x, y \in L$ be such that $x \in \mathfrak{L}(a,b)$ and $P(x\backslash y) \in \mathfrak{L}(a,b)$. Then $P(a,b\backslash x) = 1$ and $P(a,b,x\backslash y) = 1$. Using (p1), (p8), (p11) and (p14), we get

$$\begin{aligned} 1 = P(a,b,x\backslash y) &= P(a\backslash P(P(b\backslash x)\backslash P(b\backslash y))) \\ &= P(P(a,b\backslash x)\backslash P(a,b\backslash y)) \\ &= P(1\backslash P(a,b\backslash y)) = P(a,b\backslash y), \end{aligned}$$

and so $y \in \mathfrak{L}(a,b)$. Hence $\mathfrak{L}(a,b)$ is a deductive system of L. □

Obviously, if D is a deductive system of L, then D contains $\mathfrak{L}(a,b)$ for all $a, b \in D$.

Theorem 3.11. *Let D be a nonempty subset of L. Then D is a deductive system of L if and only if $\mathfrak{L}(a,b) \subseteq D$ for all $a, b \in D$.*

Proof. Necessity is straightforward. Assume that $\mathfrak{L}(a,b) \subseteq D$ for all $a, b \in D$. Note that $1 \in \mathfrak{L}(a,b) \subseteq D$ for all $a, b \in D$. Let $x, y \in L$ be such that $x \in D$ and $P(x\backslash y) \in D$. Since $P(x\backslash P(P(x\backslash y)\backslash y)) = 1$, it follows that $y \in \mathfrak{L}(x, P(x\backslash y)) \subseteq D$. Hence D is a deductive system of L. □

Theorem 3.12. *If D is a deductive system of L, then*

$$D = \cup\{\mathfrak{L}(a,b) \mid a, b \in D\}.$$

Proof. Let D be a deductive system of L and $x \in D$. Since $x \in \mathfrak{L}(x,1)$, we have

$$D \subseteq \cup\{\mathfrak{L}(x,1) \mid x \in D\} \subseteq \cup\{\mathfrak{L}(a,b) \mid a, b \in D\}.$$

Now let $x \in \cup\{\mathfrak{L}(a,b) \mid a, b \in D\}$. Then there exist $y, z \in D$ such that $x \in \mathfrak{L}(y,z)$. It follows from Theorem 3.11 that $x \in D$ which means that $\cup\{\mathfrak{L}(a,b) \mid a, b \in D\} \subseteq D$. This completes the proof. □

Corollary 3.13. *If D is a deductive system of L, then*

$$D = \cup\{\mathfrak{L}(a,1) \mid a \in D\}.$$

Note that if $\mathfrak{A}$ is a nonempty family of deductive systems of L, then $D = \cap\mathfrak{A}$ is also a deductive system of L (see [4]).

Denote by $\mathbb{DS}(L)$ the set of all deductive systems of L. If $A \subseteq L$, we denote by

$$\langle A\rangle = \cap\{D \in \mathbb{DS}(L) \mid A \subseteq D\},$$

which is called the *deductive system generated* by A. If $A = \{a_1, a_2, \cdots, a_n\}$, we use $\langle a_1, a_2, \cdots, a_n\rangle$ instead of $\langle\{a_1, a_2, \cdots, a_n\}\rangle$. The deductive system generated by one element $a \in L$ will be denoted by $\langle a\rangle$, and it is easy to verify that $\langle a\rangle = \{x \in L \mid P(a\backslash x) = 1\}$, which is called a *principle deductive system.*

Proposition 3.14. *For any subsets A and B of L, the following hold:*

(i) $\langle\{1\}\rangle = \{1\}$ *and* $\langle\emptyset\rangle = \{1\}$,

(ii) $\langle L\rangle = L$,

(iii) $A \subseteq B$ *implies* $\langle A\rangle \subseteq \langle B\rangle$,

(iv) $(\forall x, y \in L)\ (x \le y \Rightarrow \langle y\rangle \subseteq \langle x\rangle)$,

(v) *if A is a deductive system of L, then $\langle A\rangle = A$.*

Proof. Straightforward. □

Let $A \subseteq L$ and construct

(q1) $A_1 = A \cup \{1\}$,

(q2) if A_k is defined, put

$$A_{k+1} := \{y \in L \mid x \in A_k \text{ and } P(x\backslash y) \in A_k\}.$$

Then $1 \in A_k$ for all $k = 1, 2, \cdots$. Let $x \in A_k$. Then $P(1\backslash x) = x \in A_k$, and so $x \in A_{k+1}$. Therefore we have a chain

