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ON THREE CONDITIONS ON A PERTURBED CANTOR SET

In-Soo Baek

Abstract. We study three conditions which seem similar but a little different on a perturbed Cantor set. Since they give different conditions on a perturbed Cantor set, we have another results corresponding to the conditions. We compare the conditions and give different examples which provide different results.

1. Introduction

In probabilistic case, the condition on a random Cantor set([2, 8]) plays an important role to give its exact fractal dimensions. To produce a more general result for a random Cantor set, we need a more general condition for a random Cantor set to be applied to some formula to give its Hausdorff or packing dimension. We need to compare these conditions more efficiently in deterministic case for applying these to probabilistic case. They can be easily compared by the uniform boundedness(cf [6]) of the contraction ratios. The most generalized form of a Cantor set([3]) is a deranged Cantor set which is a perturbed Cantor set([1]) in local sense. So we can apply the results in a perturbed Cantor set to that of the most generalized Cantor set sometimes. The three conditions to be discussed here are already used for some purpose in many publications.

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2. Preliminaries

We define a deranged Cantor set([3]). Let I_ϕ =[0,1]. We can obtain the left subinterval $I_{\tau,1}$ and the right subinterval $I_{\tau,2}$ of I_τ deleting middle open subinterval of I_τ inductively for each $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \cdots$. Consider $E_n = \cup_{\tau\in\{1,2\}^n} I_\tau$. Then $\{E_n\}$ is a decreasing sequence of closed sets. For each n, we put $\mid I_{\tau,1} \mid / \mid I_\tau \mid = c_{\tau,1}$ and $\mid I_{\tau,2} \mid / \mid I_\tau \mid = c_{\tau,2}$ for all $\tau \in \{1,2\}^n$,where $\mid I \mid$ denotes the diameter of I. We call $F = \bigcap_{n=0}^{\infty} E_n$ a deranged Cantor set. We note that if $c_{\tau,1} = a_{n+1}$ and $c_{\tau,2} = b_{n+1}$ for all $\tau \in \{1,2\}^n$ for each n then $F = \bigcap_{n=0}^{\infty} E_n$ is called a perturbed Cantor set([1]). We recall the s-dimensional Hausdorff measure of F :

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F),$$

where $H^s_\delta(F) = \inf\{\sum_{n=1}^{\infty} \mid U_n \mid^s : \{U_n\}_{n=1}^{\infty}$ is a δ-cover of $F\}$, and the Hausdorff dimension([8]) of F :

$$\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\}(= \inf\{s > 0 : H^s(F) = 0\}).$$

Also we recall the s-dimensional packing measure of F:

$$p^s(F) = \inf\{\sum_{n=1}^{\infty} P^s(F_n) : \bigcup_{n=1}^{\infty} F_n = F\},$$

where $P^s(E) = \lim_{\delta\to 0} P^s_\delta(E)$ and $P^s_\delta(E) = \sup\{\sum_{n=1}^{\infty} \mid U_n \mid^s : \{U_n\}$ is a δ-packing of E $\}$, and the packing dimension([8]) of F:

$$\dim_p(F) = \sup\{s > 0 : p^s(F) = \infty\}(= \inf\{s > 0 : p^s(F) = 0\}).$$

3. Main results

Now we introduce the three conditions on a perturbed Cantor set having contraction ratios a_n, b_n for each $n \in \mathbb{N}$ as follows:

For a real number $s \in (0,1)$

I. $\lim_{n\to\infty} \prod_{k=1}^{n} (a_k^s + b_k^s)$ is in $(0, \infty)$.

II. $\lim_{n\to\infty} s_n = s$ where $a_n^{s_n} + b_n^{s_n} = 1$.
III. $\lim_{n\to\infty} x_n = s$ where $\prod_{k=1}^{n}(a_k^{x_n} + b_k^{x_n}) = 1$.

Proposition 1. *Condition I is equivalent to the condition that* $\lim_{n\to\infty}\sum_{k=1}^{n}\log(a_k^s + b_k^s)$ *is in* $\mathbb{R}$.

