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$Max_R(M)$ AND ZARISKI TOPOLOGY

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Abstract. Let R be a commutative ring and let M be an R-module. Let $X = Spec_R(M)$ be the prime spectrum of M with Zariski topology. In this paper, by using the topological properties of X, we will obtain some conditions under which $Max_R(M) = Spec_R(M)$.

1. Introduction

Throughout this paper R denotes a commutative ring with identity and all modules are unitary and the notation " $\subset$ " denotes the strict inclusion. Further $\mathbb{Z}$ denotes the ring of integers.

The set of all prime ideals of R is called the prime spectrum of R and is denoted by $Spec(R)$. For a subset J of R, the variety of J relative to R, denoted by $V^R(J)$, is defined by

$$V^R(J) = \{p \in Spec(R) : J \subseteq p\}.$$

Set $Z_R = \{V^R(J) : J \subseteq R\}$. Then the elements of the set Z_R satisfy the axioms for closed sets in a topological space $Spec(R)$. The resulting topology is called the Zariski topology on $Spec(R)$. Set $X^R = Spec(R)$. Now let $f \in R$ and $X^R_f = X^R \setminus V^R(f) = \{p \in X^R : f \notin p\}$. Then the sets X^R_f form a basis of open sets for the Zariski topology on $Spec(R)$.

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Let M be an R-module. A proper submodule P of M is said to be prime if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P :_R M)$ (see [5]). The prime spectrum of M is denoted by $Spec(M)$ (or $Spec_R(M)$) and defined as

$$Spec(M) = \{P : P \textit{ is a prime submodule of } M\}.$$

Let N be a submodule of M. Then the variety of N is denoted by $V(N)$ and defined (see [4]) as

$$V(N) = \{P \in Spec(M) : (P :_R M) \supseteq (N :_R M)\}.$$

Further $V^*(N)$ is defined to be

$$V^*(N) = \{P \in Spec(M) : P \supseteq N\}.$$

The elements of the set

$$Z_M = \{V(N) : N \textit{ is a submodule of } M\}$$

satisfy the axioms for closed sets in a topological space $X = Spec(M)$ (see [4]). The resulting topology is called the Zariski topology relative to M.

Now let M be an R-module and let $Spec(M)$ (resp. $Max_R(M)$) be the set of all prime (resp. maximal) submodules of M with Zariski topology. Further let $\bar{R} = R/Ann_R(M)$. In this paper, among other results, we will obtain some conditions under which $Max_R(M) = Spec_R(M)$ (see [3.4], [3.7], and [3.11]). Also it is shown that (see 3.2), if the natural map $f : Spec_R(M) \to Spec(\bar{R})$, defined by $P \mapsto (P :_R M)/Ann_R(M)$, is surjective and the Noetherian ring $\bar{R}$ has no idempotent other than 0 and 1, then $Spec_R(M) = Max_R(M)$ if and only if $Spec_R(M)$ is a single set.

Throughout the remainder of this paper, for an R-module M, X denotes the $Spec_R(M)$ and $f : Spec_R(M) \to Spec(\bar{R})$ (resp. $g : Max_R(M) \to Max(\bar{R})$), defined by $P \mapsto (P :_R M)/Ann_R(M)$, (resp. $Q \mapsto (Q :_R M)/Ann_R(M)$) is called the natural map of X (resp. $Max_R(M)$).

2. Auxiliary results

Definition 2.1 (see [5]). Let M be an R-module. A proper submodule P of M is said to be prime if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P :_R M)$.

Remark 2.2. Let M be an R-module.

(a) Let K be a prime submodule of M. Then $(K :_R M)$ is a prime ideal of R (see [5]).

(b) If N be a maximal submodule of M, then N is a prime submodule of M and $(N :_R M)$ is a maximal ideal of R (see [5]).

(c) Let K be a submodule of M such that $(K :_R M)$ is a maximal ideal of R. Then K is a prime submodule of M (see [3]).

