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NEUTRAL SUBTRACTION ALGEBRAS

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Abstract. Neutral subtraction algebras and neutral ideals are introduced, and related properties are investigated.

1. Introduction

B. M. Schein [6] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [7] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [4] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [5] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In

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this paper, we introduce the notion of neutral subtraction algebras and neutral ideals, and investigate several properties.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [4, 5]):

(a1) $(x - y) - y = x - y$.

(a2) $x - 0 = x$ and $0 - x = 0$.

(a3) $(x - y) - x = 0$.

(a4) $x - (x - y) \leq y$.

(a5) $(x - y) - (y - x) = x - y$.

(a6) $x - (x - (x - y)) = x - y$.

(a7) $(x - y) - (z - y) \leq x - z$.

(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1. [4] A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

- $0 \in A$
- $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Lemma 2.2. [5] *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

3. Neutral subtraction algebras

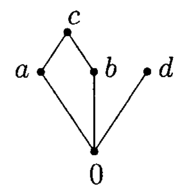
Definition 3.1. Let X be a subtraction algebra. An element $a \in X$ is said to be *neutral* if it satisfies

$$(\forall x \in X)(a \neq x \Rightarrow a - x = a, \ x - a = x).$$

Let $N(X)$ denote the set of all neutral elements of a subtraction algebra X, and we call $N(X)$ the *neutral part* of X. Obviously $0 \in N(X)$. Note that any non-zero element x of a subtraction algebra X such that $x \leq y$ for some $y \in X$ (or, $y \leq x$ for some $y(\neq 0) \in X$) can not be a neutral element of X. Hence we know that if a subtraction algebra X forms a chain, then $N(X) = \{0\}$.

Example 3.2. (1) Consider a subtraction algebra $X = \{0, a, b, c, d\}$ with the following Cayley table and Hasse diagram.

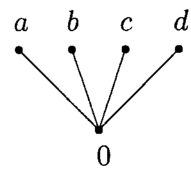
$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0



Then $N(X) = \{0, d\}$.

(2) Consider a subtraction algebra $X = \{0, a, b, c, d\}$ with the following Cayley table and Hasse diagram.

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0



Then $N(X) = \{0, a, b, c, d\} = X$.

Based on the above example, we give the notion of neutral subtraction algebras.

Definition 3.3. A subtraction algebra X is said to be *neutral* if it satisfies:

$$(\forall x, y \in X)\,(x \neq y \;\Rightarrow\; x - y = x),$$

or equivalently $N(X) = X$.

Note that the subtraction algebra X in Example 3.2(2) is a neutral subtraction algebra, but the subtraction algebra X in Example 3.2(1) is not a neutral subtraction algebra. Note also that the subtraction algebra of order 2 is neutral.

Theorem 3.4. *The neutral part of a subtraction algebra X is an ideal of X.*

Proof. Obviously $0 \in N(X)$. Let $x, y \in X$ be such that $x - y \in N(X)$ and $y \in N(X)$. Now $y \in N(X)$ implies $x = x - y \in N(X)$. Hence $N(X)$ is an ideal of X. □

Note that a subalgebra of a subtraction algebra X may not be an ideal of X. In fact, consider the subtraction algebra $X = \{0, a, b, c, d\}$ as in Example 3.2(1). Then $\{0, c\}$ is a subalgebra of X which is not an ideal of X since $a - c = 0 \in \{0, c\}$ and $a \notin \{0, c\}$.

We now give a condition for a subalgebra to be an ideal.

Theorem 3.5. *Every subalgebra of a neutral subtraction algebra is an ideal.*

Proof. Let A be a subalgebra of a neutral subtraction algebra X. Then $0 = x - x \in A$ for every $x \in A$. Let $x, y \in X$ be such that $x - y \in A$ and $y \in A$. Since x is a neutral element, it follows that $x = x - y \in A$ so that A is an ideal of X. □

Theorem 3.6. *Let B be a subset of a subtraction algebra X such that $0 \in B \subseteq N(X)$. Then B is an ideal of X.*

Proof. Let $x, y \in X$ be such that $x - y \in B$ and $y \in B$. Since y is neutral, it follows that $x = x - y \in B$. Hence B is an ideal of X. □

Corollary 3.7. *In a neutral subtraction algebra, every subset containing the zero element 0 is an ideal.*

Proof. Straightforward. □

Corollary 3.8. *If X is a neutral subtraction algebra which has n nonzero elements, then X has 2^n numbers of ideals.*

Proof. Straightforward. □

Definition 3.9. An ideal I of a subtraction algebra X is said to be *neutral* if $N(X) \subseteq I$.

Example 3.10. In Example 3.2(1), the set $I_1 = \{0, d\}$, $I_2 = \{0, b, d\}$, $I_3 = \{0, a, d\}$ and X itself are neutral ideals of X. But $J_1 = \{0, a\}$, $J_2 = \{0, b\}$, and $J_3 = \{0, a, b, c\}$ are ideals of X which are not neutral.

Note that in a neutral subtraction algebra, there are no proper neutral ideals.

Theorem 3.11. *Any intersection of neutral ideals is a neutral ideal.*

Proof. Straightforward. □

Question. If X is a subtraction algebra in which all non-zero elements do not comparable each other, then is X neutral?

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