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PRIME IDEALS IN SUBTRACTION ALGEBRAS

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Abstract. Prime elements and $\wedge$-irreducible elements are introduced, and related properties are investigated.

1. Introduction

B. M. Schein [8] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [9] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [5] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [4], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [6] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In

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this paper, we introduce the notion of prime elements and $\wedge$-irreducible elements in subtraction algebras, and investigate several properties.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [5, 6]):

(a1) $(x - y) - y = x - y$.

(a2) $x - 0 = x$ and $0 - x = 0$.

(a3) $(x - y) - x = 0$.

(a4) $x - (x - y) \leq y$.

(a5) $(x - y) - (y - x) = x - y$.

(a6) $x - (x - (x - y)) = x - y$.

(a7) $(x - y) - (z - y) \leq x - z$.

(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1. [5] A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

(i) $0 \in A$

(ii) $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Lemma 2.2. [6, Lemma 3.2] *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

Definition 2.3. Let X be a subtraction algebra and let S be a subset of X. Defined the ideal *generated* by S to be the intersection of all ideals of X which contain S, and denoted by $\langle S \rangle$.

Clearly, a *principal* ideal $\langle a \rangle$ generated by a is $\{x \in X | x \leq a\}$ (see [5, Theorem 3.4]). For any subset A, B of a subtraction algebra X, we define

$$A \wedge B = \{x \wedge y | x \in A, y \in B\},$$

where $x \wedge y = glb\{x, y\}$.

Definition 2.4. [6] Let X be a subtraction algebra and A an ideal of X. A is said to be *prime* if for any $a, b \in X$, $a \wedge b \in A$ implies $a \in A$ or $b \in A$. A is said to be $\wedge$-*irreducible* if for any ideals B, C of X, $A = B \wedge C$ implies $A = B$ or $A = C$.

Lemma 2.5. [6, Theorem 3.7] *Let P be an ideal of a subtraction algebra X. Then the following are equivalent.*

(i) *P is a prime ideal of X.*

(ii) *For any ideals A and B of X, $A \wedge B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

3. Main Results

Lemma 3.1. *Let X be a subtraction algebra and $a, b \in X$. Then $a \leq b$ if and only if $\langle a \rangle \subseteq \langle b \rangle$.*

Proof. Straightforward. □

Theorem 3.2. *Let X be a subtraction algebra and $a, b \in X$. Then $\langle a \rangle \wedge \langle b \rangle = \langle a \wedge b \rangle$.*

Proof. Let $x \in \langle a \rangle \wedge \langle b \rangle$, then $x = a_1 \wedge b_1$ for some $a_1 \in \langle a \rangle$ and $b_1 \in \langle b \rangle$. Since $a_1 \leq a$ and $b_1 \leq b$, we have $x = a_1 \wedge b_1 \leq a \wedge b$, which implies $x \in \langle a \wedge b \rangle$.

Conversely, let $y \in \langle a \wedge b \rangle$, then $y \leq a \wedge b$. But $y \leq a \wedge b \leq a$ and $y \leq a \wedge b \leq b$. Hence $y \in \langle a \rangle$ and $y \in \langle b \rangle$. If $y \in \langle a \rangle$, then there exists t_1 (say) such that $y = t_1 \in \langle a \rangle$. Similarly, $y = t_2 \in \langle b \rangle$ and so $y = y \wedge y = t_1 \wedge t_2 \in \langle a \rangle \wedge \langle b \rangle$. This completes the proof. □

Definition 3.3. Let $a (\neq 0)$ and b be elements of a subtraction algebra X. We define a as a $\wedge$-*factor* of b if $b = a \wedge c$ for some $c \in X$. A non-zero element $p \in X$ is called a *prime element* if p is a $\wedge$-factor of $a \wedge b$ implies p is a $\wedge$-factor of a or p is a $\wedge$-factor of b.

Clearly, a is a $\wedge$-factor of b if and only if $b \leq a$.

We call a subtraction algebra *principal* if all its ideals are principal.

