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SOME RESULTS ON PP AND PF-MODULES

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Abstract. For a commutative ring with unity R, it is proved that R is a PF-ring if and only if the annihilator, $ann_R(a)$, for each $a \in R$ is a pure ideal in R. Also it is proved that the polynomial ring, $R[x]$, is a PF-ring if and only if R is a PF-ring. Finally, we prove that M as an R-module is PF-module if and only if $M[x]$ is a PF $R[x]$-module. Also M is a PP R-module if and only if $M[x]$ is a PP $R[x]$-module.

1. Introduction

All rings in this paper are commutative with identity and all modules are unitary. An ideal I of a ring R is called pure if for any $x \in I$, there exists $y \in I$ such that $xy = x$.

An ideal I of a ring R is called semiprime if $\sqrt{I} = I$.

A ring R is called fully semiprime if $\sqrt{I} = I$ for every ideal I of R or $\frac{R}{I}$ has no non-zero nilpotent.

For example regular Von Neumann rings are fully semiprime, or so does every Noetherian ring in which every ideal has no embedded prime ideal.

An R-module M is called a PF-module if every principal submodule xR is a flat R-module, for example every Domain.

Also M is called a PP-module if every principal submodule xR is projective. One can easily show that xR is projective if and only if the

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annihilator, $ann_R(x)$, is generated by an idempotent element. (See [3], [6])

2. PP and PF-modules

First, we state a proposition characterizing flat R-modules elementwise. This is a well known result in commutative ring theory. (see [5])

Proposition 1. An R-module M is a flat R-module if and only if for any pair of finite subsets $\{x_1, x_2, \cdots, x_n\}$ and $\{a_1, a_2, \cdots, a_n\}$ of M and R respectively, such that $\sum_{i=1}^{n} x_i a_i = 0$, there exists elements $z_1, \cdots, z_k \in M$ and $b_{ij} \in R, i = 1, 2, \cdots, k$ such that $\sum_{i=1}^{n} b_{ij} a_i = 0, j = 1, 2, \cdots, k$ and $x_i = \sum_{j=1}^{k} z_j b_{ji}, i = 1, 2, \cdots, n.$

In the following Theorem we establish that M is a PF-module if and only if $ann_R(x)$ for each $x \in M$ is a pure ideal.

Theorem 1. Let M be an R-module. Then M is a PF-module if and only if $ann_R(x)$ for each $x \in M$ is a pure ideal.

Proof. Let $x \in M$ and $ann_R(x)$ be a pure ideal. Let $x_1, x_2, \cdots, x_n \in xR$ and $a_1, a_2, \cdots, a_n \in R$ with $\sum_{i=1}^{n} a_i x_i = 0$. Then there exist $r_1, r_2, \cdots, r_n \in R$ such that $x_i = r_i x, i = 1, 2, \cdots, n$ so $\sum_{i=1}^{n} a_i r_i x = 0$. Hence $\sum_{i=1}^{n} a_i r_i \in ann_R(x)$. Since $ann_R(x)$ is a pure ideal, there exists $t \in ann_R(x)$ such that $(\sum_{i=1}^{n} a_i r_i)t = \sum_{i=1}^{n} a_i r_i$ or $\sum_{i=1}^{n} a_i r_i(t-1) = 0$. Now take, $-x \in xR$ and $r_1(t-1), r_2(t-1), \cdots, r_n(t-1) \in R$, these elements satisfy $a_1 r_1(t-1) + a_2 r_2(t-1) + \cdots + a_n r_n(t-1) = 0$ and $x_i = -x r_i(t-1) = -x r_i t + x r_i = 0 + x r_i$ (because $t \in ann_R(x)$).

Conversely, let $m \in M$ and $b \in ann_R(m)$, then $mb = 0$. Since mR is a flat R-module, there exist $z_1, z_2, \cdots, z_k \in mR$, $b_{1j} \in R, j = 1, 2, \cdots, k$ such that $b_{1j} b = 0$, $j = 1, 2, \cdots, k$ and

$$\begin{aligned} m &= z_1 b_{11} + z_2 b_{12} + \cdots + z_k b_{1k} = m r_1 b_{11} + m r_2 b_{12} + \cdots + m r_k b_{1k} \\ &= m(r_1 b_{11} + \cdots + r_k b_{1k}) \end{aligned}$$

because $z_i = mr_i$ for some $r_i \in R$. Hence $1 - r_1b_{11} - r_2b_{12} - \cdots - r_kb_{1k} \in ann_R(m)$. Also $b(1 - b_{11} - b_{12} - \cdots - b_{1k}) = b$. Therefore, $ann_R(m)$ is a pure ideal.

