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INTUITIONISTIC FUZZY SUBGROUPS

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Abstract. We discuss various types of sublattice of the lattice of intuitionistic fuzzy subgroups of a given group. We prove that a special class of intuitionitic fuzzy normal subgroups constitutes a modular sublattice of the lattice of intuitionistic fuzzy subgroups.

0. Introduction

In 1965, Zadeh[21] introduced the concept of a fuzzy set. After that time, it has been a tremendous interest in the subject due to its diverse application ranging from engineering and computer science to social behaviour studices. In particular, several researchers[1,8,19,20] have applied the notion of fuzzy sets to group theory.

In 1986, Atanassov[2] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. Since then, Çoker and his colleagues[6,7,10], Lee and Lee[18], and Hur and his colleagues[13] applied the notion of intuitionistic fuzzy sets to topology. In particular, Hur and his colleagues[15] applied one to topological group. Moreover, many reaserchers[3,4,11,12,14,16,17] applied the concept of intuitionistic fuzzy sets to algebra.

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In this paper, we discuss some interesting sublattices of the lattice of intuitionistic fuzzy subgroups of group. As a main result of our paper, we prove the set of all intuitionistic fuzzy normal subgroups with finite range and attaining the same value at the identity element of the group forms a modular sublattice of the lattice of intuitionistic fuzzy subgroups. In fact, it is an intuitionistic fuzzy version of a well-known result of classical lattice theory. Finally, though a lattice diagram we exhibit the interrelationship of the sublattices of the lattice of intuitionistic fuzzy subgroups discussed here.

1. Preliminaries

We will list some concepts and some results needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as I. And for an ordinary subset A of a set X, we will denote the characteristic function of A as χ_A.

Definition 1.1[2,6]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set*(in short, IFS) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership(namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in a set X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 1.2[2]. Let X be a nonempty sets and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be an IFSs in X. Then

(1) $A \subset B$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ if and only if $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\ A] = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3[6]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\wedge \mu_{A_i}, \vee \nu_{A_i})$.

(2) $\bigcup A_i = (\vee \mu_{A_i}, \wedge \nu_{A_i})$.

Definition 1.4[11]. Let $(X, \cdot)$ be a groupoid and let $A \in$ IFS(X). Then A is called an *intuitionistic fuzzy subgroupoid*(in short, IFGP) of X if for any $x, y \in X$, $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

Definition 1.5[12]. Let G be a group and let $A \in$ IFS(G). Then A is called an *intuitionistic fuzzy subgroup*(in short, IFG) of G if it satisfies the following conditions:

(i) A is an IFGP of G.

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \geq \nu_A(x)$ for each $x \in G$.

We will denote the set of all IFGs of G as IFGG.

Result 1.A[12, Proposition 2.3]. Let G be a group and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset$ IFG(G). Then $\bigcap_{\alpha \in \Gamma} A_\alpha \in$ IFG(G).

Result 1.B[12, Proposition 2.6]. Let G be a group and let $A, B \in \mathrm{IFG}(G)$. Then $A(x^{-1}) = A(x)$, $\mu_A(x) \le \mu_A(e)$ and $\nu_A(x) \ge \nu_A(e)$ for each $x \in G$, where e is the identity element of G.

Result 1.C[12, Proposition 2.2]. Let G be a group and let $A \subset G$. Then A is a subgroup of G if and only if $(\chi_A, \chi_{A^c}) \in \mathrm{IFG}(G)$.

Definition 1.6[11]. Let A be an IFS in a set X and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \le 1$. Then the set $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \ge \lambda$ and $\nu_A(x) \le \mu\}$ is called a (λ, μ)-*level subset* of A.

