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FUZZY EQUIVALENCE RELATIONS AND FUZZY PARTITIONS

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Abstract. By using the new concepts of fuzzy equivalence relations and fuzzy partitions which Dib and Youssef introduced, we obtains fuzzy analogues of many results concerning ordinary equivalence relations and partitions. Also, we give some examples.

0. Introduction

The notion of cartesian product plays an important role in the usual theory of relations and functions. A number of authors[1,3,5,6] have worked with fuzzy relations without referring to what may be called *fuzzy Cartesian product.*

In 1991, Dib and Youssef[2] introduced the concept of fuzzy Cartesian product and they defined a fuzzy relation as a subset of the fuzzy Cartesian product. This definition is different from all know definitions of fuzzy relations. Moreover, they defined and studied fuzzy equivalence relations and fuzzy equivalence classes (see Result 3.B and 3.C).

In this paper, we obtain fuzzy analogues of many results concerning ordinary equivalence relations and partitions in the new point of view which Dib and Youssef introduced. And we give some examples.

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1. Preliminaries

The totally ordered set $I = [0, 1]$ is a distributive but not complemented lattice under the operations of infimum $\wedge$ and supremum $\vee$. On $J = I \times I$ we define a partial order $\leq$, in terms of the partial order on I, as follows: For every (r_1, r_2), $(s_1, s_2) \in J$,

(i) $(r_1, r_2) \leq (s_1, s_2)$ if and only if $r_1 \leq s_1$, $r_2 \leq s_2$ whenever $s_1 \neq 0$ and $s_2 \neq 0$,

(ii) $(0, 0) = (s_1, s_2)$ whenever $s_1 = 0$ or $s_2 = 0$.

It is clear that J is a distributive but not complemented vector lattice. The operations of infimum and supremum in J are given respectively by: For every (r_1, r_2), $(s_1, s_2) \in J$,

$$(r_1, r_2) \wedge (s_1, s_2) = (r_1 \wedge s_1, r_2 \wedge s_2)$$

and

$$(r_1, r_2) \vee (s_1, s_2) \leq (r_1 \vee s_1, r_2 \vee s_2)$$

where the equality holds in the last relation when $r_i \neq 0 \neq s_i$.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping if* $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings, where $f(x) = (f_1(x), f_2(x))$ for each $x \in X$. Now we modify the definition of J-fuzzy set introduced by Dib and Youssef.

Definition 1.1. Let X be a nonempty set. A complex mapping $A = (\mu_A, \eta_A) : X \to J$ is called a *J-fuzzy set* (in short, *fuzzy set*) in X, where $A(x) = (\mu_A(x), \eta_A(x))$ for each $x \in X$, In particular, $\varnothing$ and X denote the *J-fuzzy empty set* and *J-fuzzy whole set* in X defined by $\varnothing(x) = (0, 0)$ and $X(x) = (1, 1)$ for each $x \in X$, respectively.

The notation $\{(x, A(x)) : x \in X\}$ or simply $\{(x, r)\}$, where $r = A(x)$, will be used to denote a fuzzy set in X (see [7]). Similarly, a J-fuzzy set in X, a fuzzy set in $X \times Y$ and a J-fuzzy set in $X \times Y$ will be denoted respectively by $\{(x,(r_1,r_2))\}$, $\{((x,y),r)\}$ and $\{((x,y),(r_1,r_2))\}$. To each fuzzy set $\{(x,r_1)\}$ in X and fuzzy set $\{(y,r_2)\}$ in Y there corresponds a J-fuzzy set $\{((x,y),(r_1,r_2))\}$ in $X \times Y$. Throughout this paper, the notation $(x,r) \in A$ means that $A(x) = r$, where $A \in I^X$, and X, Y, Z, etc denote ordinary sets.

Definition 1.2. Let X be a nonempty set and let $A = (\mu_A, \eta_A), B = (\mu_B, \eta_B) \in J^X$. Then we have

(1) $A \subset B$ if and only if $\mu_A \leq \mu_B$ and $\eta_A \leq \eta_B$.

(2) $A = B$ if and only if $A \subset B$ and $B \subset A$.

(3) $A^c = (1 - \mu_A, 1 - \eta_A)$.

