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DIFFERENTIAL EQUATIONS ON WARPED PRODUCTS (II)

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Abstract. In this paper, we consider the problem of achieving a prescribed scalar curvature on warped product manifolds according to fiber manifolds with zero scalar curvature.

1. Introduction

One of the well-known problems in differential geometry is that of whether a given smooth function on a compact Riemannian manifold is necessarily the scalar curvature of some metric. In order to study these kinds of problems, we need some analytic methods in differential geometry, because they have the forms of differential equations.

In recent work, some authors have considered the problem of scalar curvature functions on a warped product manifold and obtained partial results about the existence and nonexistence of a warped metric with some prescribed scalar curvature function (cf. [5], [6],[7],[8], [9]).

In this paper, using the upper solution and lower solution methods, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M = B \times_f F$ in terms of its warping function f and the scalar curvatures of B and F. Using upper solution and lower solution

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methods, we treat the existence of a warping function f such that the resulting metric admits the prescribed scalar curvature function.

In this paper, we extend the results of Theorem 2.6 in [5]. That is, we show that if $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that

$$\frac{4n}{n+1}Bt^\beta \geq R(t) \geq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad for \quad t \geq t_0,$$

where $t_0 > a$, $0 < \beta$, $0 < c < 1$ and B are positive constants, then equation (2.4) has a positive solution on $[a,\infty)$.

2. Main Results

Let (N,g) be a Riemannian manifold of dimension n and let $f : [a,\infty) \to R^+$ be smooth function, where a is a positive number. The Lorentzian warped product of N and $[a,\infty)$ with warping function f is defined to be the product manifold $([a,\infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g. \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N,g). Then the scalar curvature $R(t,x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{2.2}$$

for $t \in [a,\infty)$ and $x \in N$ (for details, cf. [3] or [4]). If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u'' - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian n-manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [4], we have the following proposition.

PROPOSITION 2.1. *If the scalar curvature of the fiber manifold N is arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $[a,\infty)$ such that the resulting Lorentzian warped product metric on $[a,\infty)\times_f N$ produces positive constant scalar curvature.*

However, the results of [4] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature.

Remark 2.2. Theorem 5.5 in [10] implies that all timelike geodesics are future complete on $[a,+\infty)\times_{f(t)} N$ if and only if $\int_{t_0}^{+\infty}\frac{f(t)}{\sqrt{1+f(t)^2}}dt = +\infty$ and Remark 2.58 in [1] implies that all null geodesics are future complete if and only if $\int_{t_0}^{+\infty} f(t)dt = +\infty$ (See also Theorem 4.1 and Remark 4.2 in [2]). □

We assume that the fiber manifold N of $M=[a,\infty)\times_f N$ has a positive scalar curvature, where a is a positive number. If we let $u(t)=t^\alpha$, where $\alpha\in(0,1)$ is a constant, then we have

$$R(t,x) > -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} \geq -\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}, \quad t>a.$$

By Proposition 2.4 in [5], we have the following:

PROPOSITION 2.3. *If $R(g)=0$, then there is no positive solution to equation (2.3) with*

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad for \quad t\geq t_0,$$

where $c>1$ and $t_0>a$ are constants.

If N has a zero scalar curvature, then equation (2.3) becomes

$$\frac{4n}{n+1}u''(t) - R(t,x)u(t) = 0. \tag{2.4}$$

If $R(t,x)$ is the function of only t-variable, then we have the following theorem.

PROPOSITION 2.4. *Suppose that $R(g) = 0$ and $R(t,x) = R(t) \in C^\infty([a,\infty))$. Assume that for $t > t_0$, there exist an upper solution $u_+(t)$ and a lower solution $u_-(t)$ of equation (2.4) such that $0 < u_-(t) \leq u_+(t)$. Then there exists a solution $u(t)$ of equation (2.4) such that for $t > t_0$ $0 < u_-(t) \leq u(t) \leq u_+(t)$.*

Proof. See the proof of Theorem 2.5 in [5]. □

Theorem 2.5. *Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a function such that*

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \leq \frac{4n}{n+1} d \times t^\alpha \quad \text{for} \quad t > t_0,$$

where $t_0 > a, 0 < c < 1, d > 0$ and $\alpha > 0$ are constants. Then equation (2.4) has a positive solution on $[a,\infty)$.