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots.$$

Put $D = \cup\{A_k \mid k = 1, 2, \cdots\}$. Obviously $1 \in D$. Let $x, y \in L$ be such that $x \in D$ and $P(x\backslash y) \in D$. Then $x \in A_p$ and $P(x\backslash y) \in A_q$ for some natural numbers p and q. Without loss of generality we may choose $k = \max\{p, q\}$. Then $x \in A_k$ and $P(x\backslash y) \in A_k$. Thus $y \in A_{k+1} \subseteq D$, i.e., D is a deductive system of L containing A. On the other hand, if D^* is a deductive system of L containing A, then D^* contains also A_k for $k = 1, 2, \cdots$. Hence D is the least deductive system of L containing A. Therefore we have the following theorem.

Theorem 3.15. *Let $A \subseteq L$ and let $A_1, A_2, \cdots$ be the sets determined by* (q1) *and* (q2)*. Then* $\langle A \rangle = \cup\{A_k \mid k = 1, 2, \cdots\}$.

Proposition 3.16. *In a BL-algebra L, we have the following property:*

$$(\forall x, y, z \in L)(P(z, x\backslash y) = 1,\ P(z\backslash x) = 1 \Rightarrow P(z\backslash y) = 1).$$

Proof. Let $x, y, z \in L$ be such that $P(z, x\backslash y) = 1$ and $P(z\backslash x) = 1$. Then $P(x\backslash y) \in \langle z \rangle$ and $x \in \langle z \rangle$. Since $\langle z \rangle$ is a deductive system of L, it follows from (ds2) that $y \in \langle z \rangle$, that is, $P(z\backslash y) = 1$ □

Lemma 3.17. *Let $D \in \mathbb{DS}(L)$ and $a_1, a_2, \cdots, a_n \in D$. If $P(a_n, \cdots, a_1\backslash x) \in D$, then $x \in D$.*

Proof. Straightforward. □

Theorem 3.18. *If A is a nonempty subset of L, then*

$$\langle A \rangle = \{x \in L \mid P(a_n, a_{n-1}, \cdots, a_1\backslash x) = 1 \text{ for some } a_1, \cdots, a_n \in A\}.$$

Proof. Denote

$$D := \{x \in L \mid P(a_n, a_{n-1}, \cdots, a_1\backslash x) = 1 \text{ for some } a_1, \cdots, a_n \in A\}.$$

We first show that D is a deductive system of L. Obviously $1 \in D$. Let $x, y \in L$ be such that $x \in D$ and $P(x\backslash y) \in D$. Then there exist $a_1, \cdots, a_m, b_1, \cdots, b_n \in A$ such that

(1) $$P(a_m, a_{m-1}, \cdots, a_1\backslash x) = 1,$$

(2) $$P(b_n, b_{n-1}, \cdots, b_1\backslash P(x\backslash y)) = 1.$$

Using (p10), (2) implies that

$$(3) \qquad P(x, b_n, \cdots, b_2 \backslash P(b_1 \backslash y)) = 1.$$

It follows from (1) and (p13) that

$$(4) \quad 1 = P(a_m, a_{m-1}, \cdots, a_1 \backslash x) \leq P(a_m, a_{m-1}, \cdots, a_1, b_n, \cdots, b_1 \backslash y)$$

so that $P(a_m, a_{m-1}, \cdots, a_1, b_n, \cdots, b_1 \backslash y) = 1$. Hence $y \in D$ and D is a deductive system of L. Clearly $A \subseteq D$. Let E be a deductive system containing A and let $x \in D$. Then there exist $c_1, c_2, \cdots, c_n \in A$ such that

$$P(c_n, \cdots, c_1 \backslash x) = 1,$$

and so $P(c_n, \cdots, c_1 \backslash x) \in E$. Using Lemma 3.17, we have $x \in E$ and thus $D \subseteq E$. This completes the proof. □

Theorem 3.19. *Let $D \in \mathbb{DS}(L)$ and $a \in L$. Then*

$$\langle D \cup \{a\} \rangle = \{x \in L \mid P(a^n \backslash x) \in D \text{ for some natural number } n\},$$

where $P(a^n \backslash x) = P(a, a, \cdots, a \backslash x)$ in which a occurs n-times.