Proof. It follows from that the log function is a continuous function. □

Remark 1. A self-similar Cantor set F with contraction ratios a, b satisfies the condition I. In this case, the real number s which is the solution of the equation $a^s + b^s = 1$ gives that $\lim_{n\to\infty}\prod_{k=1}^{n}(a_k^s + b_k^s) = 1$.

Proposition 2. *If a perturbed Cantor set satisfies the condition I, then it satisfies the condition II.*

Proof. Fix a real number $s \in (0,1)$. Assume that a perturbed Cantor set satisfies the condition I. By the above Proposition we see that the real sequence $(\sum_{k=1}^{n}\log(a_k^s + b_k^s))$ is a convergent sequence. Since the sequence $(\sum_{k=1}^{n}\log(a_k^s + b_k^s))$ is a real convergent sequence, it is a Cauchy sequence. Therefore the real sequence $(\log(a_n^s + b_n^s))$ converges to 0 as $n \to \infty$. Since the contraction ratios and gap ratios are uniformly bounded away from 0, we may assume that there is a number $\beta \in (0,1)$ such that $0 < a_n, b_n < \beta$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} s_n \neq s$ where $a_n^{s_n} + b_n^{s_n} = 1$. Then there exists a positive number $\epsilon > 0$ such that $s_n \notin (s-\epsilon, s+\epsilon)$ for infinitely many $n \in \mathbb{N}$. If $s_n > s + \epsilon$ for infinitely many $n \in \mathbb{N}$, we have

$$a_n^s + b_n^s = a_n^{s_n + (s - s_n)} + b_n^{s_n + (s - s_n)} \geq \beta^{s - s_n} \geq \beta^{-\epsilon} > 1.$$

If $s_n < s - \epsilon$ for infinitely many $n \in \mathbb{N}$, we have

$$a_n^s + b_n^s = a_n^{s_n + (s - s_n)} + b_n^{s_n + (s - s_n)} \leq \beta^{s - s_n} \leq \beta^{\epsilon} < 1.$$

If $s_n \notin (s-\epsilon, s+\epsilon)$ for infinitely many $n \in \mathbb{N}$, then $s_n > s+\epsilon$ for infinitely many $n \in \mathbb{N}$ or $s_n < s-\epsilon$ for infinitely many $n \in \mathbb{N}$. In either case, $a_n^s + b_n^s \notin (\beta^\epsilon, \beta^{-\epsilon})$ for infinitely many $n \in \mathbb{N}$, which means the sequence $(a_n^s + b_n^s)$ cannot converge to 1. □

Proposition 3. *If a perturbed Cantor set satisfies the condition II, then it satisfies the condition III.*

Proof. It follows from the fact([5]) that

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} s_n.$$

□

We have examples showing that the three conditions are quite different. The following example shows that the condition III does not imply the condition II.

Example 1. Consider a perturbed Cantor set whose contraction ratios $a_{2n-1} = b_{2n-1} = \frac{1}{4\sqrt{2}}$ and $a_{2n} = b_{2n} = \frac{1}{2\sqrt{2}}$ where for each $n \in \mathbb{N}$. This perturbed Cantor set satisfies the condition III. Precisely, $x_{2n-1} = \frac{4n-2}{8n-3}$ and $x_{2n} = \frac{1}{2}$ for each $n \in \mathbb{N}$, which means $\lim_{n\to\infty} x_n = \frac{1}{2}$ where $\prod_{k=1}^{n}(a_k^{x_n} + b_k^{x_n}) = 1$. However this perturbed Cantor set does not satisfy the condition III. Precisely, $s_{2n-1} = \frac{2}{5}$ and $s_{2n} = \frac{2}{3}$ for each $n \in \mathbb{N}$, which means $\lim_{n\to\infty} s_n$ does not exist where $a_n^{s_n} + b_n^{s_n} = 1$.

The following example shows that the condition II does not imply the condition I.