2. Lattices of intuitionistic fuzzy subgroups.

In this section, we study the lattice structure of the set of intuitionistic fuzzy subgroups of a given group. From Definitions 1.1, 1.2 and 1.3, we can see that for a set X, $\mathrm{IFS}(S)$ forms a complete lattice under the usual ordering of intuitionistic fuzzy inclusion $\subset$, where the inf and the sup are the intersection and the union of intuitionistic fuzzy sets, respectively. To construct the lattice of intuitionistic fuzzy subgroups, we define the inf of a family A_α of intuitionistic fuzzy subgroups to be the intersection $\cap A_\alpha$. However, the sup is defined as the intuitionistic fuzzy subgroup generated by the union $\cup A_\alpha$ and denoted by $(\cup A_\alpha)$. Thus we have the following result.

Proposition 2.1. Let G be a group. Then $\mathrm{IFG}(G)$ forms a complete lattice under the usual ordering of intuitionistic fuzzy set inclusion $\subset$.

Proof. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be any subset of $\mathrm{IFG}(G)$. Then, by Result 1.A, $\bigcap_{\alpha\in\Gamma} \in \mathrm{IFG}(G)$. Moreover, it is clear that $\bigcap_{\alpha\in\Gamma} A_\alpha$ is the largest intuitionistic fuzzy subgroup contained in A_α for each $\alpha \in \Gamma$. So

$\bigwedge_{\alpha\in\Gamma} A_\alpha = \bigcap_{\alpha\in\Gamma} A_\alpha$. On the other hand, by Result 1.??, $(\bigcup_{\alpha\in\Gamma} A_\alpha)$ is the least intuitionistic fuzzy subgroup containing A_α for each $\alpha \in \Gamma$. So $\bigvee_{\alpha\in\Gamma} A_\alpha = (\bigcup_{\alpha\in\Gamma} A_\alpha)$. Hence IFG$(G)$ is a complete lattice. □

Next we contruct certain sublattice of the lattice IFG(G). In fact, these sublattices reflect certain peculiarities of the intuitionistic fuzzy setting. For a group G, let $\mathrm{IFG}_f(G) = \{A \in \mathrm{IFG}(G) : \mathrm{Im}\, A \text{ is finite }\}$ and let $\mathrm{IFG}_{(s,t)}(G) = \{A \in \mathrm{IFG}(G) : A(e) = (s,t)\}$, where e is the identity of G. Then it is clear that $\mathrm{IFG}_f(G)$[resp. $\mathrm{IFG}_{(s,t)}(G)$] is a sublattice of IFG(G). Moreover, $\mathrm{IFG}_f(G) \cap \mathrm{IFG}_{(s,t)}(G)$ is also a sublattice of IFG(G).

Now to obtain our main results, we start with following lemma.

Lemma 2.2. Let G be a group and let $A, B \in \mathrm{IFG}(G)$. Then for each $(\lambda,\mu) \in I \times I$ with $\lambda + \mu \leq 1$, $A^{(\lambda,\mu)} \cdot B^{(\lambda,\mu)} \subset (A \circ B)^{(\lambda,\mu)}$.

Proof. Let $z \in A^{(\lambda,\mu)} \cdot B^{(\lambda,\mu)}$. Then there exist $x_0, y_0 \in G$ such that $z = x_0 y_0$. Thus $\mu_A(x_0) \geq \lambda$, $\nu_A(x_0) \leq \mu$ and $\mu_A(y_0) \geq \lambda$, $\nu_A(y_0) \leq \mu$. So

$$\mu_{A\circ B}(z) = \bigvee_{z=xy}[\mu_A(x) \wedge \mu_B(y)] \geq \mu_A(x_0) \wedge \mu_B(y_0) \geq \lambda$$

and

$$\nu_{A\circ B}(z) = \bigwedge_{z=xy}[\nu_A(x) \vee \nu_B(y)] \leq \nu_A(x_0) \vee \nu_B(y_0) \leq \mu.$$

Thus $z \in (A \circ B)^{(\lambda,\mu)}$. Hence $A^{(\lambda,\mu)} \cdot B^{(\lambda,\mu)} \subset (A \circ B)^{(\lambda,\mu)}$. □

The following is the converse of Lemma 2.2.