Proof. Denote

$$G = \{x \in L \mid P(a^n \backslash x) \in D \text{ for some natural number } n\}.$$

In order to prove G is a deductive system of L, let $x, y \in L$ be such that $x \in G$ and $x \leq y$. Then there exist a natural number n and $u \in D$ such that $P(a^n \backslash x) = u$. Using (p10), we see that

$$(5) \qquad P(a^n, u \backslash x) = P(u, a^n \backslash x) = u \rightsquigarrow P(a^n \backslash x) = u \rightsquigarrow u = 1.$$

Using (p13), we know that $x \leq y$ implies

$$1 = P(a^n, u \backslash x) \leq P(a^n, u \backslash y),$$

and so $1 = P(a^n, u \backslash y) = u \rightsquigarrow P(a^n \backslash y)$, that is, $u \leq P(a^n \backslash y)$. It follows from (ds4) that $P(a^n \backslash y) \in D$ so that $y \in G$. This shows that G satisfies (ds4). Let $x, y \in G$. Noticing that $x \leq P(y \backslash (x \odot y))$, we get $P(y \backslash (x \odot$

$y)) \in G$, and so there exist a natural number n and $v \in D$ such that $v = P(a^n, y\backslash(x \odot y)) = P(y, a^n\backslash(x \odot y))$. It follows that

$$1 = P(v, y, a^n\backslash(x \odot y)) = P(y, a^n, v\backslash(x \odot y)) = y \rightsquigarrow P(a^n, v\backslash(x \odot y)),$$

that is,

$$y \leq P(a^n, v\backslash(x \odot y)). \tag{6}$$

Since $y \in G$, there exist a natural number m and $w \in D$ such that $P(a^m\backslash y) = w$, which implies that

$$P(a^m, w\backslash y) = P(w, a^m\backslash y) = w \rightsquigarrow P(a^m\backslash y) = w \rightsquigarrow w = 1. \tag{7}$$

It follows from (6), (p13) and (p10) that

$$\begin{aligned} 1 = P(a^m, w\backslash y) \ &\leq \ P(a^m, w, a^n, v\backslash(x \odot y)) \\ &= P(w, v, a^m, a^n\backslash(x \odot y)) \\ &= P(w, v, a^{m+n}\backslash(x \odot y)) \end{aligned} \tag{8}$$

so that $1 = P(w, v, a^{m+n}\backslash(x \odot y))$. Since $v, w \in D$, it follows from (ds4) that $P(a^{m+n}\backslash(x \odot y)) \in D$. This means that $x \odot y \in G$. Thus G satisfies condition (ds3), and we have proved that G is a deductive system of L. As $P(a\backslash a) = 1 \in D$, $a \in G$. Let $x \in D$. Then $x \leq P(a\backslash x)$ by (p5). If we use condition (ds4), then $P(a\backslash x) \in D$ and so $x \in G$. This shows that $D \cup \{a\} \subseteq G$. Finally, let $H \in \mathbb{DS}(L)$ be such that $D \cup \{a\} \subseteq H$. If $x \in G$, then there exists a natural number k such that $P(a^k\backslash x) \in D \subseteq H$. Applying $a \in H$ and using condition (ds3), we have

$$P(a, a^{k-1}\backslash x) \odot a = P(a^k\backslash x) \odot a \in H. \tag{9}$$

Since $P(a, a^{k-1}\backslash x) \odot a \leq P(a^{k-1}\backslash x)$, we get $P(a^{k-1}\backslash x) \in H$ by (ds4). Repeating the procedure above, we conclude that $x \in H$. This proves that $G \subseteq H$, and thus G is the least deductive system containing D and a. This completes the proof. □

4. Concluding remarks

In this paper, we discussed how to generate a deductive system by a set, and investigated some related properties. We gave a characterization of a deductive system. The results of this paper will be devoted to study of MV-algebras, lattice implication algebras, Łukasiewicz' logic, Gödel's logic and the product logic, which are different extensions of basic logic. Moreover, it will be devoted to the problem to reveal the logical content of various methods from fuzzy logic which play a specific role in fuzzy control and expert systems, e.g. Zadeh's compositional rule of inference, generalized modus ponens, min-composition, generalized quantification, etc. Some important issues for future work are: (i) developing the properties of a deductive system, (ii) defining new deductive systems which are related to given deductive systems, and (iii) finding useful results on the new structures.

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