Al-Ezeh in [1] proved the following several propositions(2,3,4) as Lemmas and Theorem.

Proposition 2. Let $I_1, \cdots, I_n$ be a finite set of pure ideals of a ring R. Then $J = \bigcap_{j=1}^{n} I_j$ is a pure ideal.

Proposition 3. Let R be a reduced ring, and let $h(x) = \sum_{i=0}^{n} h_i x^i \in R[x]$. If $\sum_{i=0}^{m} a_i x^i \in ann_{R[x]}(h(x))$, then $a_i h_j = 0$ for each $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

Proposition 4. Let R be a PF-ring, then R is reduced.

Definition 1. Let R be a ring. Then $x \in R$ is called a regular element if there exists $y \in R$ such that $x = xyx$. The ring R is called regular provided that all the elements of R are regular.

Definition 2. Let R be a ring. Then R is called fully semiprime if $\sqrt{I} = I$ for every ideal I of R.

Corollary 1. Let R be a fully semiprime ring. Then R is a semiprime ring.

Proof. Obvious.

Lemma 1. For a ring R, the following conditions are equivalent.

a) R is fully semiprime.

b) R is regular.

c) If $x^2 \in I$ then $x \in I$ for all $x \in R$ and all ideals I of R.

Proof. $(a) \Longrightarrow (c)$: If $x^2 \in I$, then $x \in \sqrt{I} = I$.

$(c) \Longrightarrow (a)$: Let $x \in \sqrt{I}$, then $x^n \in I$ for some $n \in N$ and hence $x^2 \in I$. Therefore $x \in I$.

$(b) \Longrightarrow (c)$: Let $x^2 \in I$, then $Rx^2 \subseteq I$ and since $Rx = Rx^2$, $x \in I$.

$(c) \Longrightarrow (b)$: It is enough to show that $I \subseteq I^2$. Let $x \in I$, then $x^2 \in I^2 = J$. Therefore $x \in J = I^2$.

Lemma 2. Let R be a regular ring. Then,

a) every ideal I of R is pure.

b) R is reduced.

Proof. (a) Let $x \in I$. Hence there exists $y \in R$ such that $x = xyx$. Since $yx \in I, I$ is pure.

(b) Let a be a non-zero nilpotent element of R. Let n be the least positive integer greater than 1 such that $a^n = 0$. Hence $a \in ann_R(a^{n-1})$. Since $ann_R(a^{n-1})$ is pure (by (a)), there exists $b \in ann_R(a^{n-1})$ with $ab = a$.

Now $0 = ba^{n-1} = a^{n-1}$, since $ba = a$. This is a contradiction. Thus R is reduced.

Lemma 3. Let M be an R-module, $h(x) = h_0 + h_1x + \cdots + h_nx^n \in M[x], f(x) = a_0 \in R[x]$ and $f(x) \in ann_{R[x]}(h(x))$. Then $a_0 \in \sqrt{\bigcap_{j=0}^{n} ann(h_j)}$.

Proof. Obvious.

Lemma 4. Let M be a an R-module, $h(x) = h_0 + h_1x + \cdots + h_nx^n \in M[x], f(x) = a_0 + a_1x \in R[x]$ and $f(x) \in ann_{R[x]}(h(x))$. Then $a_i \in \sqrt{\bigcap_{j=0}^{n} ann(h_j)}$, $i = 0, 1$.

Proof. Since $f(x)h(x) = 0$. Hence we have the following:

$$
\begin{aligned}
&a_0h_0 = 0 && (0)\\
&a_0h_1 + a_1h_0 = 0 && (1)\\
&a_0h_2 + a_1h_1 = 0 && (2)\\
&a_0h_3 + a_1h_2 = 0 && (3)\\
&\quad\vdots && \vdots\\
&a_0h_{n-1} + a_1h_{n-2} = 0 && (n-1)\\
&a_1h_n + a_1h_{n-1} = 0 && (n)\\
&a_1h_n = 0 && (n+1).
\end{aligned}
$$

Now by (0) and (1), $a_0^2h_1 = 0$. And the conditions $a_0^2h_1 = 0$ and (2) imply that $a_0^3h_2 = 0$. If we continue this procedure, ultimately we shall have $a_0^{n+1}h_n = 0$, hence, $a_0 \in \sqrt{ann(h_j)}, j = 0, 1, 2, \cdots, n$. The conditions $(n+1)$ and (n) imply that $a_1^2h_{n-1} = 0$. Also $a_1^2h_{n-1} = 0$ and $(n-1)$ imply that $a_1^3h_{n-2} = 0$, and hence $a_1^nh_1 = 0$. Therefore $a_1 \in \sqrt{ann(h_j)}, j = 0, 1, 2, \cdots, n$.