Proof. Since $R(g) = 0$, put $u_+(t) = t^{\frac{1}{2}}$. Then $u''_+(t) = \frac{-1}{4} t^{\frac{1}{2}-2}$. Hence

$$\begin{aligned} &\frac{4n}{n+1} u''_+(t) - R(t)u_+(t) = \frac{4n}{n+1}\frac{-1}{4} t^{\frac{1}{2}-2} - R(t)t^{\frac{1}{2}} \\ =&\frac{4n}{n+1} t^{\frac{1}{2}}[\frac{-1}{4}t^{-2} - \frac{n+1}{4n}R(t)] \leq \frac{4n}{n+1} t^{\frac{1}{2}-2}[-\frac{1}{4} + \frac{c}{4}] \leq 0 \end{aligned}$$

Therefore $u_+(t)$ is our upper solution. And put $u_-(t) = e^{-t^\beta}$, where β is a positive constant and will be determined later. Then

$$u''_-(t) = e^{-t^\beta}[\beta^2 \times t^{2\beta-2} - \beta(\beta-1)t^{\beta-2}].$$

Hence

$$\begin{aligned} &\frac{4n}{n+1} u''_-(t) - R(t)u_-(t) = \frac{4n}{n+1}(e^{-t^\beta}[\beta^2 \times t^{2\beta-2} - \beta(\beta-1)t^{\beta-2}]) - R(t)e^{-t^\beta} \\ \geq&\frac{4n}{n+1} e^{-t^\beta} \times t^{\beta-2}[\beta^2 \times t^\beta - \beta(\beta-1) - d \times t^{\alpha-\beta+2}] \geq 0 \end{aligned}$$

for large $\beta > \alpha + 2$ and large t. Since $t > t_0 > a$, we can see that for large t, $u_-(t)$ is a lower solution and $0 < u_-(t) < u_+(t)$. By Proposition 2.4, equation (2.4) has a positive solution $u(t)$ such that $0 < u_-(t) \leq u(t) \leq u_+(t)$. □

Corollary 2.6. *Suppose that* $R(g) = 0$. *Assume that* $R(t,x) = R(t) \in C^\infty([a,\infty))$ *is a function such that*

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \leq 0 \quad \textit{for} \quad t > t_0,$$

where $t_0 > 0$ *and* $0 < c < 1$ *are constants. Then equation (2.4) has a positive solution on* $[a,\infty)$ *and on* M *the resulting Lorentzian warped product metric is a future geodesically complete metric of non-positive scalar curvature outside a compact set.*

Proof. Since $R(g) = 0$ and $R(t,x) \leq 0$, the lower solution $u_-(t) = c_-$ is a small positive constant and the upper solution $u_+(t) = t^{\frac{1}{2}}$ as in Theorem 2.5. Therefore equation (2.4) has a positive solution $u(t) = f^{\frac{n+1}{2}}(t)$ such that $0 < u_- \leq u(t) \leq u_+(t)$. Hence

$$\int_{t_0}^{\infty} \frac{f(t)}{\sqrt{1+f(t)^2}}dt = \int_{t_0}^{\infty} \frac{u(t)^{\frac{2}{n+1}}}{\sqrt{1+u(t)^{\frac{4}{n+1}}}}dt \geq \int_{t_0}^{\infty} \frac{c_-^{\frac{2}{n+1}}}{\sqrt{1+c_-^{\frac{4}{n+1}}}}dt \to \infty.$$

and

$$\int_{t_0}^{\infty} f(t)dt = \int_{t_0}^{\infty} u(t)^{\frac{2}{n+1}}dt \geq \int_{t_0}^{\infty} c_-^{\frac{2}{n+1}}dt \to \infty.$$

□

We consider some special cases about scalar curvatures on $M = [a,\infty) \times_f N$ with $R(g) = 0$.

Example 2.7 we consider the Lorentzian warped product manifold with

$$R(t,x) = R(t) = -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t > t_0,$$

where $0 < c < 1$ is a constant.

