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INTUITIONISTIC FUZZY CONGRUENCES CONTAINED IN $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$ AND INTUITIONISTIC FUZZY IDEMPOTENT SEPARATING CONGRUENCES

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Abstract. First, we obtain some results of intuitionistic fuzzy congruences on a regular semigroups contained in $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$. Secondly, we introduce the concept of intuitionistic fuzzy idempotent separating congruences on a semigroup and investigate some of it's properties

0.Introduction

The theory of fuzzy sets proposed by Zadeh[25] in 1965 has achieved a greate success in vavious fields. After that time, some authors [1, 18-20, 22-24] applied this concept to congruences.

With the research of fuzzy sets, in 1986, Atanassov[2] introduced the concept of intuitionistic fuzzy sets which are effective to deal with vagueness. The notion of the intuitionistic fuzzy sets is a generalization of one of the fuzzy sets. Recently, Çoker and his colleagues [6, 7, 10], and Lee and Lee [21] introduced the concept of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets and investigate some of it's properties. Some researchers [3, 4, 12-14] applied to algebra using intuitionistic fuzzy sets. In 1966, Bustince and Burillo [5] introduced the

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concept of intuitionistic fuzzy relations and investigated some properties of the composition of intuitionistic fuzzy relations. In 2003, Deschrijver and Kerre [9] studied some properties of the composition of intuitionistic fuzzy relations. In particular, Hur and his colleagues [16] investigated various properties of intuitionistic fuzzy equivalences. Also Hur and his colleagues [15, 17] introduced the concept of intuitionistic fuzzy congruences on a semigroup and a lattice and obtained many results, respectively.

In this paper, first, we obtain some results of intuitionistic fuzzy congruences on a regular semigroups contained in $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$. Secondly, we introduce the concept of intuitionistic fuzzy idempotent separating congruences on a semigroup and investigate some of it's properties.

1. Preliminaries

In this section, we list some basic concepts and well-known results which are needed in the later sections.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I. For any ordinary relation R on a set X, we will denote the characteristic function of R as χ_R.

Definition 1.1 [2, 6]. Let X be a nonempty set. A complex mapping $A = (\mu_A,\ \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$

to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2 [2]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\ ]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3 [6]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4 [5]. Let X be a set. Then a complex mapping $R = (\mu_R,\ \nu_R) : X \times X \to I \times I$ is called an *intuitionistic fuzzy relation* (in short, *IFR*) on X if $\mu_R(x,y) + \nu_R(x,y) \leq 1$ for each $(x,y) \in X \times X$, i.e., $R \in$ IFS $(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 1.5 [5]. Let $R \in$ IFR(X). Then the *inverse* of R, R^{-1} is defined as by $R^{-1}(x,y) = R(y,x)$ for any $x, y \in X$.

Definition 1.6 [5, 9]. Let X be a set and let $R, Q \in \text{IFR}(X)$. Then the *composition* of R and Q, $Q \circ R$, is defined as follows : for any $x, y \in X$,

$$\mu_{Q\circ R}(x,y) = \bigvee_{z\in X} [\mu_R(x,z) \wedge \mu_Q(z,y)]$$

and

$$\nu_{Q\circ R}(x,y) = \bigwedge_{z\in X} [\nu_R(x,z) \vee \nu_Q(z,y)].$$

Definition 1.7 [5]. An Intuitionistic fuzzy relation R on a set X is called an *intuitionsitic fuzzy equivalence relation* (in short, IFER) on X if it satisfies the following conditions :

(i) it is *intuitionsitic fuzzy reflexive*, i.e., $R(x, x) = (1, 0)$ for each $x \in X$.

(ii) it is *intuitionsitic fuzzy symmetric*, i.e., $R^{-1} = R$.

(iii) it is *intuitionsitic fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as IFE(X).

Let R be an intuitionistic fuzzy equivalence relation on a set X and let $a \in X$. We define a complex mapping $Ra : X \to I \times I$ as follows : for each $x \in X$

$$Ra(x) = R(a, x).$$

Then clearly $Ra \in \text{IFS}(X)$. The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class* of R containing $a \in X$. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set of* X *by* R and denoted by X/R.

Result 1.A [16, Theorem 2.15]. Let R be an intuitionistic fuzzy equivalence relation on a set X. Then the followings hold :

(1) $Ra = Rb$ if and only if $R(a, b) = (1, 0)$ for any $a, b \in X$.

(2) $R(a, b) = (0, 1)$ if and only if $Ra \cap Rb = 0_\sim$ for any $a, b \in X$.

(3) $\bigcup_{a \in X} Ra = 1_{\sim}$.

(4) There exists the surjection $p : X \to X/R$ defined by $p(x) = Rx$ for each $x \in X$.

Definition 1.8 [17]. An IFR R on a groupoid S is said to be:

(1) *intuitionistic fuzzy left compatible* if $\mu_R(x,y) \leq \mu_R(zx,zy)$ and $\nu_R(x,y) \geq$

$\nu_R(zx,zy)$, for any $x,y,z \in S$.

(2) *intuitionistic fuzzy right compatible* if $\mu_R(x,y) \leq \mu_R(xz,yz)$ and $\nu_R(x,y) \geq$

$\nu_R(xz,yz)$, for any $x,y,z \in S$.

(3) *intuitionistic fuzzy compatible* if $\mu_R(x,y) \wedge \mu_R(z,t) \leq \mu_R(xz,yt)$ and

$\nu_R(x,y) \vee \nu_R(z,t) \geq \nu_R(xz,yt)$, for any $x,y,z,t \in S$.