(d) For every non-zero R- module M the natural map of X is surjective in each of the following cases (see [4, 3.5]).

  (1) M is a finitely generated R-module.

  (2) M is a faithfully flat R-module.

  (3) M is a ring S containing R as a subring (with the same identity) such that the spectral map $\theta : Spec(S) \to Spec(R)$ defined by $P \mapsto P \cap R$ is surjective. (For example S is an integral extension of R).

(e) The natural map of X is surjective if and only if $pM_p \neq M_p$ for every $p \in V(Ann_R(M))$ (see [4, 3.3]).

Remark 2.3. Let M be an R-module and let N and L be submodules of M. Then we have the following.

(a) If $(N :_R M) = (L :_R M)$, then $V(N) = V(L)$. The converse is also true if both N and L are prime (see [4, Section 2, Result 1]).

(b) $V(N) = V((N :_R M)M)$ and $V(IM) = V^*(IM)$ for every ideal I of R (see [4, Section 2, Result 3]). Also

$$V(0) = Spec_R(M) \textit{ and } V(M) = \emptyset,$$

$$V(N) \cup V(L) = V(N \cap L).$$

Further for any index set Λ (see [4, page 419]),

$$\cap_{\lambda \in \Lambda} V(N_\lambda) = V(\sum_{\lambda \in \Lambda}(N_\lambda : M)M).$$

Also we have

$$\cap_{\lambda \in \Lambda} V^*(N_\lambda) = V^*(\sum_{\lambda \in \Lambda}(N_\lambda)).$$

Remark 2.4. Let M be a non-zero R-module, where $R \neq 0$. If M is finitely generated, then the natural map of $Max(M)$ is surjective so that $Max(M) \neq \emptyset$ (see [5, Theorem 2]).

Remark 2.5.

(a) Let M be an R-module. For each $f \in R$, we define $X_f = X \setminus V(fM)$. Then every X_f is an open set of X. Further the set $B = \{X_f : f \in R\}$ is a basis for the Zariski topology on X and for any $f, g \in R$, $X_{fg} = X_f \cap X_g$ (see [4, 4.2 and 4.3]).

(b) Let M be an R-module such that the natural map of X is surjective. Then for any $f \in R$, X_f is quasi compact (see [4, 4.4]). In particular $X_1 = Spec_R(M)$ is quasi compact.

Remark 2.6 (see [4, 6.3]). Let M be an R-module whose $Spec_R(M)$ may be $\emptyset$. Then $Spec_R(M)$ is a T_1 space if and only if $Max_R(M) = Spec_R(M)$, where $Max_R(M)$ is the set of all maximal submodules of M.

Remark 2.7 (see [4, 6.5]). Let M be an R-module such that the natural map of X is surjective. Then X is T_0 space if and only if X is homeomorphic to $Spec(R/Ann_R(M))$.

Remark 2.8. Let R be a Noetherian ring. Then the following are equivalent (see [1, Chap. 8, Theorem 8.5 and Exc. 2]).

(a) $Spec(R) = Max(R)$.

(b) $Spec(R)$ is discrete and finite.

(c) $Spec(R)$ is discrete.

(d) R is an Artinian ring.

Remark 2.9. Let M be an R-module. Then M is said to be a top module with Zariski topology, or a top module for short, if for every submodule N and L of M there exists a submodule E of M such that $V^*(N) \cup V^*(L) = V^*(E)$ (see [3]). By 2.3 (a), for a top module M, the natural map of $Spec_R(M)$ is injective.

Remark 2.10. Let R be a ring. Then R is a Jacobson ring if every prime ideal in R is an intersection of maximal ideals of R. For example every Artinian ring is a Jacobson ring. Also if R is a Jacobson ring, then $Max(R)$ is a dense subspace in topological space $Spec(R)$ (see [1, Chap. 5, Exc. 23 and 26]).