(4) $A \cup B = (\mu_A \vee \mu_B, \eta_A \vee \eta_B)$.

(5) $A \cap B = (\mu_A \wedge \mu_B, \eta_A \wedge \eta_B)$.

Definition 1.3. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be an arbitrary family of J-fuzzy sets in a set X, where $A_\alpha = (\mu_{A_\alpha}, \eta_{A_\alpha})$ for each $\alpha \in \Gamma$ and Γ is an index set, Then we have

(1) $\bigcap_{\alpha\in\Gamma} A_\alpha = (\bigwedge \mu_{A_\alpha}, \bigwedge \eta_{A_\alpha})$.

(2) $\bigcup_{\alpha\in\Gamma} A_\alpha = (\bigvee \mu_{A_\alpha}, \bigvee \eta_{A_\alpha})$.

¿From Definitions 1.1, 1.2 and 1.3, we can easily see that the properties of $(J^X, \cup, \cap, ^c, \phi, X)$ and $(I^X, \cup, \cap, ^c, \phi, X)$ are the same.

Definition 1.4[2]. Let X and Y be two ordinary sets. Then the collection of all J-fuzzy sets in $X \times Y$ is called the *fuzzy Cartesian product* of X and Y and is denoted by $X \overline{\underline{\times}} Y$. Hence $X \overline{\underline{\times}} Y = J^{X\times Y}$.

The *fuzzy Cartesian product* of a fuzzy set $A = \{(x,r)\}$ in X and a fuzzy set $B = \{(y,s)\}$ in Y is the J-fuzzy set $A \underline{\times} B$ in $X \times Y$ defined by:

$$A \underline{\times} B = \{((x,y),(r,s)) : x \in X, y \in Y\} \equiv \{((x,y),(r,s))\}.$$

It is clear that $A\underline{\times}B \in X\overline{\underline{\times}}Y$ for each $A \in I^X$ and $B \in I^Y$. The above definitions can be generalized for any finite number of sets. Furthermore, the above definitions can be generalized in an obvious way by replacing the unit interval I by an arbitrary completely distributive lattice.

Definition 1.5[2]. ρ is called a *fuzzy relation from X to Y* if $\rho \subset X\overline{\underline{\times}}Y$. In particular, ρ is called a *fuzzy relation in X* if $\rho \subset X\overline{\underline{\times}}X$.

It is clear that $X\overline{\underline{\times}}Y$ is itself a fuzzy relation from X to Y. Any collection of $A\underline{\times}B$, where $A \in I^X$ and $B \in I^Y$, is a fuzzy relation from X to Y.

The fuzzy cartesian product $X\overline{\underline{\times}}X$ is called the *universal fuzzy relation in X*. The fuzzy relation $\emptyset\underline{\times}\emptyset = \emptyset$ is called the *empty fuzzy relation.* Between these two extreme cases, lies the *identity fuzzy relation*, denoted by Δ_X, where Δ_X is the fuzzy relation in X whose members are the J-fuzzy sets $\{((x,x),(r,r)) : x \in X \text{ and } r \in I\}$.

Definition 1.6[2]. Let ρ_1 and ρ_2 be fuzzy relations from X to Y.

(1) We say that ρ_1 *is contained in* ρ_2 if whenever $((x,y),(r_1,r_2)) \in A \in \rho_1$, there exists $B \in \rho_2$ such that $((x,y),(r_1,r_2)) \in B$. In this case, we write $\rho_1 \subset \rho_2$.

(2) We say that ρ_1 and ρ_2 *are equal* if $\rho_1 \subset \rho_2$ and $\rho_2 \subset \rho_1$. In this case, we write $\rho_1 = \rho_2$.

To each J-fuzzy set $C = \{((x,y),(r,s))\}$ in $X \times Y$ we associate a J-fuzzy set C^{-1} in $Y \times X$ defined by $C^{-1} = \{((y,x),(s,r))\}$.

Definition 1.7[2]. Let ρ be a fuzzy relation from X to Y. Then the *inverse* of ρ, denoted ρ^{-1}, is the fuzzy relation from Y to X defined by $\rho^{-1} = \{C^{-1} : C \in \rho\}$.

