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ON THE EXISTENCE OF STABLE MINIMAL HYPERSURFACES OF THE THREE DIMENSIONAL CRITICAL POINT EQUATION

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Abstract. On a compact oriented smooth 3-dimensional manifold (M, g), we consider the critical point equation(CPE) defined as $z_g = {s'_g}^*(f)$. Under CPE, it is shown in [5] that every stable minimal hypersurface in M is contained in $\varphi^{-1}(0)$ for $\varphi \in \ker {s'_g}^*$. We study analytic and geometric conditions under which the stable minimal hypersurface in M does not exist.

1. Introduction and notations

It is important but still unsolved whether there exists a metric of constant Ricci curvature on a given manifold. One of the natural approaches to get a metric of constant Ricci curvature on a compact n-dimensional oriented manifold M^n is the following.

Let $\mathcal{M}_1$ be the set of Riemannian metrics on M^n with volume one. We look into the total scalar curvature functional $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ given by

$$\mathcal{S}(g) = \int_{M^n} s_g \, \mathrm{dvol}_g,$$

where s_g is scalar curvature function of g. It is well known that any compact manifold carries many metrics with constant scalar curvature. Let $\Xi = \{g \in \mathcal{M}_1 \,|\, s_g \text{ is constant}\}$. A standard variational technique

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tells us that the metric g is critical point of $\mathcal{S}$ restricted to Ξ if and only if there is a function f on M^n satisfying

$$z_g = s_g'^*(f). \tag{1.1}$$

Here, we used the following well-known notations.

· r_g is the Ricci tensor.
· $z_g = r_g - (s_g/n)g$ is the traceless Ricci tensor.
· $s_g'^*(f) = D_g df - \Delta_g f g - f r_g$ is the L^2 adjoint operator of the linearization of the scalar curvature s_g' given by

$$s_g'(h) = \Delta_g \mathrm{tr}\, h + \delta_g{}^* \delta_g h - g(h, r_g).$$

We call the equation (1.1) the *critical point equation*(CPE). Note that $f \equiv 0$ leads that g is an Einstein metric. Now the following conjecture may arise naturally.

Conjecture 1.1. *If there is a non-zero function f satisfying* (1.1), *then the metric g is Einstein.*

We assume that (1.1) has a non-zero solution f on M^3 throughout this article. We refer to [1] for details on Conjecture 1.1. As a remark, M. Obata showed that the metric g in Conjecture 1.1 should be isometric to the standard sphere $\mathbb{S}^n$(see [7]). But unfortunately, it is even not known yet whether the metric is homeomorphic to $\mathbb{S}^n$ or not. Related with this topological observation, we are interested in the stable minimal hypersurfaces in a 3-dimensional manifold M^3. Since the existence of stable minimal hypersurfaces in M^3 corresponds to the existence of non-zero elements of $H_2(M^3, \mathbb{Z})$, the existence of stable minimal hypersurfaces would be a topological obstruction for M^3 to have an Einstein metric. Now, let φ be a function on M^3 satisfying the following equation

$$0 = D_g d\varphi - \Delta_g \varphi g - \varphi r_g. \tag{1.2}$$

It can be regarded that $\varphi \in \ker s_g'^*$. In fact, the existence of a non-zero solution φ of (1.2) is equivalent to the existence of at least two non-zero solutions of (1.1), because if f is a solution of (1.1) then $f+\varphi$ is another solution of (1.1). In other words, φ can be obtained as $f_1 - f_2$ for two different solutions f_1 and f_2 of (1.1). In [5], S. Hwang proved that every compact stable minimal hypersurface in M^3 should be contained in the set $\varphi^{-1}(0)$ for some non-zero function φ satisfying (1.2). Note that such a set $\varphi^{-1}(0)$ is a totally geodesic submanifold of M^3(see [3]).

In this article, we study the relation between the conditions under which the compact stable minimal hypersurface may exist and analytic properties of the function f. We show that the conditions depend on the number of components of $\varphi^{-1}(0)$ and the values of f. Throughout this paper, M denotes a 3-dimensional manifold unless otherwise stated, and we will use the following notations.