Lemma 5. Let M be an R-module, $h(x) = h_0 + h_1x + \cdots + h_nx^n \in M[x], f(x) = a_0 + a_1x + a_2x^2 \in R[x]$ and $f(x) \in ann_{R[x]}(h(x))$, that is $f(x)h(x) = 0$. Then $a_i \in \sqrt{\bigcap_{j=0}^{n} ann(h_j)}, i = 0, 1, 2$.

Proof. Let $(a_0 + a_1x + a_2x^2)(h_0 + h_1x + \cdots + h_nx^n) = 0$, then we have the following conditions.

$$
\begin{aligned}
&a_0h_0 = 0 && (0)\\
&a_0h_1 + a_1h_0 = 0 && (1)\\
&a_0h_2 + a_1h_1 + a_2h_0 = 0 && (2)\\
&a_0h_3 + a_1h_2 + a_2h_1 = 0 && (3)\\
&\quad\vdots && \vdots\\
&a_0h_n + a_1h_{n-1} + a_2h_{n-2} = 0 && (n)\\
&a_0h_n + a_2h_{n-1} = 0 && (n+1)\\
&a_2h_n = 0 && (n+2).
\end{aligned}
$$

Now by (0) and (1), $a_0^2h_1 = 0$, and the conditions $a_0^2h_1 = 0$ and (2) imply that $a_0^3h_2 = 0$. If we continue this procedure, ultimately we shall have $a_0^{n+1}h_n = 0$, hence, $a_0 \in \sqrt{ann(h_j)}, j = 0, 1, 2, \cdots, n$. The conditions $(n+2)$ and $(n+1)$ imply that $a_2^2h_{n-1} = 0$. Also the conditions $a_2^2h_{n-1} = 0$ and (n) and $(n+2)$ imply that $a_2^3h_{n-2} = 0$, if we continue this procedure, we shall have, $a_2^{n+1}h_0 = 0$. Hence $a_2 \in \sqrt{ann(h_j)}, j = 0, 1, 2, \cdots, n$.

Now by (1) we have $a_0a_1h + a_1^2h_0 = 0$, hence

$$\begin{aligned} a_1^2h_0 &= -a_0a_1h_1 = -a_0(a_1h_1) = -a_0(-a_0h_2 - a_2h_0) \\ (by(2)) &= a_0^2h_2. \end{aligned}$$

Hence $a_1^2h_0 = a_0^2h_2$ and then $a_1^3h_0 = a_0^2a_1h_2 = a_0^2(-a_0h_3 - a_2h_1) = -a_0^3h_3$ (by(3) and since $a_0^2h_1 = 0$). So $a_1^3h_0 = -a_0^3h_3$. If we continue this, then we shall have $a_1^nh_0 = -a_0^nh_n$ and hence $a_1^{n+1}h_0 = -a_0^na_1h_n = a_0^na_2h_{n-1} = 0$ (by $(n+1)$ and since $a_0^nh_{n-1} = 0$). Therefore $a_1 \in \sqrt{ann(h_0)}$. Similar argument shows that $a_1 \in \sqrt{ann(h_j)}, j = 1, 2, \cdots, n$, we continue similar to last state.

Theorem 2. Let M be an R-module and $h(x) = h_0 + h_1x + \cdots + h_nx^n \in M[x]$. Then $N[x] \subseteq ann_{R[x]}(h(x)) \subseteq \sqrt{N}[x]$, where N is the annihilator of the submodule generated by $h_0, h_1, \cdots, h_n$, that is $N = ann_R(h_0, h_1, \cdots, h_n) = \bigcap_{i=1}^{n} ann(h_i)$. Moreover if $f(x) = a_0 + a_1x + \cdots + a_mx^m \in ann_{R[x]}(h(x))$ then $a_i \in \sqrt{\bigcap_{j=0}^{n} ann(h_j)}, i = 0, 1, 2, \cdots, m$.

