Honam Mathematical J. **28** (2006), No. 3, pp. 333–342

A CHARACTERIZATION OF AN SN-MATRIX RELATED WITH L-MATRIX

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Abstract. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A. A matrix A is an SN*-matrix* provided there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\tilde{A}$ is $\mathcal{S}$, for each $\tilde{A} \in \mathcal{Q}(A)$. We have a characterization of an SN-matrix related with L-matrix and we analyze the structure of an SN-matrix.

1. Introduction

The *sign* of a real number a is defined by

$$\operatorname{sign}(a) = \begin{cases} -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0, \text{ and} \\ 1 & \text{if } a > 0. \end{cases}$$

A *sign pattern* is a $(0, 1, -1)$-matrix. The *sign pattern of a matrix* A is the matrix obtained from A by replacing each entry by its sign. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A. The *zero pattern* of a matrix A is the $(0, 1)$ matrix obtained from A by replacing each nonzero entry by 1.

Received July 12, 2006. Revised August 23, 2006.

2000 Mathematics Subject Classification : 05C50.

Key words and phrases : zero pattern, conformal contraction, signed null-space.

† This work was supported by a grant from 2005 Research Fund of Andong National University.

A vector is *mixed* if it has both a positive entry and a negative entry. A matrix is *row-mixed* if each of its rows is mixed. A vector is *balanced* if it is the zero vector or is mixed. The notion of a row-balanced matrix is defined analogously. A *signing* is a nonzero, diagonal $(0, 1, -1)$-matrix. A signing is *strict* if each of its diagonal entries is nonzero. A matrix B is *strictly row-mixable* provided there exists a strict signing D such that BD is row-mixed.

Let A be an m by n matrix and b an m by 1 vector. The linear system $Ax = b$ has *signed solutions* provided there exists a collection $\mathcal{S}$ of n by 1 sign patterns such that the set of sign patterns of the solutions to $\widetilde{A}x = \tilde{b}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. This notion generalizes that of a sign-solvable linear system (see [1] and references therein). The linear system, $Ax = b$, is *sign-solvable* provided each linear system $\widetilde{A}x = \tilde{b}$ ($\widetilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$) has a solution and all solutions have the same sign pattern. Thus, $Ax = b$ is sign-solvable if and only if $Ax = b$ has signed solutions and the set $\mathcal{S}$ has cardinality 1.

A matrix A is an *SN-matrix* provided $Ax = 0$ has signed solutions. Thus, A is an SN-matrix if and only if there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. An *$L$-matrix* is a matrix, A, with the property that each matrix in $\mathcal{Q}(A)$ has linearly independent rows. A square L-matrix is a *sign-nonsingular*, or *SNS*-matrix for short. A *totally L-matrix* is an $m \times n$ matrix such that each m by m submatrix is an SNS-matrix. It is known that totally L-matrices are SN-matrices[3].

Some properties of SN-matrices have been studied in [3, 4, 5, 6]. In this paper, we show that if a strictly row-mixable m by n SN-matrix is not conformally contractible, then it is permutation equivalent to

$$\begin{bmatrix} I_k & B \\ O & C \end{bmatrix}$$

where $2 \leq k \leq m$.

We use the following standard notations throughout the paper. If k is a positive integer, then $\langle k\rangle$ denotes the set $\{1, 2, \ldots, k\}$. Let A be an $m \times n$ matrix. If α is a subset of $\{1, 2, \ldots, m\}$ and β is a subset of $\{1, 2, \ldots, n\}$, then $A[\alpha|\beta]$ denotes the submatrix of A determined by the rows whose indices are in α and the columns whose indices are in β. The submatrix complementary to $A[\alpha|\beta]$ is denoted by $A(\alpha|\beta)$. In particular, $A(\alpha|-)$ denotes the submatrix obtained from A by deleting the columns whose indices are in α. We write $\text{diag}(d_1, d_2, \ldots, d_n)$ for the n by n diagonal matrix whose (i,i)-entry is d_i. Let $J_{m,n}$ denote the m by n matrix all of whose entries are 1 and let $\mathbf{e}_i$ denote the column vector all of whose entries are 0 except for the ith entry which is 1. O denotes a zero matrix.

2. Main Results

The following results will be useful.