Lemma 2.3. Let G be a group and let $A, B \in \mathrm{IFG}(G)$. If Im A and Im B are finite, then for each $(\lambda,\mu) \in I \times I$ with $\lambda + \mu \leq 1$,

$$(A \circ B)^{(\lambda,\mu)} \subset A^{(\lambda,\mu)} \cdot B^{(\lambda,\mu)}.$$

Proof. Let $z \in (A \cdot B)^{(\lambda,\mu)}$. Then

$$\mu_{A\circ B}(z) = \bigvee_{z=xy}[\mu_A(x) \wedge \mu_B(y)] \geq \lambda$$

and

$$\nu_{A\circ B}(z) = \bigwedge_{z=xy}[\nu_A(x) \vee \nu_B(y)] \leq \mu.$$

Since Im A and Im B are finite, there exist $x_0, y_0 \in G$ with $z = x_0y_0$ such that

$$\bigvee_{z=xy}[\mu_A(x) \wedge \mu_B(y)] = \mu_A(x_0) \wedge \mu_B(y_0) \geq \lambda$$

and

$$\bigwedge_{z=xy}[\nu_A(x) \vee \nu_B(y)] = \nu_A(x_0) \vee \nu_B(y_0) \leq \lambda.$$

Thus $\mu_A(x_0) \geq \lambda$, $\nu_A(x_0) \leq \mu$ and $\mu_B(y_0) \geq \lambda$, $\nu_B(y_0) \leq \mu$. So $x_0 \in A^{(\lambda,\mu)}$ and $y_0 \in B^{(\lambda,\mu)}$, i.e., $z = x_0y_0 \in A^{(\lambda,\mu)} \cdot B^{(\lambda,\mu)}$. Hence $(A \circ B)^{(\lambda,\mu)} \subset A^{(\lambda,\mu)} \cdot B^{(\lambda,\mu)}$. This completes the proof. □

The following is the immediate result of Lemmas 2.2 and 2.3.

Proposition 2.4. Let G be a group and let $A, B \in$ IFG(G). If Im A and Im B are finite, then for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$,

$$(A \circ B)^{(\lambda,\mu)} = A^{(\lambda,\mu)} \cdot B^{(\lambda,\mu)}.$$

Definition 2.5[12]. Let G be a group and let $A \in$ IFG(G). Then A is called an *intuitionistic fuzzy normal subgroup* (in short, IFNG) of G if $A(xy) = A(yx)$ for any $x, y \in G$.

We will denote the set of all IFNGs of G as IFNG(G). It is clear that if G is abelian, then every IFG of G is an IFNG of G.

Result 2.A[16, Proposition 2.13]. Let G be a group, let $A \in$ IFNG(G) and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then $A^{(\lambda,\mu)} \lhd G$, where $A^{(\lambda,\mu)} \lhd G$ means that $A^{(\lambda,\mu)}$ is a normal subgroup of G.

Result 2.B[16, Proposition 2.18]. Let G be a group and let $A \in$ IFG(G). If $A^{(\lambda,\mu)} \lhd G$ for each $(\lambda, \mu) \in$ Im A, Then $A \in$ IFNG(G).

The following is the immediate result of Results 2.A and 2.B.

Theorem 2.6. Let G be a group and let $A \in \mathrm{IFG}(G)$. Then $A \in \mathrm{IFNG}(G)$ if and only if for each $(\lambda, \mu) \in \mathrm{Im}\, A$, $A^{(\lambda,\mu)} \lhd G$.

Result 2.C[12, Proposition 3.3]. Let G be a group and let $A \in \mathrm{IFNG}(G)$. If $B \in \mathrm{IFG}(G)$, then $B \circ A \in \mathrm{IFG}(G)$.

The following is the immediate result of Result 1.A and Definition 2.5.

Proposition 2.7. Let G be a group and let $A, B \in \mathrm{IFNG}(G)$. Then $A \cap B \in \mathrm{IFNG}(G)$.

It is well-known that the set of all normal subgroups of a group forms a sublattice of the lattice of its subgroups. As an intuitionistic fuzzy analog of this classical result we obtain the following result.