Proof. Assume that $f(x) = \sum_{i=0}^{m} a_ix^i \in N[x]$. Then $a_i \in N, i = 0, 1, \cdots, m$. Hence $a_ih_j = 0, i = 0, 1, \cdots, m, j = 0, 1, 2, \cdots, n$ and therefore $f(x) \in ann_{R[x]}(h(x))$.

Now assume that $f(x) \in ann_{R[x]}(h(x))$. To prove $f(x) \in \sqrt{N}[x]$, one can use Lemmas 3,4 and 5 and induction.

Corollary 2. Let R be a regular ring and M be an R-module and

$f(x) = a_0 + a_1x + \cdots + a_mx^m \in ann_{R[x]}(h(x))$ where $h(x) = h_0 + h_1x + \cdots + h_nx^n \in M[x]$. Then:

a) $ann_{R[x]}(h(x)) = N[x]$ where N is the annihilator of the submodule generated by $h_0, h_1 \cdots h_n$, that is $N = ann_R(h_0, \cdots, h_n) = \bigcap_{i=0}^{n} ann(h_i)$.

b) $a_ih_j = 0, i = 0, 1, \cdots, m, j = 0, 1, \cdots, n$.

Proof. Since R is a regular ring, by Lemma 1, $\sqrt{N} = N$ and by Theorem 2 we have $ann_{R[x]}(h(x)) = \sqrt{N}[x] = N[x]$ and $a_i \in \sqrt{N} = N, i = 0, 1, \cdots, m$. Therefore $a_ih_j = 0, i = 0, 1, \cdots, m, j = 0, 1, \cdots, n$.

Theorem 3. Let R be a regular ring and M an R-module. Then $M[x]$ is a PF $R[x]$-module.

Proof. Since R is a regular ring, all R-modules are flat. Therefore M is a PF R-module. Now by Theorem 1, it is enough to show that $ann_{R[x]}(h(x))$ is a pure ideal in $R[x]$ for each $h(x) \in M[x]$.

Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in ann_{R[x]}(h(x))$ where $h(x) = h_0 + h_1x + \cdots + h_nx^n$. By Corollary 2, $a_i \in N = \bigcap_{j=0}^{n} ann(h_j), i = 0, 1, \cdots, m$. By Lemma 2(a) and Proposition 2, N is pure. Hence there exists $b_0, b_1, \cdots, b_m \in N$ such that $a_ib_i = a_i, i = 0, 1, \cdots, m$. Now our aim is to find $c(x) \in N[x]$ such that $c(x)f(x) = f(x)$. We construct this element inductively.

First, $a_0b_0 = a_0$. Consider

$$
\begin{aligned}
& (a_0 + a_1x)(b_0 + b_1 - b_1b_0) \\
= \; & a_0b_0 + a_0b_1 - a_0b_0b_1 + a_1b_0x + a_1b_1x - a_1b_0b_1x \\
= \; & a_0 + a_0b_1 - a_0b_1 + a_1b_0x + a_1x - a_1b_0x \\
= \; & a_0 + a_1x.
\end{aligned}
$$

Let $c_1 = b_0 + b_1 - b_1 b_0$. Then

$$\begin{aligned} & (a_0 + a_1 x + a_2 x^2)(c_1 + b_2 - c_1 b_2) \\ = \ & (a_0 + a_1 x)c_1 + b_2(1 - c_1)(a_0 + a_1 x) + a_2 c_1 x^2 + a_2 b_2 x^2 - a_2 b_2 c_1 x^2 \\ = \ & a_0 + a_1 x + a_2 c_1 x^2 + a_2 b_2 x^2 - a_2 c_1 x^2 \\ = \ & a_0 + a_1 x + a_2 x^2 \end{aligned}$$

Similarly, $c_2 = c_1 + b_2 - c_1 b_2, \cdots$

$$c_m = c_{m-1} + b_m - c_{m-1} b_m \text{ and}$$

$$(a_0 + a_1 x + \cdots + a_i x^i)c_i = a_0 + a_1 x + \cdots + a_i x^i$$

$$i = 0, 1, 2, \cdots, m. \text{ Moreover } c_0, c_1, \cdots, c_m \in N.$$

Thus there exists $c = c_m \in N \subset N[x]$ with $cf(x) = f(x)$.
M.W. Evans introduced the following proposition in [3].

Proposition 5. If R is a ring and M is an R-module, then the following conditions are equivalent

a) M is a PP-module.

b) For each $x \in M$ $ann_R(x) = eR$ for some idempotent $e \in R$.