Definition 1.9 [17]. An IFER R on a groupoid S is called an:

(1) *intuitionistic fuzzy left congruence* (in short, $IFLC$) if it is intuitionistic fuzzy left compatible.

(2) *intuitionistic fuzzy right congruence* (in short, $IFRC$) if it is intuitionistic fuzzy right compatible.

(3) *intuitionistic fuzzy congruence* (in short, IFC) if it is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFLCs and IFRCs] on a groupoid S as IFC(S) [resp. IFLC(S) and IFRC(S)].

Result 1.B [17, Theorem 2.8]. Let R be relation on a groupoid S. Then $R \in C(S)$ if and only if $(\chi_R, \chi_{R^c}) \in \text{IFC}(S)$.

Definition 1.10 [12]. Let $(X, \cdot)$ be a groupoid and let $A, B \in \text{IFS}(X)$. Then the *intuitionistic fuzzy product* of A and B, $A \circ B$ is defined as follows : for any $x \in X$

$$(A \circ B)(x) = \begin{cases} (\bigvee_{yz=x}[\mu_A(y) \wedge \mu_B(z)], \bigwedge_{yz=x}[\nu_A(y) \vee \nu_B(z)]) \\ (0,1) \qquad \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

Let R be an intuitionistic fuzzy congruence on a semigroup S and let $a \in S$. The intuitionistic fuzzy set Ra in S is called an *intuitionistic fuzzy congruence class of* R *containing* $a \in S$ and we will denote the set of all intuitionistic fuzzy congruence classes of R as S/R.

Result 1.C [17, Proposition 2.19]. Let G be a group and let e be the identity element of G. If $R, Q \in$ IFC(G), then $Re \circ Qe = Qe \circ Re$.

Definition 1.11 [14]. Let G be a group, let $A \in$ IFG(G) and let $x \in G$. We define two complex mappings

$$Ax = (\mu_{Ax}, \nu_{Ax}) : G \to I \times I$$

and

$$xA = (\mu_{xA}, \nu_{xA}) : G \to I \times I$$

as follows respectively : for each $g \in G$,

$$Ax(g) = A(gx^{-1}) \text{ and } xA(g) = A(x^{-1}g).$$

Then Ax [resp.xA] is called the *intuitionistic fuzzy right* [resp.*left*] *coset* of G determined by x and A.

2. Intuitionistic fuzzy congruences contained in $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$

If a is an element of a semigroup S, the smallest left [resp. right] ideal containing a is $Sa \cup \{a\}$ [resp. $aS \cup \{a\}$], (called the *principal left* [resp. *right*] *ideal generated by* a) and denoted by S^1a [resp. aS^1], where S^1 is the monoid defined as follows :

$$S^1 = \begin{cases} S & \text{if } S \text{ has the identity} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Definition 2.1 [11]. The equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ on a semigroup S are defined as follows, respectively :

(1) $\mathcal{L} = \{(a,b) \in S \times S : S^1a = S^1b\}$.

(2) $\mathcal{R} = \{(a,b) \in S \times S : aS^1 = bS^1\}$.

(3) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

(4) $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$.

It is clear that $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c}) \in \text{IFE}(S)$.

Result 2.A [11, Corollary II.2.6]. If e ia an idempotent in a semigroup S, then He is a subgroup of S. Hence any $\mathcal{H}-$ class contains at most one idempotent.

Definition 2.2. Let R be an intuitionistic fuzzy relation on a semigroup S. We define a complex mapping $R^\circ = (\mu_{R^\circ}, \nu_{R^\circ}) : S \times S \to I \times I$ as follows : for any $x, y \in S$,

$$R^\circ(x,y) = \Big(\bigwedge_{s,t \in S^1} \mu_R(sxt, syt), \bigvee_{s,t \in S^1} \nu_R(sxt, syt)\Big).$$

It is clear that $R^\circ \in \text{IFR}(S)$.

Proposition 2.3. Let S be a semigroup and let $R, P, Q \in \text{IFR}(S)$. Then

(1) $R^\circ \subset R$.

(2) $(R^\circ)^{-1} = (R^{-1})^\circ$.

(3) If $P \subset Q$, then $P^\circ \subset Q^\circ$.

(4) $(R^\circ)^\circ = R^\circ$.

(5) $(P \cap Q)^\circ = P^\circ \cap Q^\circ$.

(6) $R = R^\circ$ if and only if R is intuitionistic fuzzy left and right compatible.

proof. The proofs of (1), (2) and (3) are clear.

(4) By (1) and (3), it is clear that $(R^\circ)^\circ \subset R^\circ$. Let $x, y \in S$. Then

$$\begin{aligned}\mu_{(R^\circ)^\circ}(x,y) &= \bigwedge_{s,t\in S^1} \mu_{R^\circ}(sxt, syt) = \bigwedge_{s,t\in S^1} \bigwedge_{a,b\in S^1} \mu_R(a(sxt)b, a(syt)b) \\ &= \bigwedge_{s,t\in S^1} \bigwedge_{a,b\in S^1} \mu_R((as)x(tb), (as)y(tb)) \\ &\geq \bigwedge_{as,tb\in S^1} \mu_R((as)x(tb), (as)y(tb)) = \mu_{R^\circ}(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_{(R^\circ)^\circ}(x,y) &= \bigvee_{s,t\in S^1} \nu_{R^\circ}(sxt, syt) = \bigvee_{s,t\in S^1} \bigvee_{a,b\in S^1} \nu_R(a(sxt)b, a(syt)b) \\ &= \bigvee_{s,t\in S^1} \bigvee_{a,b\in S^1} \nu_R((as)x(tb), (as)y(tb)) \\ &\leq \bigvee_{as,tb\in S^1} \nu_R((as)x(tb), (as)y(tb)) = \nu_{R^\circ}(x,y).\end{aligned}$$

Thus $R^\circ \subset (R^\circ)^\circ$. Hence $(R^\circ)^\circ = R^\circ$.