Theorem 2.8. Let G be a group and let $\mathrm{IFN}_{f(s,t)}(G) = \{A \in \mathrm{IFNG}(G) :$ $\mathrm{Im}\, A$ is finite and $A(e) = (s,t)\}$. Then $\mathrm{IFN}_{f(s,t)}(G)$ is a sublattice of $\mathrm{IFG}_f(G) \cap \mathrm{IFG}_{(s,t)}(G)$. Hence $\mathrm{IFN}_{f(s,t)}(G)$ is a sublattice of $\mathrm{IFG}(G)$.

Proof. Let $A, B \in \mathrm{IFN}_{f(s,t)}(G)$. Then, by Result 2.C, $A \circ B \in \mathrm{IFG}(G)$. Let $z \in G$. Then

$$\mu_{A\circ B}(z) = \bigvee_{z=xy}[\mu_A(x) \wedge \mu_B(y)] \geq \mu_A(z) \wedge \mu_B(e)$$
$$= \mu_A(z) \wedge \mu_A(e) \quad (\text{Since } A(e) = (s,t) = B(e))$$
$$= \mu_A(z) \quad (\text{By Result 1.B})$$

and

$$\nu_{A\circ B}(z) = \bigwedge_{z=xy}[\nu_A(x) \vee \nu_B(y)] \leq \nu_A(z) \vee \nu_B(e)$$
$$= \nu_A(z) \vee \nu_A(e) = \nu_A(z).$$

Thus $A \subset A \circ B$. Similarly, we have $B \subset A \circ B$. Let $C \in \mathrm{IFG}(G)$ such that $A \subset C$ and $B \subset C$. Let $z \in G$. Then

$$\mu_{A\circ B}(z) = \bigvee_{z=xy}[\mu_A(x) \wedge \mu_B(y)]$$

$$\begin{aligned}&\leq \bigvee_{z=xy}[\mu_C(x) \wedge \mu_C(y)] \quad \text{(Since } A \subset C \text{ and } B \subset C\text{)}\\ &\leq \mu_C(xy) \quad \text{(Since } C \in \mathrm{IFG}(G)\text{)}\\ &= \mu_C(z)\end{aligned}$$

and

$$\begin{aligned}&\nu_{A\circ B}(z) = \bigwedge_{z=xy}[\nu_A(x) \vee \nu_B(y)] \geq \bigwedge_{z=xy}[\nu_C(x) \vee \nu_C(y)]\\ &\geq \nu_C(xy) = \nu_C(z).\end{aligned}$$

Thus $A \circ B \subset C$. So $A \circ B = A \vee B$.

Now let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Since $A, B \in \mathrm{IGNG}(G)$, $A^{(\lambda,\mu)} \lhd G$ and $B^{(\lambda,\mu)} \lhd G$. Then $A^{(\lambda,\mu)} \circ B^{(\lambda,\mu)} \lhd G$. By Proposition 2.4, $(A \circ B)^{(\lambda,\mu)} \lhd G$. Thus, by Theorem 2.6, $A \circ B \in \mathrm{IFNG}(G)$. So $A \vee B \in \mathrm{IFN}_{f(s,t)}(G)$. From Proposition 2.6, it is clear that $A \wedge B \in \mathrm{IFNG}(G)$. Thus $A \wedge B \in \mathrm{IFN}_{f(s,t)}(G)$. Hence $\mathrm{IFN}_{f(s,t)}(G)$ is a sublattice of $\mathrm{IFG}_f \cap \mathrm{IFG}_{(s,t)}(G)$, and therefore of $\mathrm{IFG}(G)$. This complete the proof. □

The relationship of different sublattice of the lattice of intuitionistic fuzzy subgroup discussed herein can be visualized by the lattice diagram in Figure 1.