3. Main results

Proposition 3.1. Let M be an R-module such that the natural map of X is surjective. Set $\bar{R} = R/Ann_R(M)$. Then the following hold.

(a) If $Spec_R(M) = Max_R(M)$, then $Spec(\bar{R}) = Max(\bar{R})$.

(b) If $Spec_R(M) = Max_R(M)$, then $Spec_R(M)$ is a T_4 space.

Proof. (a) Let $\bar{p} \in Spec(\bar{R})$ where $Ann_R(M) \subseteq p \in Spec(R)$. Since the natural map of $Spec_R(M)$ is surjective, there exists $P \in Spec_R(M)$ such that $(P :_R M) = p$. But by assumption $P \in Max_R(M)$. Now by 2.2 (b), it follows that $\bar{p} \in max(\bar{R})$.

(b) Assume that $Spec_R(M) = Max_R(M)$. Then $Spec_R(M)$ is T_1 space by 2.6. Hence by 2.7, $Spec_R(M)$ is homeomorphic to $Spec(\bar{R})$. Thus $Spec(\bar{R})$ is a T_1 space. Now by [1, Chap. 3, Exe. 11], $Spec(\bar{R})$ is a T_1 space if and only if it is a Hausdorff topological space. Hence by the above arguments, $Spec_R(M)$ is a Hausdorff topological space. But by

[6, Chap.3, Exc.3.5], every Hausdorff and compact topological space is a T_4 space. Now the result follows from 2.5 (b).

Theorem 3.2. Let M be an R-module such that the natural map of X is surjective. Let $\bar{R} = R/Ann_R(M)$ be a Noetherian ring which contains no idempotent other than 0 and 1, then $Spec_R(M) = Max_R(M)$ if and only if $Spec_R(M)$ is a single set.

Proof. Assume that $Spec_R(M) = Max_R(M)$. Then $Spec_R(M)$ is a T_4 space by 3.1 (b). Also $Spec(\bar{R}) = Max(\bar{R})$ by 3.1 (a). Hence $Max(\bar{R})$ is finite by 2.8. But $Spec_R(M)$ is homeomorphic to $Spec(R/Ann_R(M))$ by 2.7. Thus $Spec_R(M)$ is also a finite set. Further $Spec_R(M)$ is connected by [4, 3.8]. But every connected and T_3 space is uncountable or single. Therefor, $Spec_R(M)$ has just one element. Conversely if $Spec_R(M)$ has just one element, then $Spec_R(M)$ is T_1 space so that $Spec_R(M) = Max_R(M)$ by 2.6.

Example 3.3. Since for every faithfully flat R-module M, the natural map of $Spec_R(M)$ is surjective by 2.2 (d). Hence by 3.1 (a), we have $Spec_{\mathbb{Z}}(M) \neq Max_{\mathbb{Z}}(M)$ for every faithfully flat $\mathbb{Z}$-module M.

The following theorem extends 2.8.

Theorem 3.4. Let M be a Noetherian R-module. Then the following are equivalent.

(a) $Spec_R(M) = Max_R(M)$.
(b) $Spec_R(M)$ is discrete and finite.
(c) $Spec_R(M)$ is discrete.
(d) M is an Artinian cyclic R-module.

Proof. (a) $\Rightarrow$ (b) Let $Spec_R(M) = Max_R(M)$. Set $\bar{R} = R/Ann_R(M)$ (note that M is a finitely generated R-module and $\bar{R}$ is a Noetherian ring). Then $Spec(\bar{R}) = Max(\bar{R})$ by 2.2 (d) and 3.1 (a). Hence $Spec(\bar{R})$

is discrete and finite by 2.8. But by 2.6 and 2.7, $Spec_R(M)$ is homeomorphic to $Spec(\bar{R})$. Hence $Spec_R(M)$ is discrete and finite.

(b) $\Rightarrow$ (c) This is clear.