Theorem 3.4. *Let X be a principal subtraction algebra and $q \in X$. Then q is a prime element of X if and only if $\langle q \rangle$ is a prime ideal of X.*

Proof. Suppose that $q (\neq 0)$ is a prime element of X. Let A and B be ideals of X. Assume that $A \wedge B \subseteq \langle q \rangle$. Since X is principal, there exist $a, b \in X$ such that $\langle a \rangle = A$ and $\langle b \rangle = B$. So $\langle a \rangle \wedge \langle b \rangle \subseteq \langle q \rangle$. By Theorem 3.2, we have $\langle a \wedge b \rangle \subseteq \langle q \rangle$. Using Lemma 3.1, we have $a \wedge b \leq q$ and hence $a \wedge b = q \wedge (a \wedge b)$. This implies that q is a $\wedge$-factor of $a \wedge b$ and by assumption q is a $\wedge$-factor of a or q is a $\wedge$-factor of b. Therefore,

$a \leq q$ or $b \leq q$, and so $A = \langle a \rangle \subseteq \langle q \rangle$ or $B = \langle b \rangle \subseteq \langle q \rangle$. It follows from Lemma 2.5 that $\langle q \rangle$ is a prime ideal.

Conversely, suppose that $\langle q \rangle$ is a prime ideal and q is a $\wedge$-factor of $a \wedge b$ for $a, b \in X$. Then $a \wedge b \leq q$. By Lemma 3.1 and Theorem 3.2, we have $\langle a \rangle \wedge \langle b \rangle \subseteq \langle p \rangle$. Since $\langle q \rangle$ is a prime ideal, it follows from Lemma 2.5 that $\langle a \rangle \subseteq \langle q \rangle$ or $\langle b \rangle \subseteq \langle q \rangle$, so that by Lemma 3.1 $a \leq q$ or $b \leq q$. Consequently, q is a $\wedge$-factor of a or q is a $\wedge$-factor of b, ending the proof. □

Definition 3.5. Let X be a subtraction algebra and $q \in X$. q is said to be $\wedge$-*irreducible* if for all $x, y \in X$, $q = x \wedge y$ implies $q = x$ or $q = y$.

Theorem 3.6. *Let X be a principal subtraction algebra and $q \in X$. Then q is a $\wedge$-irreducible element of X if and only if $\langle q \rangle$ is a $\wedge$-irreducible ideal of X.*

Proof. Suppose that $\langle q \rangle$ is a $\wedge$-irreducible ideal of X. Let $x, y \in X$. Assume that $q = x \wedge y$. Then $\langle q \rangle = \langle x \wedge y \rangle = \langle x \rangle \wedge \langle y \rangle$. Since $\langle q \rangle$ is a $\wedge$-irreducible ideal, we have $\langle q \rangle = \langle x \rangle$ or $\langle q \rangle = \langle y \rangle$. Hence $q = x$ or $q = y$.

Conversely, suppose that q is a $\wedge$-irreducible element of X. Let A and B be ideals of X. Assume that $\langle q \rangle = A \wedge B$. Since X is principal, there exist $a, b \in X$ such that $\langle a \rangle = A$ and $\langle b \rangle = B$. Therefore $\langle q \rangle = \langle a \rangle \wedge \langle b \rangle = \langle a \wedge b \rangle$. This implies $q = a \wedge b$, and so $q = a$ or $q = b$. Hence $\langle q \rangle = \langle a \rangle = A$ or $\langle q \rangle = \langle b \rangle = B$. The proof is completed. □

A non-empty subset S of a subtraction algebra X is said to be $\wedge$-*closed* if $x \wedge y \in S$ whenever $x, y \in S$.

Lemma 3.7. [3, Theorem 3.9] *Let X be a subtraction algebra, S a non-empty subset of X and A an ideal of X. If S is $\wedge$-closed such that $A \cap S = \emptyset$, then there is a maximal ideal P of X such that $A \subseteq P$ and $P \cap S = \emptyset$. Moreover P is a prime ideal.*

Since $\{a\}$ is $\wedge$-closed for any $a \in X$, by Lemma 3.7 we have:

Corollary 3.8. *Let X be a subtraction algebra. If A is an ideal of X and $a \notin A$ with $a \neq 0$, then there is a prime ideal P_a such that $A \subseteq P_a$ and $a \notin P_a$.*

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