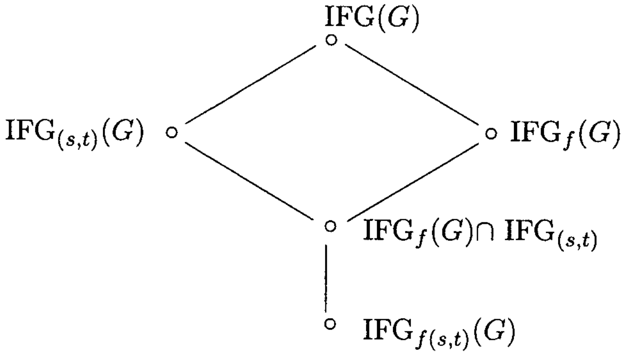

Figure 1.

It is also well-known[5, Theorem I.11] that the sublattice of normal subgroups of a group is modular. As an intuitionistic fuzzy version to the classical theoretic result, we prove that $\mathrm{IFN}_{(s,t)}(G)$ forms a modular lattice.

Result 2.D[17, Lemma 2.2]. Let G be a group and let $A \in \mathrm{IFG}(G)$. If for any $x, y \in G$, $\mu_A(x) < \mu_A(y)$ and $\nu_A(x) > \nu_A(y)$, then $A(xy) = A(x) = A(yx)$.

Definition 2.9[5,9]. A lattice $(L, \wedge, \vee)$ is said to be *modular* if for any $x, y, z \in L$ with $x \leq z$[resp. $x \geq z$], $x \vee (y \wedge z) = (x \vee y) \wedge z$[resp. $x \wedge (y \vee z) = (x \wedge y) \vee z$].

In any lattice L, it is well-known[9, Lemma I.4.9] that for any $x, y, z \in L$ if $x \leq z$[resp. $x \geq z$], then $x \vee (y \wedge z) \leq (x \vee y) \wedge z$[resp. $x \wedge (y \vee z) \geq (x \wedge y) \vee z$]. The inequality is called the *modular inequality.*

Theorem 2.10. The lattice $\mathrm{IFN}_{f(s,t)}(G)$ is modular.

Proof. Let $A, B, C \in \mathrm{IFN}_{f(s,t)}(G)$ such that $A \supset C$. Then, by the modular inequality, $(A \wedge B) \vee C \subset A \wedge (B \vee C)$. Assume that $A \wedge (B \vee C) \not\subset (A \wedge B) \vee C$, i.e., there exists $z \in G$ such that

$$\mu_{A\wedge(B\vee C)}(z) > \mu_{(A\wedge B)\vee C}(z) \quad \text{and} \quad \nu_{A\wedge(B\vee C)}(z) < \nu_{(A\wedge B)\vee C}(z).$$

Since $\mathrm{Im}\, B$ and $\mathrm{Im}\, C$ are finite, there exist $x_0, y_0 \in G$ with $z = x_0 y_0$ such that

$(B \vee C)(z) = (B \circ C)(z)$ (By the process of the proof of Theorem 2.7)

$= (\bigvee_{z=xy}[\mu_B(x) \wedge \mu_C(y)], \bigwedge_{z=xy}[\nu_B(x) \vee \nu_C(y)])$

$= (\mu_B(x_0) \wedge \mu_C(y_0), \nu_B(x_0) \vee \nu_C(y_0)).$

Thus $[A \wedge (B \vee C)](z) = (\mu_A(z) \wedge (\mu_B(x_0) \wedge \mu_C(y_0)), \nu_A(z) \vee (\nu_B(x_0) \vee \nu_C(y_0)))$.(*)

On the other hand,

$\mu_{(A\wedge B)\vee}(z) = \bigvee_{z=xy}[\mu_{A\wedge B}(x) \wedge \mu_C(y)]$

$\geq \mu_{A\wedge B}(x_0) \wedge \mu_C(y_0) = \mu_A(x_0) \wedge \mu_B(x_0) \wedge \mu_C(y_0) \quad (**)$

and

$$\nu_{(A\wedge B)\vee}(z) = \bigwedge_{z=xy}[\nu_{A\wedge B}(x) \vee \nu_C(y)]$$
$$\leq \nu_{A\wedge B}(x_0) \vee \nu_C(y_0) = \nu_A(x_0) \vee \nu_B(x_0) \vee \nu_C(y_0) \quad (**')$$