(5) By (3), $(P \cap Q)^\circ \subset P^\circ$ and $(P \cap Q)^\circ \subset Q^\circ$. Thus $(P \cap Q)^\circ \subset P^\circ \cap Q^\circ$. Now let $x, y \in S$. Then

$$\begin{aligned}\mu_{P^\circ \cap Q^\circ}(x,y) &= \mu_{P^\circ}(x,y) \wedge \mu_{Q^\circ}(x,y) \\ &= \bigwedge_{s,t\in S^1} \mu_P(sxt, syt) \wedge \bigwedge_{s',t'\in S^1} \mu_Q(s'xt', s'yt') \\ &\leq \bigwedge_{a,b\in S^1} [\mu_P(axb, ayb) \wedge \mu_Q(axb, ayb)] \\ &= \bigwedge_{a,b\in S^1} \mu_{P\cap Q}(axb, ayb) = \mu_{(P\cap Q)^\circ}(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_{P^\circ \cap Q^\circ}(x,y) &= \nu_{P^\circ}(x,y) \vee \nu_{Q^\circ}(x,y) \\ &= \bigvee_{s,t\in S^1} \nu_P(sxt,syt) \vee \bigvee_{s',t'\in S^1} \nu_Q(s'xt',s'yt') \\ &\geq \bigvee_{a,b\in S^1} [\nu_P(axb,ayb) \vee \nu_Q(axb,ayb)] \\ &= \bigvee_{a,b\in S^1} \nu_{P\cap Q}(axb,ayb) = \nu_{(P\cap Q)^\circ}(x,y).\end{aligned}$$

Thus $P^\circ \cap Q^\circ \subset (P\cap Q)^\circ$. Hence $(P\cap Q)^\circ = P^\circ \cap Q^\circ$.

(6) ($\Rightarrow$) : Suppose $R = R^\circ$ and let $x,y,z \in S$. Then

$$\mu_R(x,y) = \mu_{R^\circ}(x,y) = \bigwedge_{s,t\in S^1} \mu_R(sxt,syt) \leq \mu_R(zx,zy)$$

and

$$\nu_R(x,y) = \nu_{R^\circ}(x,y) = \bigvee_{s,t\in S^1} \nu_R(sxt,syt) \geq \nu_R(zx,zy).$$

Similarly, we have

$$\mu_R(x,y) \leq \mu_R(xz,yz) \text{ and } \nu_R(x,y) \geq \nu_R(xz,yz).$$

Hence R is intuitionistic fuzzy left and right compatible.

($\Leftarrow$) : Suppose R is intuitionistic fuzzy left and right compatible and let $x,y \in S$. Then

$$\mu_{R^\circ}(x,y) = \bigwedge_{s,t\in S^1} \mu_R(sxt,syt) \geq \mu_R(x,y)$$

and

$$\nu_{R^\circ}(x,y) = \bigvee_{s,t\in S^1} \nu_R(sxt,syt) \leq \nu_R(x,y).$$

Thus $R \subset R^\circ$. Hence, by (1), $R = R^\circ$. $\square$

Theorem 2.4. Let S be a semigroup and let $R \in \mathrm{IFE}(S)$. Then R° is the largest intuitionistic fuzzy congruence on S contained in R.

proof. Let $x \in S$. Then

$$R^\circ(x,x) = (\bigwedge_{s,t \in S^1} \mu_R(sxt, sxt), \bigvee_{s,t \in S^1} \nu_R(sxt, sxt)) = (1,0).$$

Thus R° is intuitionistic fuzzy reflexive. Since R is intuitionistic fuzzy symmetric, by Proposition 2.2 (2), $(R^\circ)^{-1} = (R^{-1})^\circ = R^\circ$. Thus R° is intuitionistic fuzzy symmetric. Now let $x, y \in S$. Then