Example 2. Consider a perturbed Cantor set whose contraction ratios $a_n = b_n = \frac{e^{\frac{2}{n}}}{4}$ where for each $n \in \mathbb{N}$. This perturbed Cantor set satisfies the condition II. Precisely, $(a_n^{\frac{1}{2}} + b_n^{\frac{1}{2}}) = e^{\frac{1}{n}} \to 1$ as $n \to \infty$. By the arguments of the proof of Proposition 2, if $(a_n^{\frac{1}{2}} + b_n^{\frac{1}{2}}) \to 1$ as $n \to \infty$, then $\lim_{n\to\infty} s_n = \frac{1}{2}$ where $a_n^{s_n} + b_n^{s_n} = 1$. However this perturbed Cantor set does not satisfy the condition I. Precisely, $\lim_{n\to\infty}\sum_{k=1}^{n}\log(a_k^{\frac{1}{2}}+b_k^{\frac{1}{2}})=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}=\infty$.

From the distortion condition of contraction ratios([6]), we have the equivalence of Hausdorff measure and lower Cantor measure([7]). Similarly packing measure and upper Cantor measure are equivalent except for some singular behaviour in a perturbed Cantor set([7]). First we introduce the distortion condition of contraction ratios and perturbed Cantor set has the condition.

Remark 2. ([6]) In the perturbed Cantor set, we note that

$$\frac{c_{\tau,l_1}c_{\tau,l_1,l_2}\cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1}c_{\sigma,l_1,l_2}\cdots c_{\sigma,l_1,l_2,\cdots,l_m}}=1$$

for all $\tau,\sigma\in\{1,2\}^k$where k is any non-negative integer.

Remark 3. ([7, 9]) Let F be a quasi-perturbed Cantor set([7]), that is, for all $\tau,\sigma\in\{1,2\}^k$where k is any non-negative integer,

$$\frac{c_{\tau,l_1}c_{\tau,l_1,l_2}\cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1}c_{\sigma,l_1,l_2}\cdots c_{\sigma,l_1,l_2,\cdots,l_m}}\geq B$$

for all m where $B>0$. Then Hausdorff measure H^s and lower Cantor measure h^s are equivalent and the packing measure p^s and the upper Cantor measure q^s are equivalent except that if $p^s(F)=0$ then $q^s(F)=0$. Here $h^s(F)=\liminf_{n\to\infty}\sum_{\sigma\in\{1,2\}^n}\mid I_\sigma\mid^s$ and $q^s(F)=\limsup_{n\to\infty}\sum_{\sigma\in\{1,2\}^n}\mid I_\sigma\mid^s$ for $s\in(0,1)$ and a deranged Cantor set F([6]). Since a perturbed Cantor set is clearly a quasi-perturbed Cantor set, Hausdorff measure H^s and lower Cantor measure h^s are equivalent and the packing measure p^s and the upper Cantor measure q^s are equivalent except that if $p^s(F)=0$ then $q^s(F)=0$. Further $h^s(F)=\liminf_{n\to\infty}\prod_{k=1}^{n}(a_k^s+b_k^s)$ and $q^s(F)=\limsup_{n\to\infty}\prod_{k=1}^{n}(a_k^s+b_k^s)$.

Proposition 4. *If a perturbed Cantor set F satisfies the condition I, that is, for a real number $s\in(0,1)$*

$$\lim_{n\to\infty}\prod_{k=1}^{n}(a_k^s+b_k^s)$$

is in $(0,\infty)$, *then* $0 < H^s(F) \le p^s(F) < \infty$.

Proof. It is immediate from the above Remarks. □

Remark 4. Let F be a perturbed Cantor set. Then

$$\liminf_{n\to\infty} \prod_{k=1}^{n} (a_k^s + b_k^s)$$

and $H^s(F)$ are equivalent, that is, they have the same value which is one of the three values $0, \infty$ and positive finite real number. Similarly

$$\limsup_{n\to\infty} \prod_{k=1}^{n} (a_k^s + b_k^s) = \infty \Longleftrightarrow p^s(F) = \infty$$

and $\limsup_{n\to\infty} \prod_{k=1}^{n} (a_k^s + b_k^s) = 0$ implies $p^s(F) = 0$. But we cannot guarantee that $p^s(F) = 0$ implies $\limsup_{n\to\infty} \prod_{k=1}^{n} (a_k^s + b_k^s) = 0$.

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In-Soo Baek
Department of Mathematics
Pusan University of Foreign Studies
Pusan 608-738, Korea
E-mail: isbaek@pufs.ac.kr