Theorem A([3]) *Let A be a strictly row-mixable, m by n matrix. Then A is an SN-matrix if and only if A has term rank m and its mth compound is signed.*

A real matrix is an SN-matrix if and only if its sign pattern is an SN-matrix. Hence all matrices we consider from now on are assumed to be $(0, 1, -1)$-matrices.

Let A be an m by n $(0, 1, -1)$-matrix. The matrix B is *conformally contractible* to A provided there exists an index k such that the rows and columns of B can be permuted so that B has the form

$$\left[\begin{array}{c|c|c} A[\langle m\rangle|\langle n\rangle \setminus \{k\}] & x & y \\ \hline 0 \ \cdots \ 0 & 1 & -1 \end{array}\right],$$

where $x = [x_1, \ldots, x_m]^T$ and $y = [y_1, \ldots, y_m]^T$ are $(0, 1, -1)$ vectors such that $x_i y_i \geq 0$ for $i = 1, 2, \ldots, m$, and the sign pattern of $x + y$ is the kth column of A. We say that the zero pattern of A is obtained from the zero pattern of B by a contraction. More precisely, let $A = [a_{ij}]$ be an m by n $(0, 1)$-matrix such that the row p of A contains exactly two 1's, say $a_{pr} = a_{ps} = 1$ whenever $r \neq s$. Let u be the m by 1 $(0, 1)$ column vector whose ith entry is 1 if and only if $a_{ir} = 1$ or $a_{is} = 1$. Let B the $m - 1$ by $n - 1$ matrix obtained from A by replacing column s by u and then deleting row p and column r. We say that B is the matrix obtained from A by the contraction of columns r and s on row p. It is known that if B is conformally contractible to A, then A is an SN-matrix if and only if B is an SN-matrix[3].

Theorem B([3], [6]) *Let an m by n matrix A have a k by $k+1$ submatrix B whose complementary submatrix in A has term rank $m - k$. If there is a matrix B^* obtained from B by replacing some nonzero entries with 0's if necessary such that $J_{2,3}$ is the zero pattern of a matrix obtained from B^* by a sequence of conformal contractions, then A is not an SN-matrix.*

Every m by $m + 2$ totally L-matrix has at least two columns each of which contains exactly one nonzero entry[1]. Such a property can be also found in a matrix which generalizes an m by $m + 2$ totally L-matrix as shown in the following.

Proposition 1. *Let A be a strictly row-mixable m by n SN-matrix. If each row of A has at least three nonzero entries, then A has at least two columns each of which contains exactly one nonzero entry.*

Proof. Without loss of generality, we may assume that A has no zero column. We prove it by induction on m. Trivially we have the result for

$m = 1, 2$. Let $m \geq 3$. A can be rearranged as

$$A = \begin{bmatrix} B & O \\ b & c \\ O & C \end{bmatrix},$$

where matrices B and C (possibly with no rows) are strictly row-mixable SN-matrices, and vectors b and c are nonzero(see [2] or [3]). Also

$$\begin{bmatrix} B \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ C \end{bmatrix}$$

are SN-matrices. Let α, γ and β, δ be subsets of $< m >$ and $< n >$ respectively such that $A[\alpha|\beta] = \begin{bmatrix} B \\ b \end{bmatrix}$ and $A[\gamma|\delta] = \begin{bmatrix} c \\ C \end{bmatrix}$. Let $|\alpha| = k$, $|\beta| = s$, $|\gamma| = l$ and $|\delta| = t$. Then $k + l - 1 = m$ and $s + t = n$.

Let $k > 1$ and $l > 1$. If $A[\alpha|\beta]$ has one of the unit vectors $\pm\mathbf{e}_k$ as a column, then we can assume that $A[\alpha|\beta]$ is of the form