(c)$\Rightarrow$ (d) By 2.7, $Spec_R(M)$ is homeomorphic to $Spec(\bar{R})$. Hence $Spec(\bar{R})$ is discrete topological space. This implies that $R/Ann_R(M)$ is an Artinian ring by 2.8. It follows that M is an Artinian R- module. But by [4, 6.6], M is a multiplication R-module. This implies that M is an Artinian cyclic R-module by [5, Page. 3750, Cor.2].

(d)$\Rightarrow$ (a) Since M is an Artinian cyclic R-module, $M \cong \bar{R}$ and $\bar{R}$ is an Artinian ring. Hence $Spec(\bar{R}) = Max(\bar{R})$ by 2.8. This implies that $Spec_R(M) = Max_R(M)$.

Example 3.5. Let M be a finitely generated $\mathbb{Z}$-module. Then $Spec_{\mathbb{Z}}(M) = Max_{\mathbb{Z}}(M)$ if and only if $M \cong \mathbb{Z}_n$ for some $n \in \mathbb{N}$.

Proof. Suppose that $Spec_{\mathbb{Z}}(M) = Max_{\mathbb{Z}}(M)$. Then M is a cyclic $\mathbb{Z}$-module by 3.4. Hence $M \cong \mathbb{Z}/Ann_{\mathbb{Z}}(M)$. But by 3.1 (a), $Ann_{\mathbb{Z}}(M) \neq 0$. Hence $M \cong \mathbb{Z}_n$ for some $n \in \mathbb{N}$. The reverse implication is clear.

Theorem 3.6. Let M be a top R-module. Set $\bar{R} = R/Ann_R(M)$. If $Spec(\bar{R}) = Max(\bar{R})$, then $Spec_R(M) = Max_R(M)$. In particular, for every top module M over an Artinian ring R, $Spec_R(M) = Max_R(M)$.

Proof. Since $Spec(\bar{R}) = Max(\bar{R})$, then $Spec_R(\bar{R}) = Max_R(\bar{R})$. Hence $Spec(\bar{R})$ is a T_1 space by 2.6. But by [4, 3.1], the natural map of $Spec_R(M)$ is a continuous map. Also by 2.9, the natural map of X is one to one. This implies that $Spec_R(M)$ is a T_1 space. It follows that $Spec_R(M) = Max_R(M)$ by 2.6.

Corollary 3.7. Let M be a top R-module such that the natural map of X is surjective. Then the following are equivalent.

(a) $Spec_R(M) = Max_R(M)$.
(b) $Spec(\bar{R}) = Max(\bar{R})$.
(c) $Spec_R(M)$ is a T_4 space.

Proof. Use 3.1 (a), 3.1 (b), 3.6 and 2.6.

Theorem 3.8. Let M be a top R-module. Then the following are equivalent.

(a) The natural map of $Max_R(M)$ is surjective.
(b) $pM \neq M$ for every $p \in (Max(R) \cap V(Ann_R(M)))$.
(c) $pM \in Max_R(M)$ for every $p \in (Max(R) \cap V(Ann_R(M)))$.
(d) $pM \in Spec_R(M)$ for every $p \in (Max(R) \cap V(Ann_R(M)))$.

Proof. (a) $\Rightarrow$ (b) Let $p \in (Max(R) \cap V(Ann_R(M)))$. Since the natural map of $Max_R(M)$ is surjective, there exists $P \in Max_R(M)$ such that $(P :_R M) = p$. Hence we have

$$pM = (P :_R M)M \subseteq P \subset M.$$

(b) $\Rightarrow$ (c) Let $p \in (Max(R) \cap V(Ann_R(M)))$. Since $pM \neq M$, we have $(pM :_R M) = p$. Now let $pM \subseteq Q \subset M$. Then $p \subseteq (Q :_R M) \subset R$. Hence $(Q :_R M) = (pM :_R M) = p \in Max(R)$. Therefore, Q and pM are prime by 2.2 (c). Now since M is a top module, $Q = pM$ by 2.9. Hence pM is maximal submodule of M.