By $(*)$, $(**)$ and $(**')$,

$$\mu_A(z) \wedge \mu_B(x_0) \wedge \mu_C(y_0) > \mu_A(x_0) \wedge \mu_B(x_0) \wedge \mu_C(y_0)$$

and

$$\nu_A(z) \vee \nu_B(x_0) \vee \nu_C(y_0) < \nu_A(x_0) \vee \nu_B(x_0) \vee \nu_C(y_0).$$

Then

$$\mu_A(z), \mu_B(x_0), \mu_C(y_0) > \mu_A(x_0) \wedge \mu_B(x_0) \wedge \mu_C(y_0)$$

and

$$\nu_A(z), \nu_B(x_0), \nu_C(y_0) < \nu_A(x_0) \vee \nu_B(x_0) \vee \nu_C(y_0).$$

Thus $\mu_A(x_0) \wedge \mu_B(x_0) \wedge \mu_C(y_0) = \mu_A(x_0)$ and $\nu_A(x_0) \vee \nu_B(x_0) \vee \nu_C(y_0) = \nu_A(x_0)$. So $\mu_A(z) > \mu_A(x_0)$, $\nu_A(z) < \nu_A(x_0)$ and $\mu_C(y_0) > \mu_A(x_0)$, $\nu_C(y_0) < \nu_A(x_0)$. By Result 1.B, $\mu_A(x_0^{-1}) = \mu_A(x_0) < \mu_A(x_0y_0)$ and $\nu_A(x_0^{-1}) = \nu_A(x_0) > \nu_A(x_0y_0)$. By Result 2.D, $A(x_0) = A(y, x, x_0^{-1}) = A(y_0)$. Thus $\mu_C(y_0) > \mu_A(y_0)$ and $\nu_C(y_0) < \nu_A(y_0)$. This contradicts the fact that $A \supset C$. So $A \wedge (B \vee C) \subset (A \wedge B) \vee C$. Hence $A \wedge (B \vee C) = (A \wedge B) \vee C$. Therefore $\text{IFN}_{f(s,t)}(G)$ is modular. This completes the proof. □

We discuss some interesting facts concerning a special class of intuitionistic fuzzy subgroups that attain the value $(1, 0)$ at the identity element of G.

Lemma 2.11. Let A be a subset of a group G. Then

$$((\chi_A, \chi_{A^c})) = (\chi_{<A>}, \chi_{<A>^c}),$$

where $< A >$ is the subgroup generated by A.

Proof. Let $\mathcal{B} = \{B \in \text{IFG}(G) : (\chi_A, \chi_{A^c}) \subset B\}$, let $B \in \mathcal{B}$ and let $x \in A$. Then

$$\chi_A(x) = 1 \leq \mu_B(x) \text{ and } \chi_{A^c}(x) = 0 \geq \nu_B(x).$$

Thus $\mu_B(x) = 1$ and $\nu_B(x) = 0$. Since $B \in \mathrm{IFG}(G)$, $B = 1_\sim$ for any composite of elements of A. So $(\chi_{<A>}, \chi_{<A>^c}) \subset B$. Hence $(\chi_{<A>}, \chi_{<A^c>}) \subset \cap \mathcal{B}$. By Result 1.C, $(\chi_{<A>}, \chi_{<A>^c}) \in \mathrm{IFG}(G)$. Moreover, $(\chi_{<A>}, \chi_{<A>^c}) \in \mathcal{B}$. Therefore $(\chi_{<A>}, \chi_{<A>^c}) = \cap \mathcal{B} = ((\chi_A, \chi_{A^c}))$. □

The following can be easily seen.

Lemma 2.12. Let A and B subgroups of a group G. Then

(1) $A \lhd G$ if and only if $(\chi_A, \chi_{A^c}) \in \mathrm{IFNG}(G)$.

(2) $(\chi_A, \chi_{A^c}) \circ (\chi_B, \chi_{B^c}) = (\chi_{A \cdot B}, \chi_{(A \cdot B)^c})$.