$$\begin{bmatrix} B' & O \\ b' & 1 \end{bmatrix}.$$

If b' has at least two nonzero entries, then $A[\alpha|\beta]$ is strictly row-mixable. Since each row of $A[\alpha|\beta]$ has at least three nonzero entries, by induction $A[\alpha|\beta]$ and hence A has at least two columns each of which contains exactly one nonzero entry. Let b' have at most one nonzero entry. Since B' satisfies the hypothesis, B' has at least two columns each of which contains exactly one nonzero entry. Thus B has at least two columns each of which contains exactly one nonzero entry. Therefore we are done. Similarly we have the result in the case that $A[\gamma|\delta]$ has one of the unit vectors $\pm\mathbf{e}_1$ as a column. Assume that neither $A[\alpha|\beta]$ has the unit vectors $\pm\mathbf{e}_k$ nor $A[\gamma|\delta]$ has the unit vectors $\pm\mathbf{e}_1$ as columns. Since b is nonzero, the k by $s+1$ matrix B^* obtained from $A[\alpha|\beta]$ by adding $\mathbf{e}_k$ as the last column is strictly row-mixable. Since B is an SN-matrix, B^* is also an SN-matrix. Applying the similar method shown above to B^*, we can derive that $A[\alpha|\beta]$ has at least one column which contains exactly one nonzero entry. Similarly $A[\gamma|\delta]$ also has at least one column which contains exactly one nonzero entry. Hence we have the result.

Let $k = 1$. If $s \geq 2$, then we are done. Hence we may assume that $s = 1$ and $A = [a_{ij}]$ is of the form

$$\begin{bmatrix} 1 & c \\ O & C \end{bmatrix}$$

where $\begin{bmatrix} c \\ C \end{bmatrix}$ has no unit vectors $\pm\mathbf{e}_1$ as columns.

Since C is a strictly row-mixable $m-1$ by $n-1$ SN-matrix, by induction hypothesis, we may assume that C is of the form

$$\begin{bmatrix} C_1 & S_1 \\ O & C_{22} \end{bmatrix}$$

where C_1 is a block diagonal matrix such that the zero patterns of its diagonal blocks are of the form $(1, 1, \ldots, 1)$ and each column of $\begin{bmatrix} S_1 \\ C_{22} \end{bmatrix}$ has at least two nonzero entries. Suppose that all entries of c corresponding to the columns of $\begin{bmatrix} C_1 \\ O \end{bmatrix}$ are nonzero. If C_{22} is vacuous, we can find a submatrix of $\begin{bmatrix} c \\ C \end{bmatrix}$ whose zero pattern is permutation equivalent to one of these matrices:

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ \hline 1 & 1 & 1 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 1 & 1 & * \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 1 & * & * \\ \hline 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{ccc|cc} 1 & 1 & 1 & * & * \\ \hline 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

where $*$ is 0 or 1. It is impossible by Theorem B. Let C_{22} be non-vacuous. Since C_{22} is a strictly row-mixable SN-matrix, we may assume that C_{22} is of the form

$$\begin{bmatrix} C_2 & S_2 \\ O & C_{33} \end{bmatrix}$$

where $C_2 = [O\ C_2']$, and C_2' is a block diagonal matrix such that the zero patterns of its diagonal blocks are of the form $(1, 1, \ldots, 1)$ and each column of $\begin{bmatrix} S_2 \\ C_{33} \end{bmatrix}$ has at least two nonzero entries. Continuing this process, we can assume that C_{ii} is of the form

$$\begin{bmatrix} C_i & S_i \\ O & C_{i+1,i+1} \end{bmatrix}$$

where $C_i = [O\ C_i']$, and C_i' is a block diagonal matrix such that the zero patterns of its diagonal blocks are of the form $(1, 1, \ldots, 1)$ and each column of $\begin{bmatrix} S_i \\ C_{i+1,i+1} \end{bmatrix}$ has at least two nonzero entries for $i = 2, \ldots, q-1$. Thus we may assume that C is of the form