· $B = f^{-1}(-1)$
· $\Gamma = \varphi^{-1}(0)$
· $\Sigma =$ compact stable minimal hypersurface in M.

2. Analysis on Γ

It is easily checked that $\Delta_g \varphi = -\frac{s_g}{2}$, i.e., φ is an eigenfunction. So Γ is the boundary of the nodal domains of φ, and there are no critical points of φ on Γ (see [3]). Furthermore we have the following lemma which tells that $||d\varphi||$ is constant on each component Γ_i of Γ.

Lemma 2.1. *Let $\Gamma = \cup_{i=1}^{I} \Gamma_i$, where Γ_i is a component of Γ. Then*

$$||d\varphi||\Big|_{\Gamma_i} \equiv c_i,$$

where c_i is a positive constant.

Proof. The proof follows basically the argument in [4]. For $\xi_i \in T\Gamma_i$, (1.2) gives

$$\xi_i \langle d\varphi, d\varphi\rangle = 2\langle D_{\xi_i} d\varphi, d\varphi\rangle = -s_g \varphi \langle \xi_i, d\varphi\rangle + 2r_g \varphi \langle \xi_i, d\varphi\rangle = 0.$$

The last equality comes from the fact that $\varphi = 0$ on $\Gamma_i \subset \Gamma$, and it completes the proof. □

Note that c_i and c_j may be different to each other for different i and j in the case that Γ has more than two components. Next, we recall the following lemma which is useful to analyze the properties of Γ.

Lemma 2.2. ([4])

$$\int_\Gamma f||d\varphi|| = 0.$$

Proof. Let $M_{0,\varphi} = \{x \in M^3 \,|\, \varphi(x) < 0\}$. Then the boundary of $M_{0,\varphi}$ is Γ. The following equation can be easily deduced from the facts $\Delta\varphi = -(s_g/2)\varphi$ and $\Delta f = -(s_g/2)f$.

$$\int_{M_{0,\varphi}} f\Delta\phi = \int_{M_{0,\varphi}} \phi\Delta f \tag{2.1}$$

Since $d\varphi \neq 0$ at any point in Γ, Green's formula with a unit normal vector field $d\varphi/||d\varphi||$ on Γ gives the following two equations

$$\int_{M_{0,\varphi}} f\Delta\phi = \int_\Gamma f\,||d\varphi|| - \int_{M_{0,\varphi}} \langle df, d\varphi\rangle \tag{2.2}$$

$$\int_{M_{0,\varphi}} \phi\Delta f = \int_\Gamma \phi\,\langle df, \frac{d\varphi}{||d\varphi||}\rangle - \int_{M_{0,\varphi}} \langle d\varphi, df\rangle = -\int_{M_{0,\varphi}} \langle d\varphi, df\rangle. \tag{2.3}$$

By combining (2.1), (2.2), and (2.3), we get

$$\int_\Gamma f\,||d\varphi|| = 0.$$

□

Now apply Lemma 2.2 on each component of Γ. As the same as Γ, Σ also need not be connected. Hence we let

$$\Sigma = \cup_{i=1}^{I}\Sigma_i,$$

$$\Gamma = \cup_{j=1}^{J}\Gamma_j,$$

where Σ_i and Γ_j are connected components of Σ and Γ respectively. Since $\Sigma \subset \Gamma$, we can arrange the indices satisfying $\Sigma_i \subset \Gamma_i$ for $1 \leq i \leq I \leq J$.

With these notations, from Lemma 2.1 and Lemma 2.2, we have the following

$$\int_{\Gamma} f||d\varphi|| = \sum \int_{\Gamma_i} f||d\varphi|| = \sum_i a_i c_i = 0, \tag{2.4}$$

where $a_i = \int_{\Gamma_i} f$ and $c_i = ||d\varphi||\Big|_{\Gamma_i}$.

By the above observations, we can formulate the following theorem which asserts a condition for the possible existence of stable minimal hypersurfaces in M.