If we use the technique of Cauchy-Euler equation, then we have the following concrete solutions of equation (2.4):

$$u_1(t) = t^{\frac{1-\sqrt{1-c}}{2}} \quad \text{and} \quad u_2(t) = t^{\frac{1+\sqrt{1-c}}{2}} \quad \text{for} \quad t > t_0.$$

The function $u_1(t)$ is between our lower solution and upper solution as in the proof of Corolly 2.6. However, the function u_2 is bounded by our lower solution of Corolly 2.6, but is not bounded by our upper solution of Corolly 2.6.

By Remark 2.2, the warped product manifold using the warping function $f(t) = u_1(t)^{\frac{2}{n+1}}$ or $f(t) = u_2(t)^{\frac{2}{n+1}}$ is future timelike and future null geodesically complete.

Example 2.8 In example 2.7, if $c = 1$, then we have two solutions of equation (2.4) :

$$u_1(t) = t^{\frac{1}{2}} \quad \text{and} \quad u_2(t) = t^{\frac{1}{2}} \ln t \quad \text{for} \quad t > t_0.$$

Therefore the resulting manifold is future timelike and future null geodesically complete.

Corollary 2.9. *Suppose $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a function such that*

$$0 \le R(t) \le \frac{4n}{n+1}\frac{C}{4}\frac{1}{t^k} \quad \textit{for} \quad t > t_0,$$

where $c > 0$ and $k > 2$ are constants. Then equation(2.4) has a positive solution on $[a,\infty)$ and on M the resulting Lorentzian warped product metric is a future geodesically complete metric of non-negative scalar curvature outside a compact set.

Proof. Since $R(t) \geq 0$ as in the proof of Theorem 2.5, put $u_+(t) = t^{\frac{1}{2}}$, which is clearly our upper solution. And put $u_-(t) = t^{-\alpha}$, where $0 < \alpha < 1$ is a constant. Then

$$\frac{4n}{n+1}u_-''(t) - R(t)u_-(t) = \frac{4n}{n+1}\alpha(\alpha+1)t^{-\alpha-2} - R(t)t^{-\alpha}$$
$$= \frac{4n}{n+1}t^{-\alpha}[\alpha(\alpha+1)t^{-2} - R(t)] \geq \frac{4n}{n+1}t^{-\alpha}[\alpha(\alpha+1)t^{-2} - \frac{C}{4}\frac{1}{t^k}] \geq 0$$

for large t. We can also see that for large t, $u_-(t)$ is a lower solution and $0 < u_-(t) < u_+(t)$. By Proposition 2.4, equation (2.4) has a positive solution $u(t)$ such that $0 < u_-(t) < u(t) < u_+(t)$. By remark 2.1 for $f(t) = u(t)^{\frac{2}{n+1}}$

$$\int_{t_0}^{\infty} \frac{f(t)}{\sqrt{1+f(t)^2}}dt \geq \int_{t_0}^{\infty} \frac{1}{\sqrt{2}}t^{-\frac{2\alpha}{n+1}}dt \to \infty.$$

and

$$\int_{t_0}^{\infty} f(t)dt \geq \int_{t_0}^{\infty} t^{-\frac{2\alpha}{n+1}}dt \to \infty.$$

since $n \geq 3$ and $-1 < -\frac{2\alpha}{n+1} < 0$. Hence the resulting Lorentzian warped product metric is a future geodesically complete metric. □

Example 2.10 We consider the Lorentzian warped product manifold with $R(t,x) = R(t) = \frac{4n}{n+1}\frac{C}{4}\frac{1}{t^2}$ for $t \geq t_0$, where C is a positive constant. Using the technique of Cauchy-Euler equation, we have the following concrete solutions of equation (2.3):

$$u_1(t) = t^{\frac{1-\sqrt{1+C}}{2}} \quad \text{and} \quad u_2(t) = t^{\frac{1+\sqrt{1+C}}{2}}.$$

The function $u_1(t)$ is bounded by our lower solution and upper solution as in the proofs of Theorem 2.5 or Corollary 2.9, but the function $u_2(t)$ is not bounded. The warped product manifold using the warping function $f(t) = u_1(t)^{\frac{2}{n+1}}$ or $f(t) = u_2(t)^{\frac{2}{n+1}}$ is future timelike geodesically complete, but if $0 < C \leq (n+1)(n+3)$, then the warped product manifold using the warping function $f(t) = u_1(t)^{\frac{2}{n+1}}$ is also future null geodesically complete.

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