$$\begin{bmatrix} C_1 & & * \\ O & \ddots & \\ & & C_q \end{bmatrix}$$

where $C_{qq} = C_q = [O\ C_q'\ S_q]$. Here C_q' is a block diagonal matrix and each column of S_q has at least two nonzero entries. S_q may be vacuous. Consider a nonzero entry a_{m,j_r} in the last row of $A = [a_{ij}]$. Since each column of $\begin{bmatrix} S_{q-1} \\ C_q \end{bmatrix}$ has at least two nonzero entries, there exists a nonzero entry a_{k_r,j_r} in A for some $k_r (\neq m)$. Then there exists another nonzero entry $a_{k_r,j_{r-1}}$ in C_t for some t. Continuing this process, we obtain a sequence of nonzero entries $a_{1,j_1}, a_{k_1,j_1}, a_{k_1,j_2}, \ldots, a_{k_r,j_{r-1}}, a_{k_r,j_r}, a_{m,j_r}$ in A. Applying the process above to another two nonzero entries in the last row of A, we have a submatrix N of A whose complementary submatrix has full term rank such that there exists a matrix N^* obtained from N by replacing some nonzero entries to zero which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$. This is impossible by Theorem B. Hence one of entries of c corresponding to the columns of $\begin{bmatrix} C_1 \\ O \end{bmatrix}$ must be zero. Thus we have the result.

Similarly we have the same result for $l = 1$. ■

Proposition 1 implies that if A is a strictly row-mixable m by n SN-matrix such that it is not conformally contractible to a matrix, then the bipartite graph of A has at least two edges as endblocks and A is permutation equivalent to a matrix of the form

$$\begin{bmatrix} I_k & B \\ O & C \end{bmatrix} \tag{1}$$

where $2 \leq k \leq m$.

It is easy to show that a row-mixable m by $m+1$ matrix A is an SN-matrix if and only if A is an S^*-matrix. A necessary and sufficient condition for an m by $m+2$ matrix A to be an SN-matrix is given in the following.

Theorem 2. *Let A be a strictly row-mixable m by $m+2 (m \geq 2)$ matrix and let each row of A have at least three nonzero entries. Then A is an SN-matrix if and only if A is a totally L-matrix.*

Proof.(*Sufficiency.*) See Proposition 3 in [3] or Corollary 2.3 in [6].
(*Necessity.*) A can not have zero columns. For, if it has a zero column, the matrix A' obtained from A by deleting the zero column must be S^*-matrix. A' and hence A has a row with exactly two nonzero entries, which is impossible. Let $\mathcal{M}$ be the set of all signings D such that AD is row-balanced. Suppose that there exists a signing in $\mathcal{M}$ with at least two zero diagonal entries. Let M be a signing in $\mathcal{M}$ such that any signing obtained from M by replacing a nonzero entry with 0 is not in $\mathcal{M}$. Then there exist permutation matrices P, Q such that

$$PAQ = \begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}$$

where A_1 is a strictly row-mixable l by $m+2-k$ matrix whose columns correspond to the nonzero columns of M and $k \geq 2$. Since A is an

SN-matrix, $\begin{bmatrix} A_1 \\ O \end{bmatrix}$ and hence A_1 is an SN-matrix. Since A_1 is strictly row-mixable, $l < m + 2 - k$. The notion of the signing M implies that any $m + 2 - k - 1$ columns of A_1 is linearly independent and hence $m + 1 - k \leq l$. Thus $l = m + 1 - k$ and hence A_1 is an S^*-matrix. Therefore A_3 is $k - 1$ by k strictly row-mixable matrix. Since A_1 is S^*-matrix and A is an SN-matrix, A_3 is an SN-matrix by Theorem A. Thus A_3 is an S^*-matrix and hence A has a row exactly two nonzero entries. This is a contradiction. Therefore every signing D in $\mathcal{M}$ has at most one zero diagonal entry. This implies that every m-square submatrix of A is an SNS-matrix. Thus A is a totally L-matrix. ■

Corollary 3. *Let*

$$A = \begin{bmatrix} I_k & B \\ O & C \end{bmatrix}$$

be a strictly row-mixable m by $m+2$ matrix and let each row of A have at least three nonzero entries. If A is an SN-matrix, then C is a totally L-matrix.

Proof. Since A is a totally L-matrix by Proposition 2, C is a totally L-matrix. ■

Let A be an SN-matrix. A is a *maximal SN-matrix* if any matrix obtained from A by replacing a zero entry with a nonzero entry is not an SN-matrix. Then we also have a result which can be found in [6].

Corollary 4. *An m by $m + 2$ totally L-matrix is a maximal SN-matrix .*

Proof. The result comes from Proposition 2 and the fact that every row of an m by $m + 2$ totally L-matrix has exactly three nonzero entries. ■

ACKNOWLEDGEMENTS. We would like to thank the referees for their careful reading of this paper and for their helpful comments.

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