(c) $\Rightarrow$ (d) This is clear by 2.2 (b).

(d) $\Rightarrow$ (a) Let $\bar{p} \in Max(\bar{R})$ so that $p \in (Max(R) \cap V(Ann_R(M)))$. Since $pM \in Spec_R(M)$, then $pM \neq M$ so that $pM \in Max_R(M)$ by part ((b) $\Rightarrow$ (c)) and $(pM :_R M) = p$.

One may think that every top R-module M with surjective natural map of $Max_R(M)$ is necessarily a multiplication R-module. The following example shows that this is not true in general.

Example 3.9. Let I be the set of all prime integers. Let M^p denote the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty) \oplus C_p$, where C_p is the cyclic group of order p. Set $M = \bigoplus_{p \in I} M^p$. Then M is a top module such that the natural map of $Max_{\mathbb{Z}}(M)$ is surjective but M is not a multiplication $\mathbb{Z}$-module.

Proof. Since the homomorphic image of every multiplication module is multiplication, then M is not multiplication by [3, 3.7]. Now let p_0 and q_0 be two distinct elements of I. Then M^{p_0} and M^{q_0} are prime-compatible (i.e. there does not exist a prime ideal P in $\mathbb{Z}$ with $Spec_P(M^{p_0})$ and $Spec_P(M^{q_0})$ both non-empty. Here for a prime ideal q of $\mathbb{Z}$, $Spec_q(M) = \{Q \in Spec_{\mathbb{Z}}(M) : (Q :_{\mathbb{Z}} M) = q\}$ (see [3, p. 98])). Otherwise, there exists a prime ideal $P = p\mathbb{Z}$ of $\mathbb{Z}$ such that $Spec_P(M^{p_0}) = \{p_0 M^{p_0}\}$ and $Spec_P(M^{q_0}) = \{q_0 M^{q_0}\}$ by 3.7. Hence

$$(p_0 M^{p_0} :_{\mathbb{Z}} M^{p_0}) = (q_0 M^{q_0} :_{\mathbb{Z}} M^{q_0}) = p\mathbb{Z} \in Max(\mathbb{Z}).$$

This implies that $p_0 = q_0$ which is a contradiction. Hence M is a top module by [3, 5.1]. Now let $q \in I$. Then by [5, 3.7],

$$qM = \bigoplus_{p \in I} qM^p = qM^q \oplus (\bigoplus_{p \neq q} M^p) \neq M.$$

It follows that $(qM :_{\mathbb{Z}} M) = q\mathbb{Z}$. It is enough to show that $qM \in Max_{\mathbb{Z}}(M)$. To see this let $qM \subseteq N \subset M$ for a submodule N of M. Then

$$q\mathbb{Z} = (qM :_{\mathbb{Z}} M) \subseteq (N :_{\mathbb{Z}} M) \subset \mathbb{Z}.$$

Hence $(qM :_{\mathbb{Z}} M) = (N :_{\mathbb{Z}} M) = q\mathbb{Z} \in Max(\mathbb{Z})$. Thus qM and N are in $Spec_{\mathbb{Z}}(M)$ by 2.2 (c). Since M is a top module, this implies that $qM = N$ and the proof is completed.

Example 3.10. Let M be a top $\mathbb{Z}$-module such that the natural map of X is surjective. Then the following are equivalent.

(a) $Spec_{\mathbb{Z}}(M) = Max_{\mathbb{Z}}(M)$.
(b) $Spec_R(M)$ is finite.
(c) M is not faithful.

Proof. (a) $\Rightarrow$ (b) Since $Spec_{\mathbb{Z}}(M) = Max_{\mathbb{Z}}(M)$, then $Spec(\mathbb{Z}/Ann_{\mathbb{Z}}(M)) = Max(\mathbb{Z}/Ann_{\mathbb{Z}}(M))$ by 3.7. It follows that $\mathbb{Z}/Ann_{\mathbb{Z}}(M) = \mathbb{Z}_n$ for some $n \in \mathbb{N}$. Hence $Spec_{\mathbb{Z}}(M)$ is homeomorphic to $Spec(\mathbb{Z}_n)$ by 2.7. Hence $Spec_R(M)$ is finite.