$$\begin{aligned}
\mu_{R^\circ \circ R^\circ}(x,y) &= \bigvee_{z \in S} [\mu_{R^\circ}(x,z) \wedge \mu_{R^\circ}(z,y)] \\
&= \bigvee_{z \in S} [(\bigwedge_{s,t \in S^1} \mu_R(sxt, szt)) \wedge (\bigwedge_{s',t' \in S^1} \mu_R(s'xt', s'zt'))] \\
&\leq \bigvee_{z \in S} (\bigwedge_{s,t \in S^1} [\mu_R(sxt, szt) \wedge \mu_R(szt, syt)]) \\
&\leq \bigwedge_{s,t \in S^1} (\bigvee_{z \in S} [\mu_R(sxt, szt) \wedge \mu_R(szt, syt)]) \\
&\leq \bigwedge_{s,t \in S^1} (\bigvee_{a \in S} [\mu_R(sxt, a) \wedge \mu_R(a, syt)]) = \bigwedge_{s,t \in S^1} \mu_{R \circ R}(sxt, syt) \\
&\leq \bigwedge_{s,t \in S^1} \mu_R(sxt, syt) \text{ (Since } R \circ R \subset R) \\
&= \mu_{R^\circ}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{R^\circ \circ R^\circ}(x,y) &= \bigwedge_{z\in S}[\nu_{R^\circ}(x,z)\vee \nu_{R^\circ}(z,y)] \\
&= \bigwedge_{z\in S}[(\bigvee_{s,t\in S^1}\nu_R(sxt,szt))\vee(\bigvee_{s',t'\in S^1}\nu_R(s'xt',s'zt'))] \\
&\geq \bigwedge_{z\in S}(\bigvee_{s,t\in S^1}[\nu_R(sxt,szt)\vee \nu_R(szt,syt)]) \\
&\geq \bigvee_{s,t\in S^1}(\bigwedge_{z\in S}[\nu_R(sxt,szt)\vee \nu_R(szt,syt)]) \\
&\geq \bigvee_{s,t\in S^1}(\bigwedge_{a\in S}[\nu_R(sxt,a)\vee \mu_R(a,syt)]) \\
&= \bigvee_{s,t\in S^1}\nu_{R\circ R}(sxt,syt)\geq \bigvee_{s,t\in S^1}\nu_R(sxt,syt)=\nu_{R^\circ}(x,y).
\end{aligned}$$

Thus R° is intuitionistic fuzzy transitive. So $R^\circ \in \mathrm{IFE}(S)$. On the other hand, by Proposition 2.2(4) and (6), R° is intuitionistic fuzzy left and right compatible. So $R^\circ \in \mathrm{IFC}(S)$. Now let $Q \in \mathrm{IFC}(S)$ such that $Q \subset R$. Then clearly $Q = Q^\circ \subset R^\circ$. Hence R° is the largest intuitionistic fuzzy congruence on S contained in R. □

It is clear that if S is a semigroup and a' is an inverse of $a \in S$, then aa' and $a'a$ are idempotent.

Lemma 2.5. Let b be a regular element of a semigroup S, let b' be any inverse of b and let $e = bb'$. If $P \in \sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$, then for each $a \in Hb$, $P(a,b) = Pe(ab')$, where $\sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c}) = \{T \in \mathrm{IFC}(S) : T \subset (\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})\}$.

Proof. Let $a \in Hb$. Then

$$\mu_P(a,b) \leq \mu_P(ab',bb') = \mu_P(ab',e) = \mu_P(e,ab') = \mu_{Pe}(ab')$$

and

$$\nu_P(a,b) \geq \nu_P(ab',bb') = \nu_P(ab',e) = \nu_P(e,ab') = \nu_{Pe}(ab').$$

Let $f = b'b$. Then clearly $a\mathcal{H}b$ and $b\mathcal{L}f$. Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $a\mathcal{L}f$. Since f is an idempotent, by Lemma 2.14 in[8], $a = af$. Moreover, $eb = bb'b = b$. Thus

$$\mu_{Pe}(ab') = \mu_P(e, ab') \leq \mu_P(eb, ab'b) = \mu_P(eb, af) = \mu_P(b, a) = \mu_P(a, b)$$

and

$$\nu_{Pe}(ab') = \nu_P(e, ab') \geq \nu_P(eb, ab'b) = \nu_P(eb, af) = \nu_P(b, a) = \nu_P(a, b).$$

Hence $P(a, b) = Pe(ab')$. □

An element a of a semigroup S is said to be *regular* [11] if $a \in aSa$, i.e., there exists an $x \in S$ such that $a = axa$. The semigroup S is said to be *regular* if for each $a \in S$, a is a regular element. Corresponding to a regular element a, there exists at lest one $a' \in S$ such that $a = aa'a$ and $a' = a'aa'$. Such an element a' is called an *inverse* of a.

Theorem 2.6. Let S be a regular semigroup. If $P, Q \in \sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$, then $P \circ Q = Q \circ P$.

Proof. Let $a, c \in S$. Then

$$\mu_{Q \circ P}(a, c) = \bigvee_{b \in S} [\mu_P(a, b) \wedge \mu_Q(b, c)] \ (*)$$

and

$$\nu_{Q \circ P}(a, c) = \bigwedge_{b \in S} [\nu_P(a, b) \vee \nu_Q(b, c)] \ (*)'$$

Suppose there exists a $b \in S$ such that $\mu_P(a, b) \wedge \mu_Q(b, c) > 0$ and $\nu_P(a, b) \vee \nu_Q(b, c) < 1$. Then $\mu_P(a, b) > 0$, $\mu_Q(b, c) > 0$ and $\nu_P(a, b) < 1$, $\nu_Q(b, c) < 1$. Since $P \subset (\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$ and $Q \subset (\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$, $\mu_P(a, b) \leq \chi_{\mathcal{H}}(a, b), \nu_P(a, b) \geq \chi_{\mathcal{H}^c}(a, b)$ and $\mu_Q(a, b) \leq \chi_{\mathcal{H}}(b, c), \nu_Q(b, c) \geq \chi_{\mathcal{H}^c}(b, c)$. Thus $\chi_{\mathcal{H}}(a, b) = \chi_{\mathcal{H}}(b, c) = 1$ and $\chi_{H^c}(a, b) = \chi_{H^c}(b, c) = 0$ i.e., $a\mathcal{H}b$ and $b\mathcal{H}c$. So a, b and c lie in the same $\mathcal{H}-$ class of S. Since S is regular, by Theorem 2.11 in [8], there exist idempotents e and f in Ra and La, respectively. Since aRe and aLf, by Lemma 2.14 in [8], $ea = af = a$, By Lemma 2.13 in [8], there exist $x, y \in S$ such that $e = ax$ and $f = ya$. Moreover, by the process of the proof of Theorem 3.6 in [24], $a', b', c' \in R_f \cap L_e$ and $ac' = (ab')(bc')$, where a', b' and c' are inverses of a, b and c, respectively. Thus

