Honam Mathematical J. **28** (2006), No. 3, pp. 395–398

ON A CERTAIN CLASS OF INTEGRAL-FUNCTIONAL EQUATIONS

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Abstract. *In this note, for any given positive integer n, we determine all the continuous solutions $f : \mathrm{R} \longrightarrow \mathrm{R}$ of the integral-functional equation*

$$f^n(x) = n \int_0^x f(t)dt.$$

Functional equations, especially integral equations, has many applications and so their studies are worthy of attention. M. Akkouchi in [1] has found all the solutions $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the functional equation

$$f(x - f(y)) = f(x) + f(f(y)) - axf(y) - bf(y) - c$$

where a, b, c are real numbers. In [2], new recursions are derived in order to calculate the solution and the derivatives of special equation

$$\phi(t) = b \int_{at-a+1}^{at} \phi(\tau)d\tau \quad (b = \frac{a}{a-1}),$$

with the real variable t and a parameter $a > 1$. This equation has applications in probability problems which are given by G. J. Wirsching. In [3] new proofs have been given on the existence and uniqueness of

Received August 16, 2006. Revised September 13, 2006.

2000 Mathematics Subject Classification :39B05, 39B22.

Key words and phrases : Integral-functional equation.

†This research was in part supported by a grant from Islamic Azad University.

the solution of the Volterra linear equation, defined by

$$x(t) + \int_0^t k(t-\tau)x(\tau)d(\tau) = f(t).$$

Using their results, the authors in [4] have found the solution of the equation in the field of Mikusinski operators which corresponds to an integro- differential equation.

Recently, P. Perry (unpublished), has found the unique continuous solution f of a Volterra integral equation given by

$$f(x) = f_0(x) - \int_x^{-\infty} k(x,y)f(y)dy$$

where f_0 is a given continous function from $\mathbb{R}$ into a complete normed linear space.

In this paper, we also consider the special Volterra integral equation, given by

$$f^n(x) = n\int_0^x f(t)dt$$

where n is a positive integer and find all its continuous solutions.

Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and n be a natural number. Then

$$f^n(x) = n\int_0^x f(t)dt \tag{1}$$

for all $x \in \mathbb{R}$ if and only if there are $0 \le a \le +\infty$ and $-\infty \le b \le 0$ such that

$$f(x) = \begin{cases} ((n-1)(x-a))^{\frac{1}{n-1}}, & (x > a) \\ 0, & (b < x < a) \\ (n-1)(x-b)^{\frac{1}{n-1}}, & (x < b) \end{cases}$$

where n is even, and

$$f(x) = \begin{cases} \pm((n-1)(x-1))^{\frac{1}{n-1}}, & (x > a) \\ 0, & (x < a) \end{cases}$$

where n is odd. Also $f(a) = f(b) = 0$ whenever $a, b \in \mathbb{R}$.

Proof. We note that (1) implies the differentiability of the function $f^n(x)$, but not necessarily of the function $f(x)$. Suppose that (1) holds. If $x_0 < x_1$ are positive numbers such that $f(x_0) = f(x_1) = 0$, then $\int_{x_0}^{x_1} f(t)dt = 0$ and by the intermediate value theorem there is $x_2 \in (x_0, x_1)$ so that $f(x_2) = 0$.

Suppose that $c > 0$ and $f(c) = 0$. If there is $c_1 \in (0, c)$ such that $f(c_1) \neq 0$ then there is $\delta > 0$ such that $f(t) \neq 0$ for all $t \in (c_1 - \delta, c_1 + \delta)$. Put

$$c_2 = inf\{t \in (c_1, c] : f(t) = 0\}$$

and

$$c_3 = \sup\{t \in [0, c_1) : f(t) = 0\}.$$

Since $f(c_3) = f(c_2) = 0$ and $c_3 < c_2$, there is $c_4 \in (c_3, c_2)$ so that $f(c_4) = 0$ and this contradicts the definition of c_2 or c_3. Hence, $f(x) = 0$ for all $x \in [0, c]$. Put $a = \sup\{t \in [0, \infty) : f(t) = 0\}$. If $a \geq 0$ is a real number and $x > a$ then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$\lim_{h \to 0} \frac{f^n(x+h) - f^n(x)}{h(f^{n-1}(x+h) + f^{n-2}(x+h)f(x) + \cdots + f(x+h)f^{n-2}(x) + f^{n-1}(x))}$$

$$= (f^n)'(x)\tfrac{1}{nf^{n-1}(x))}.$$

Hence, $f^n(x) = \int_0^x f(t)dt$ implies that

$$n\ f^{n-1}(x)f'(x) = nf(x), \quad \text{for all} \quad x > a.$$

Now, if $n = 1$ then $f'(x) = f(x)$, which implies that $f(x) = e^{x+d}$ for some constant d and all $x > a$. But by the continuity of f at a we have $e^{a+d} = 0$, a contradiction. So $f(x) = 0$ if $x \geq 0$.

If $n \neq 1$, then considering the fact that $f^{n-2}(x)f'(x) = 1$ for $x > a$ implies;

$$\int_{a+\varepsilon}^{x} f^{n-2}(t)f'(t)dt = x - a - \varepsilon \ \ \forall \varepsilon > 0;$$

thus,

$$\frac{f^{n-1}(x) - f^{n-1}(a+\varepsilon)}{n-1} = x - a - \varepsilon.$$

Letting $\varepsilon \to 0$ in the above formula, we get $f^{n-1}(x) = (n-1)(x-a)$, for $x > a$. Consequently, $f(x) = ((n-1)(x-a))^{\frac{1}{n-1}}$ if n is even and $f(x) = \pm((n-1)(x-a))^{\frac{1}{n-1}}$ if n is odd.

Let $b = inf\{t \in (-\infty, 0] : f(t) = 0\}$. Using a similar argument, one can show that if $n = 1$ then $f(x) = 0$ for all $x \leq 0$; furthermore, if $n \neq 1$ and b is a real number then $f^{n-1}(x) = (n-1)(x-b)$ for all $x < b$, which is impossible whenever n is odd. So if n is odd then $f(x) = 0$ for all $x \leq 0$, and if n is even then $f(x) = ((n-1)(x-b))^{\frac{1}{n-1}}$ whenever $x \leq b$. The converse is obvious. □

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