Proposition 2.13. Let $S(G)$ be the set of all subgroup of a group G and let $\mathrm{IFG}(S(G)) = \{(\chi_A, \chi_{A^c}) : A \in S(G)\}$. Then $\mathrm{IFG}(S(G))$ forms a sublattice of $\mathrm{IFG}_f(G) \cap \mathrm{IFG}_{(1,0)}(G)$ and hence of $\mathrm{IFG}(G)$.

Proof. Let $A, B \in S(G)$. Then it is clear that $(\chi_A, \chi_{A^c}) \cap (\chi_B, \chi_{B^c}) = (\chi_{A \cap B}, \chi_{(A \cap B)^c}) \in \mathrm{IFG}(S(G))$. By Lemma 2.12, $((\chi_A, \chi_{A^c}) \cup (\chi_B, \chi_{B^c})) = ((\chi_{A \cup B}, \chi_{(A \cup B)^c})) = (\chi_{<A \cup B>}, \chi_{<A \cup B>^c})$. Thus $(\chi_A, \chi_{A^c}) \vee (\chi_B, \chi_{B^c}) = ((\chi_A, \chi_{A^c}) \cup (\chi_B, \chi_{B^c})) \in \mathrm{IFG}(S(G))$. Moreover, $\mathrm{IFG}(S(G)) \subset \mathrm{IFG}_f(G) \cap \mathrm{IFG}_{(1,0)}(G)$. Hence $\mathrm{IFG}(S(G))$ is a sublattice of $\mathrm{IFG}_f(G) \cap \mathrm{IFG}_{(1,0)}(G)$. □

Proposition 2.13 allows us to consider the lattice of subgroups $S(G)$ of G a group G as a sublattice of the lattice of all intuitionistic fuzzy subgroups $\mathrm{IFG}(G)$ of G.

Now, in view of Theorems 2.8 and 2.10, for each fixed $(s,t) \in I \times I$ with $s + t \leq 1$, $\mathrm{IFN}_{f(s,t)}(G)$ forms a modular sublattice of $\mathrm{IFG}_f(G) \cap \mathrm{IFG}_{(s,t)}(G)$. Therefore, for $(s,t) = (1,0)$, the sublattice $\mathrm{IFN}_{f(1,0)}(G)$ is also modular. It is clear that $\mathrm{IFN}_{f(1,0)}(G) \cap \mathrm{IFG}(S(G)) = \mathrm{IFN}(N(G))$, where $N(G)$ denotes the set of all normal subgroups of G and $\mathrm{IFN}(N(G)) = \{(\chi_N, \chi_{N^c}) : N \in N(G)\}$. Moreover, $\mathrm{IFG}(N(G))$ is also modular.

The lattice structure of these sublattices can be visualized by the diagram in Figure 2,

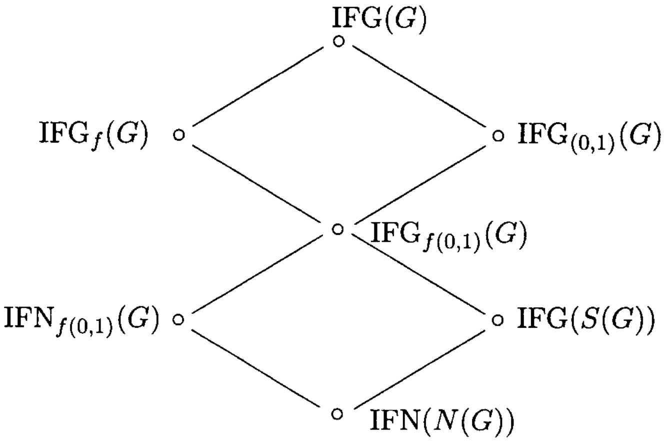

Figure 2.

By using Lemmas 2.11 and 2.12, we obtain a well-known classical result.

Corollary 2.13. Let G be a group. Then $N(G)$ forms a modular sublattice of $S(G)$.

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