$$\begin{aligned}\mu_{Pe\circ Qe}(ac') &= \bigvee_{ac'=sr} [\mu_{Pe}(s) \wedge \mu_{Qe}(r)] \ (r, s \in S) \\ &\geq \mu_{Pe}(ab') \wedge \mu_{Qe}(bc') \text{ (Since } ac' = (ab')(bc')) \\ &= \mu_P(a, b) \wedge \mu_Q(b, c) \text{ (by Lemma 2.5)}\end{aligned}$$

and

$$\begin{aligned}\nu_{Pe\circ Qe}(ac') &= \bigwedge_{ac'=sr} [\nu_{Pe}(s) \vee \nu_{Qe}(r)] \leq \nu_{Pe}(ab') \vee \nu_{Qe}(bc') \\ &= \nu_P(a, b) \vee \nu_Q(b, c).\end{aligned}$$

By $(*)$ and $(*)'$, for any $a, c \in S$,

$$\mu_{Q\circ P}(a, c) \leq \mu_{Pe\circ Qe}(ac') \text{ and } \nu_{Q\circ P}(a, c) \geq \nu_{Pe\circ Qe}(ac').$$

Since e is an idempotent in S, by Result 2.A, He is a subgroup of S. Let $P^e = P \cap (\chi_{He\times He}, \chi_{(He\times He)^c})$ and let $Q^e = Q \cap (\chi_{He\times He}, \chi_{(He\times He)^c})$. Then it is easy to see that P^e and Q^e are intuitionistic fuzzy congruences on the group He. Since $P, Q \subset (\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$, $P^e e = Pe$ and $Q^e e = Qe$. By Result 1.C, $Pe \circ Qe = Qe \circ Pe$. Thus

$$\begin{aligned}\mu_{Q\circ P}(a, c) &\leq \mu_{Pe\circ Qe}(ac') = \mu_{Qe\circ Pe}(ac') \\ &= \bigvee_{ac'=sr} [\mu_{Qe}(s) \wedge \mu_{Pe}(r)] \ (s, r \in S) \ (**)\end{aligned}$$

and

$$\begin{aligned}\nu_{Q\circ P}(a, c) &\geq \nu_{Pe\circ Qe}(ac') = \nu_{Qe\circ Pe}(ac') \\ &= \bigwedge_{ac'=sr} [\nu_{Qe}(s) \vee \nu_{Pe}(r)].\ (**)'\end{aligned}$$

Suppose there exist $s, r \in S$ such that $\mu_{Qe}(s) \wedge \mu_{Pe}(r) > 0$ and $\nu_{Qe}(s) \vee \nu_{Pe}(r) < 1$. Then $\mu_{Qe}(s) > 0$, $\mu_{Pe}(r) > 0$ and $\nu_{Qe}(s) < 1$, $\nu_{Pe}(r) < 1$. Thus $\mu_Q(e,s) > 0$, $\mu_P(e,r) > 0$ and $\nu_Q(e,s) < 1$, $\nu_P(e,r) < 1$. By the process of the proof of Theorem 3.6 in [24], there exists a $d \in S$ such that $Qe(s) = Qe(ad') = Q(a,d)$ and $Pe(r) = Pe(dc') = P(d,c)$, where d' and c' are the inverses of d and c, respectively. Thus

$$\mu_{Qe}(s) \wedge \mu_{Pe}(r) = \mu_Q(a,d) \wedge \mu_P(d,c)$$

and

$$\nu_{Qe}(s) \vee \nu_{Pe}(r) = \nu_Q(a,d) \vee \nu_P(d,c).$$

So, by $(**)$ and $(**)'$,

$$\mu_{Q\circ P}(a,c) \leq \bigvee_{ac'=sr} [\mu_{Qe}(s) \wedge \mu_{Pe}(r)] \leq \bigvee_{d\in S} [\mu_Q(a,d) \wedge \mu_P(d,c)] = \mu_{P\circ Q}(a,c)$$

and

$$\nu_{Q\circ P}(a,c) \geq \bigwedge_{ac'=sr} [\nu_{Qe}(s) \vee \nu_{Pe}(r)] \geq \bigwedge_{d\in S} [\nu_Q(a,d) \vee \nu_P(d,c) = \nu_{P\circ Q}(a,c).$$

So $Q \circ P \subset P \circ Q$. By the similar arguments, we see that $P \circ Q \subset Q \circ P$. Hence $P \circ Q = Q \circ P$. This completes the proof. □

3. Intuitionistic fuzzy idempotent separating congruences

Proposition 3.1. Let S be a regular semigroup. Let E_S be the set of idempotents of S. Then the following are equivalent: for each $R \in$ IFC(S) and each $a \in S$,

(1) $Ra \in E_{S/R}$,

(2) $Ra = Re$ for some $e \in E_S$ such that $Se \subset Sa$ and $eS \subset aS$,

(3) $Ra = Re$ for some $e \in E_S$.