Now using Proposition 5 we prove the following Theorem.

Theorem 4. Let R be a regular ring and M be an R-module. Then $M[x]$ is a PP $R[x]$-module if and only if M is a PP R-module.

Proof. It is enough to show that $ann_{R[x]}(h(x))$ is generated by an idempotent element in $R[x]$, where $h(x) = h_0 + h_1 x + \cdots + h_n x^n \in M[x]$. Since R is a regular ring, by Corollary 2, $ann_{R[x]}(h(x)) = N[x]$, where

N is the annihilator of the submodule generated by $h_0, h_1, \cdots, h_n$.

$$\begin{aligned} N &= ann_R(h_0, h_1, \cdots, h_n) \\ &= \bigcap_{i=0}^{n} ann(h_i) \\ &= \bigcap_{i=0}^{n} e_i R, e_i^2 = e_i \text{ because } M \text{ is a } PP - R - \text{module (See Proposition 5)} \\ &= e_1 e_2 \cdots e_n R \\ &= eR, \text{ where } e = e_1 e_2 \cdots e_n. \end{aligned}$$

Hence $ann_{R[x]}(h(x)) = eR[x], e^2 = e$.

Conversely, let $M[x]$ be a PP $R[x]$-module. Let $m \in M$. Since $M[x]$ is a $PP - R[x]$-module, $ann_{R[x]}(m) = g(x)R[x]$, where $g^2(x) = g(x)$. If $g(x) = b_0 + b_1 x + \cdots + b_m x^m$, then $b_0^2 = b_0$. We claim that $ann_R(m) = b_0 R$. Let $b \in ann_R(m)$. Then $bm = 0$. So $b \in g(x)R[x]$. Thus $b = (b_0 + b_1 x + \cdots + b_m x^m)(c_0 + c_1 x + \cdots + c_t x^t)$. Therefore $b = b_0 c_0$, that is $b \in b_0 R$.

For the other way around, let $b \in b_0 R$. Then $b = b_0 c_0$ for some $c_0 \in R$. Since $b_0 m = 0$, $b_0 \in ann_R(m)$. Thus $ann_R(m) = b_0 R$.

Remark. To get the analogous results in the formal power series $M[[x]]$, we assume that A_f be the ideal of M generated by the coefficients of f, that is $A_f = (a_0, a_1, \cdots)$, where $f = \sum_{i=0}^{\infty} a_i x^i \in M[[x]]$. Now if M is a Noetherian module, then A_f is finitely generated say $A_f = (a_0, a_1, \cdots, a_n)$.

Theorem 5. Let R be a Noetherian regular ring and M be a finitely generated R-module. Then we have:

a) If $h = \sum_{i=0}^{\infty} h_i x^j \in M[[x]]$ and $f = \sum_{i=0}^{\infty} a_i x^i \in ann_{R[[x]]}(h)$ then $ann_{R[[x]]}(h) = N[[x]]$ where N is the annihilator of the submodule generated by $h_0, h_1, \cdots, h_n, \cdots$, that is $N = ann_R(h_0, h_1, \cdots)$. Moreover $a_i h_j = 0, i = 0, 1, 2, \cdots, j = 0, 1, , 2 \cdots$.

b) $M[[x]]$ is a PF $R[[x]]$-module.

c) $M[[x]]$ is a PP $R[[x]]$-module if and only if M is a PP R-module.

Proof. By Remark if we set
$A_f =< a_0, a_1, \cdots >=< a_0, a_1, \cdots, a_m >$ and $A_h =< h_0, h_1, \cdots >=< h_0, \cdots, h_n >$
then the proofs of (a), (b) and (c) are similar to Corollary 2, Theorem 3 and Theorem 4 respectively.

References

[1] H. Al-Ezeh, On some propeties of the polynomial rings, Internat. J. Math. and Math. Science, 10(1987), 311-314.

[2] J. Brewer, Power series over commutative rings, Lecture notes in pure and applied mathematics, 64, Marcel Dekker, New York (1981).

[3] M. Evans, On commutative PP-rings, Pac. Math ., 41 (1972) 687-697.

[4] R. Gilmer, Mullltiplicative ideal theory, Marcel Dekker, New York, (1972).

[5] M. Larson and P. McCarthy, Multiplicative theory of ideals, Academic Press, New York and London, (1971).

[6] W. Vasconcelos, On finitely generated flat modules, Trans. Amer. Math, Soc., (1969), 505-512.

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