Theorem 2.1. *Suppose that there exists a stable minimal hypersurface in M. Then at least one of the components of Γ is contained in the set $\{x \in M | f(x) < -1\}$ and Γ intersects with the set $\{x \in M | f(x) > 0\}$.*

Proof. Let Σ is non-empty and $\Sigma_i \subset \Gamma_i$ for $1 \leq i \leq I \leq J$ as above. In fact, we know that $\Sigma_i = \Gamma_i$ by the property of the components. Since $\Sigma = \cup_{i=1}^{I} \Sigma_i$ is contained in the set $\{x \in M | f(x) < -1\}$(see [5]), the set $\cup_{i=1}^{I} \Gamma_i$ is also contained in the set $\{x \in M | f(x) < -1\}$. So the first part of the theorem is proved.

On the other hand, by (2.4) we have

$$0 = \sum_i a_i c_i = \sum_{k=1}^{I} a_k c_k + \sum_{k=I+1}^{J} a_k c_k.$$

By the above observations, we know that

$$a_i < -Area(\Gamma_i) \quad \text{for } 1 \leq i \leq I,$$
$$c_j > 0 \quad \text{for } 1 \leq j \leq J.$$

So there should exist a_k for $I+1 \leq k \leq J$ such that $a_k > 0$. Note that $I \neq J$ from this observation. Since $a_k = \int_{\Gamma_k} f$, there should be a point $p \in \Gamma_k \subset \Gamma$ satisfying $f(p) > 0$. It completes the second part of the theorem. $\square$

3. Non-existence of Γ in M

In this section, we study some conditions of Γ under which a stable minimal hypersurface does not exist in M. All of the next conditions come from the observation of Theorem 2.1.

Theorem 3.1. *If Γ has one component, then there is no stable minimal hypersurface in M.*

Proof. Suppose that there exists a stable minimal hypersurface in M. If Γ has only one component, then $\Sigma = \Gamma$. Hence $\Gamma \subset \{x \in M | f(x) < -1\}$. On the other hand, by Theorem 2.1 Γ must also intersect with the set $\{x \in M | f(x) > 0\}$ but it is impossible. It completes the proof. □

Remark 3.1. *A proof of theorem 3.1 is already known(for example, see* [4]*), but we gave another simple proof here by utilizing the facts in* [5] *and the equation (2.4).*

The following two theorems look difficult, but the proofs are easy due to Theorem 2.1.

Theorem 3.2. *If Γ is contained in $\{x \in M | f(x) \leq 0\}$, then there is no stable minimal hypersurface in M.*

Proof. The proof is done by Theorem 2.1, because there is no chance for Γ to intersect with the set $\{x \in M | f(x) > 0\}$. □

Theorem 3.3. *If Γ is contained in $\{x \in M | f(x) \geq -1\}$, then there is no stable minimal hypersurface in M.*

Proof. The proof comes from the observation that there is no place where Σ lives. □

Remark 3.2. *¿From our observation it can be concluded that, in order for M to have a stable minimal hypersurface, Γ should have more than two components and f must take both positive and negative(in fact, less than -1) values in Γ.*

References

[1] A.L. Besse, *Einstein Manifolds*, Springer-Verlag, New York, (1987).

[2] J. Chang, *Topological aspects of the three dimensional critical point equation*, Honam mathmatical J. **27** No. 3 (2005), 477-485.

[3] A.E. Fischer and J.E. Marsden, *Manifolds of Riemannian Metrics with Prescribed Scalar Curvature*, Bulletin of the AMS. **80** (1974), 479-484.

[4] S. Hwang, *The critical point equation on a three-dimensional compact manifold*, Procedings of the AMS. **131** No. 10 (2003), 3221-3230.

[5] S. Hwang, *Stable minimal hypersurfaces in a critical point equation*, Communications of the KMS. **20** No. 4 (2005), 775-779.

[6] H. B.Lawson, *Minimal varieties in real and complex geometry*, University of Montreal lecture notes, (1974).

[7] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** No. 3 (1962), 333-340.

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