Proof. (1)⇒(2) : Suppose $Ra \in E_{S/R}$. Then $Ra = Ra * Ra = Ra^2$. Let x be an inverse of a^2 in S. Then $a^2 = a^2xa^2$ and $x = xa^2x$. Let

$e = axa$. Then $e^2 = axaaxa = axa^2xa = axa = e$. Thus $e \in E_S$. So, we have :

$$\begin{aligned} Re &= Raxa = Ra * Rx * Ra = Ra^2 * Rx * Ra^2 \\ &= Ra^2xa^2 = Ra^2 = Ra. \end{aligned}$$

Let $y \in Se$. Then there exists $b \in S$ such that $y = be$. Since $e = axa, y = b(axa) = (bax)a$ and $bax \in S$. Thus $y \in Sa$. So $Se \subset Sa$. Similarly, we have $eS \subset aS$. The proofs of (2)$\Rightarrow$(3) and (3)$\Rightarrow$(1) are quite similar as the proofs. This completes the proof. $\square$

Let S be a semigroup and let E_S be the set of idempotents of S. Then a congruence R on S is said to be *idempotent separating* [11] if $R \cap (E_S \times E_S) = 1_{E_S}$, i.e., if $(e, f) \in R$ implies $e = f$, for any $e, f \in S$.

Definition 3.2. Let S be a regular semigroup and let $R \in \text{IFC}(S)$. Then R is called an *intuitionistic fuzzy idempotent separating congruence* (in short, *IFISC*) if $Re \neq Rf$ whenever $e \neq f$, i.e., $Re = Rf$ implies $e = f$ for any $e, f \in E_S$.

We will denote the set of all IFISCs on S by IFISC(S).

Example 3.3. Let $S = \{0, e, f, a, b\}$ be a set, where

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $\cdot$ be the matrix multiplication on S. Then it is clear that $(S, \cdot)$ is a regular semigroup with the following table :

	0	e	f	a	b
0	0	0	0	0	0
e	0	e	0	a	0
f	0	0	f	0	b
a	0	a	0	0	e
b	0	0	b	f	0

Moreover, $E_S = \{0, e, f\}$. Let $R = (\mu_R, \nu_R) : S \times S \to I \times I$ be the complex mapping defined as following matrix :

R	0	e	f	a	b
0	(1,0)	(0.6,0.3)	(0.8,0.1)	(0.6,0.3)	(0.8,0.1)
e	(0.6,0.3)	(1,0)	(0.6,0.3)	(0.7,0.3)	(0.6,0.3)
f	(0.8,0.1)	(0.6,0.3)	(1,0)	(0.6,0.3)	(0.9,0.1)
a	(0.6,0.3)	(0.7,0.3)	(0.6,0.3)	(1,0)	(0.6,0.3)
b	(0.8,0.1)	(0.6,0.3)	(0.9,0.1)	(0.6,0.3)	(1,0)

Then we can see that $R \in \text{IFC}(S)$. Moreover, $Rs \neq R_t$ for any $s \neq t \in E_S$. Hence $R \in \text{IFISC}(S)$. □

Result 3.A[13, Theorem V.3.2]. If S is an inverse semigroup with semilattice of idempotents E_S, then the relation

$$K = \{(a, b) \in S \times S : a^{-1}ea = b^{-1}eb \text{ for each } e \in E_S\}$$

is the greatest idempotent separating congruence on S.

The following is the immediate result of Result 3.A and Result 1.B.

Proposition 3.4. Let S be an inverse semigroup. Then (χ_K, χ_{K^c}) is an intuitionistic fuzzy idempotent separating congruence on S.

Theorem 3.5. Let R be a intuitionistic fuzzy congruence on an inverse semigroup S. Then $R \in$ IFISC(S) if and only if $R^{-1}((1,0)) \subset K$.

Proof .($\Rightarrow$) : Suppose $R \in$ IFISC(S) and let $(a,b) \in R^{-1}((1,0))$. Then, by Result 1.A(1), $R(a,b) = (1,0)$. Let $e \in E_S$. Since R is an intuitionistic fuzzy congruence on S,

$$\begin{aligned}
\mu_R(a^{-1}ea, b^{-1}eb) &\geq \bigvee_{x\in S}[\mu_R(a^{-1}ea, x) \wedge \mu_R(x, b^{-1}eb)] \\
&\geq \mu_R(a^{-1}ea, b^{-1}ea) \wedge \mu_R(b^{-1}ea, b^{-1}eb) \\
&\geq \mu_R(a^{-1}, b^{-1}) \wedge \mu_R(a,b) = \mu_R(a,b) \wedge \mu_R(a,b) = 1
\end{aligned}$$

and

$$\begin{aligned}
\nu_R(a^{-1}ea, b^{-1}eb) &\leq \bigwedge_{x\in S}[\nu_R(a^{-1}ea, x) \vee \nu_R(x, b^{-1}eb)] \\
&\leq \nu_R(a^{-1}ea, b^{-1}ea) \vee \nu_R(b^{-1}ea, b^{-1}eb) \\
&\leq \nu_R(a^{-1}, b^{-1}) \vee \nu_R(a,b) = \nu_R(a,b) \vee \nu_R(a,b) = 0.
\end{aligned}$$

Thus $R(a^{-1}ea, b^{-1}eb) = (1,0)$. By Result 1.A(1), $Ra^{-1}ea = Rb^{-1}eb$. It is clear that $a^{-1}ea, b^{-1}eb \in E_S$. Since R is idempotent separating, $a^{-1}ea = b^{-1}eb$. So $(a,b) \in K$. Hence $R^{-1}((1,0)) \subset K$.