(b) $\Rightarrow$ (c) Since the natural map of $Spec_R(M)$ is surjective, $Spec(\mathbb{Z}/Ann_{\mathbb{Z}}(M))$ is finite. It follows that $Ann_{\mathbb{Z}}(M) \neq 0$.

(c) $\Rightarrow$ (a) Since $\mathbb{Z}/Ann_{\mathbb{Z}}(M) \neq \mathbb{Z}$, then $\mathbb{Z}/Ann_{\mathbb{Z}}(M) = \mathbb{Z}_n$ for some $n \in \mathbb{N}$. Hence $Spec(\mathbb{Z}/Ann_{\mathbb{Z}}(M)) = Max(\mathbb{Z}/Ann_{\mathbb{Z}}(M))$. It follows that $Spec_{\mathbb{Z}}(M) = Max_{\mathbb{Z}}(M)$ by 3.7.

Theorem 3.11. Let R be a Jacobson ring. Let M be an R-module such that the natural map of $Max_R(M)$ is surjective. Then

(a) $Max_R(M)$ is dense in $Spec_R(M)$.
(b) $Spec_R(M) = Max_R(M)$ if and only if $Max_R(M)$ is closed subset in $Spec_R(M)$.

Proof. (a) Let $X_f \neq \emptyset$ be a basis element of $Spec_R(M)$. Choose $P \in X_f$ and set $\bar{p} = (P :_R M)/Ann_R(M)$. Then $\bar{p} \in X_{\bar{f}}^{\bar{R}}$ so that $X_{\bar{f}}^{\bar{R}} \neq \emptyset$. Hence by 2.10, $X_{\bar{f}}^{\bar{R}} \cap Max(\bar{R}) \neq \emptyset$. Choose $\bar{q} = q/Ann_R(M) \in Max(\bar{R}) \cap X_{\bar{f}}^{\bar{R}}$. Since the natural map of $Max_R(M)$ is surjective, there exists $Q \in Max_R(M)$ such that $(Q :_R M) = q$. But $\bar{q} \in X_{\bar{f}}^{\bar{R}}$ implies that $Q \in X_f$ so that $Q \in X_f \cap Max_R(M)$. Hence $Max_R(M)$ is dense in $Spec_R(M)$.

(b) This is clear by part (a).

Corollary 3.12. Let R be a Jacobson ring. Let M be a top module such that the natural map of $Max_R(M)$ is surjective. If $Max_R(M)$ is finite, then $Spec_R(M) = Max_R(M)$. The converse is true if R is a Noetherian ring and the natural map of X is surjective.

Proof. By [4, 5.1], $cl(Max_R(M)) = V(\cap_{P \in Max_R(M)} P)$. (Here for a topological space T, $cl(T)$ denotes the topological closure of T). Since M is a top R-module and $Max_R(M)$ is a finite set, we have

$$V(\cap_{P \in Max_R(M)} P) = \bigcup_{P \in Max_R(M)} V(P) = Max_R(M).$$

It follows that $cl(Max_R(M)) = Max_R(M)$. Hence $Max_R(M)$ is a closed set. Thus $Spec_R(M) = Max_R(M)$ by 3.11 (b). Now assume that R is a Noetherian ring and $Spec_R(M) = Max_R(M)$. Set $\bar{R} = R/Ann_R(M)$. By 3.1 (a), $Spec(\bar{R}) = Max(\bar{R})$. Hence $Max(\bar{R})$ is a finite set by 2.8. Since M is a top module, the natural map of X is injective. Hence $Max(M) = Spec_R(M)$ is a finite set.

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