($\Leftarrow$) : Suppose $R^{-1}((1,0)) \subset K$. Let $Re{=}Rf$ for any $e, f \in E_S$. Then $R(e,f) = (1,0)$. Thus $(e,f) \in R^{-1}((1,0))$. Since $R^{-1}((1,0)) \subset K$, $(e,f) \in K$. Since K is idempotent - separating, $e = f$. Hence $R \in$ IFISC(S). This completes the proof. □

Lemma 3.6. Let S be a regular semigroup, let $T \in IFC(S)$ and let $a \in S$. If there exists an idempotent f in S such that $\mu_T(a,f) > 0$ and $\nu_T(a,f) < 1$, then there exists an idempotent e in S such that $\mu_T(a,e) \geq \mu_T(a,f)$ and $\nu_T(a,e) \leq \nu_T(a,f)$. Moreover, e can be chosen so that $Re \subset Ra$ and $Le \subset La$, where Ra and La denote the $\mathcal{L}$-and $\mathcal{R}$-classes of S containing $a \in S$, respectively.

Proof. Let $a \in S$. Then

$$\mu_T(a^2, f) = \mu_T(a^2, f^2) \geq \mu_T(a, f) \wedge \mu_T(a, f) = \mu_T(a, f)$$

and

$$\nu_T(a^2, f) = \nu_T(a^2, f^2) \leq \nu_T(a, f) \wedge \nu_T(a, f) = \nu_T(a, f).$$

Thus :

$$\begin{aligned} \mu_T(a^2, a) &\geq \mu_{T \circ T}(a^2, a) = \bigvee_{b \in S} [\mu_T(a^2, b) \wedge \mu_T(b, a)] \\ &\geq \mu_T(a^2, f) \wedge \mu_T(f, a) = \mu_T(a, f) \ (***) \end{aligned}$$

and

$$\begin{aligned} \nu_T(a^2, a) &\leq \nu_{T \circ T}(a^2, a) = \bigwedge_{b \in S} [\nu_T(a^2, b) \vee \nu_T(b, a)] \\ &\leq \nu_T(a^2, f) \vee \nu_T(f, a) = \nu_T(a, f) \ (***)' \end{aligned}$$

Since S is a regular semigroup, there exists an inverse x of a^2 such that $a^2 = a^2xa^2$ and $x = xa^2x$. Let $e = axa$. Then clearly, e is an idempotent in S. Thus

$$\begin{aligned} \mu_T(e, a^2) &= \mu_T(axa, a^2xa^2) \\ &\geq \mu_T(a, a^2) \wedge \mu_T(x, x) \wedge \mu_T(a, a^2) \\ &= \mu_T(a, a^2) \geq \mu_T(a, f) \ (\text{By } (***)) \end{aligned}$$

and

$$\begin{aligned} \nu_T(e, a^2) &= \nu_T(axa, a^2xa^2) \\ &\leq \nu_T(a, a^2) \vee \nu_T(x, x) \vee \nu_T(a, a^2) \\ &= \nu_T(a, a^2) \leq \nu_T(a, f). \ (\text{By } (***)') \end{aligned}$$

So

$$\mu_T(a,e) \geq \mu_{T\circ T}(a,e) \geq \mu_T(e,a^2) \wedge \mu_T(a^2,a) \geq \mu_T(a,f)$$

and

$$\nu_T(a,e) \leq \nu_{T\circ T}(a,e) \leq \nu_T(e,a^2) \vee \nu_T(a^2,a) \leq \nu_T(a,f).$$

Hence $\mu_T(a,e) \geq \mu_T(a,f)$ and $\nu_T(a,e) \leq \nu_T(a,f)$. It is clear that idempotent $e = axa$ has the property that $Re \subset Ra$ and $Le \subset La$. This completes proof. □

Theorem 3.7. Let S be a regular semigroup and let $T \in \mathrm{IFC}(S)$. Then $T \in \mathrm{IFISC}(S)$ if and only if $T \in \sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$.

Proof. ($\Rightarrow$): Suppose $T \in \mathrm{IFISC}(S)$ and let $a, b \in S$. Suppose $\mu_T(a,b) > 0$ and $\nu_T(a,b) < 1$. Then

$$\mu_T(aa',ba') \geq \mu_T(a,b) > 0, \nu_T(aa',ba') \leq \nu_T(a,b) < 1$$

and

$$\mu_T(a'a,a'b) \geq \mu_T(a,b) > 0, \nu_T(a'a,a'b), \leq \nu_T(a,b) < 1,$$

where a' is an inverse of a. Since $a'a$ and aa' are idempotents, by Lemma 3.6, there exist idempotents e and f in S, respectively, such that

$$\mu_T(ba',e) \geq \mu_T(ba',aa'), \nu_T(ba',e) \leq \nu_T(ba',aa'), Re \subset Rba'$$

and

$$\mu_T(a'b,f) \geq \mu_T(a'b,a'a), \nu_T(a'b,f) \leq \nu_T(a'b,a'a), L_f \subset La'b.$$

Thus

$$\begin{aligned} \mu_T(aa',e) &\geq \mu_{T\circ T}(aa',e) = \bigvee_{c\in S}[\mu_T(aa',c) \wedge \mu_T(c,e)] \\ &\geq \mu_T(aa',ba') \wedge \mu_T(ba',e) = \mu_T(aa',ba') > 0 \end{aligned}$$

and

$$\begin{aligned}\nu_T(aa',e) &\leq \nu_{T\circ T}(aa',e)=\bigwedge_{c\in S}[\nu_T(aa',c)\vee\mu_T(c,e)]\\ &\leq \nu_T(aa',ba')\vee\nu_T(ba',e)=\nu_T(aa',ba')<1.\end{aligned}$$

By similar arguments, we have that $\mu_T(a'a,f)>0$ and $\nu_T(a'a,f)<1$. Since $T\in IFISC(S)$, $aa'=e$ and $a'a=f$. Thus $Ra=Raa'=Re\subset Rba'\subset Rb$ and $La\subset Laa'=L_f\subset Lbb'\subset Lb$. Similarly, we see that $Rb\subset Ra$ and $Lb\subset La$. So $Ra=Rb$ and $La=Lb$, i.e., $a\mathcal{R}b$ and $a\mathcal{L}b$, i.e., $a\mathcal{H}b$. Hence $\mu_T(a,b)\leq\chi_{\mathcal{H}}(a,b)=1$ and $\nu_T(a,b)\geq\chi_{\mathcal{H}^c}(a,b)=0$, i.e., $T\subset(\chi_{\mathcal{H}},\chi_{\mathcal{H}^c})$. Therefore $T\in(\chi_{\mathcal{H}},\chi_{\mathcal{H}^c})$.

($\Leftarrow$): Suppose $T\in\sum(\chi_{\mathcal{H}},\chi_{\mathcal{H}^c})$ and let $e,f\in E_S$ such that $\mu_T(e,f)>0$ and $\nu_T(e,f)<1$. Then $\chi_{\mathcal{H}}(e,f)=1$ and $\chi_{\mathcal{H}^c}(e,f)=0$. Thus $e\mathcal{H}_f$. It follows that $e=f$ from Result 1.A. Hence $T\in$ IFISC(S). This completes the proof. □

References

[1] F.A.Al-Thukair,Fuzzy congruence pairs of inverse semigroups, Fuzzy sets and systems 56(1993), 117-122.

[2] K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1986),87-96.

[3] Baldev Banerjee and Dhiren Kr. Basnet, Intuitionistic fuzzy subrings and ideals, J.Fuzzy Math. 11(1)(2003),139-155.

[4] R.Biswas, Intuitionistic fuzzy subgroups, Mathematical Forum x(1989),37-46.

[5] H.Bustince and P.Burillo, Structures on intuitionistic fuzzy relations, Fuzzy Sets and Systems 78(1996),293-303

[6] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88(1997),81-89.

[7] D. Çoker and A.Haydar Es, On fuzzy compactness in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 3(1995),899-909.

[8] A.H.Clifford and G.B.Preston, The Algebraic Theory of Semigroup, Vol.1, Math. Surveys, New York. (1961),PP.47-100.

[9] G.Deschrijver and E.E.Kerre, On the composition of intuitionistic fuzzy relations, Fuzzy Sets and Systems 136(2003), 333-361.

[10] H.Gürçay, D. Çoker and A.Haydar Es, On fuzzy continuity in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 5(1997), 365-378.

[11] J.M Howie, An Introduction to Semigroup Theory (Academic Press, New York, 1976).

[12] K.Hur, S.Y.Jang and H.W.Kang,Intuitionistic fuzzy subgroupids, International Journal of Fuzzy Logic and Intelligent Systems 3(1) (2003), 72-77.

[13] K.Hur, H.W.Kang and H.K.Song, Intuitionistic fuzzy subgroups and subrings, Honam Mathematical J.25(2)(2003),19-41.

[14] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets, Honam Math.J.26(4)(2004), 559-587.

[15] ____________________________, Intuitionistic fuzzy congruences on a lattice, J.Appl.Math.and Computing 18(1,2)(2005), 465-486.

[16] ____________________________, Intuitionistic fuzzy equivalence relations, Honam Math. J. 27(2) (2005), 163-181.

[17] K.Hur, S.Y.Jang and Y.B.Jun, Intuitionistic fuzzy congruences, Far East J.Math.Sci.
(FJMS)17(1)(2005),1-29

[18] J.P.Kim and D.R.Bae, Fuzzy congruences in groups, Fuzzy Sets and Systems 85(1997), 115-120.

[19] N.Kuroki, Fuzzy congruence and fuzzy normal subgroups, Inform.Sci.66(1992), 235-243.

[20] ________, Fuzzy congruences on inverse semigroups, Fuzzy Sets and Systems 87(1997), 335-340.

[21] S.J.Lee and E.P.Lee, The category of intuitionistic fuzzy topological spaces, Bull. Korean Math. Soc. 37(1)(2000),63-76.

[22] V.Murali, Fuzzy congruence relations, Fuzzy Sets and Systems 41(1991), 359-369.

[23] M.Samhan, Fuzzy congruences on semigroups, Inform.Sci.74(1993), 165-175.

[24] T.Yijia,Fuzzy congruences on a regular semigroup, Fuzzy sets and Systems 117 (2001), 447-453.

[25] L.A.Zadeh, Fuzzy sets, Inform. and Control 8(1965),338-353.

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