Definition 1.8[2]. Let ρ be a fuzzy relation from X to Y and let σ be a fuzzy relation from Y to Z. Then the *composition* of ρ and σ, denoted $\sigma \circ \rho$, is the fuzzy relation from X to Z whose constituting J-fuzzy sets $C \in X \underline{\overline{\times}} Z$ are defined as follows:

$((x,z),(r_1,r_3)) \in C$ if and only if there exists $(y,r_2) \in Y \times I$ such that $((x,y),(r_1,r_2)) \in A$ and $((y,z),(r_2,r_3)) \in B$ for some $A \in \rho$ and $B \in \sigma$. Hence $\sigma \circ \rho = \{C \in X \underline{\overline{\times}} Z : C$ is as defined above$\}$.

It is clear that if $\rho \subset X \underline{\overline{\times}} X$, then $\Delta_X \circ \rho \subset \rho$ and $\rho \circ \Delta_X \subset \rho$.

Result 1.A[2, Proposition in p.303]. Let $\rho, \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2$ be any fuzzy relations defined on the appropriate sets. Then we have

(1) $(\rho_1 \circ \rho_2) \circ \rho_3 = \rho_1 \circ (\rho_2 \circ \rho_3)$.

(2) $\rho_1 \subset \rho_2$ and $\sigma_1 \subset \sigma_2 \Rightarrow \rho_1 \circ \sigma_1 \subset \rho_2 \circ \sigma_2$.

(3) $\rho_1 \circ (\rho_2 \cup \rho_3) = (\rho_1 \circ \rho_2) \cup (\rho_1 \circ \rho_3)$.

(4) $\rho_1 \circ (\rho_2 \cap \rho_3) \subset (\rho_1 \circ \rho_2) \cap (\rho_1 \circ \rho_3)$.

(5) $\rho_1 \subset \rho_2 \Rightarrow \rho_1^{-1} \subset \rho_2^{-1}$.

(6) $(\rho^{-1})^{-1} = \rho$ and $(\rho_1 \circ \rho_2)^{-1} = {\rho_2}^{-1} \circ {\rho_1}^{-1}$.

(7) $(\rho_1 \cup \rho_2)^{-1} = {\rho_1}^{-1} \cup {\rho_2}^{-1}$.

(8) $(\rho_1 \cap \rho_2)^{-1} = {\rho_1}^{-1} \cap {\rho_2}^{-1}$.

2. Fuzzy equivalence relations

Definition 2.1[2]. Let ρ be a fuzzy relation in X. Then ρ is said to be:

(1) *reflexive* in X if for each $x \in X$ and $r \in I$, there exists $A \in \rho$ such that $((x,x),(r,r)) \in A$, i.e., $\Delta_X \subset \rho$.

(2) *symmetric* in X if whenever $((x,y),(r,s)) \in A \in \rho$, there exists $B \in \rho$ such that $((y,x),(s,r)) \in B$, i.e., $\rho^{-1} = \rho$.

(3) *transitive* in X if whenever $((x,y),(r,s)) \in A \in \rho$ and $((y,z),(s,t)) \in B \in \rho$, there exists $C \in \rho$ such that $((x,z),(r,t)) \in C$, i.e., $\rho \circ \rho \subset \rho$.

(4) a *fuzzy equivalence relation* in X if it is reflexive, symmetric and transitive.

We will denote the set of all fuzzy equivalence relations in X as $\mathrm{F}Rel_E(X)$. It is clear that $X\underline{\times}X$, $\Delta_X \in \mathrm{F}Rel_E(X)$.

Example 2.1. (a) Let $X = \{a, b, c, d, e\}$ and let $\rho = \triangle_X \cup \{A_1, {A_1}^{-1}, A_2, A_3\}$ be the fuzzy relation in X defined as follows:

A_1	a	b	c	d	e
a	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
b	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
c	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
d	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
e	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)

A_2	a	b	c	d	e
a	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)
b	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)
c	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)
d	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)
e	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)

A_3	a	b	c	d	e
a	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)
b	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)
c	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)
d	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)
e	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)	(t_0, t_0)

, where $(r_0, t_0) \in I \times I$ is fixed and $r_0 \neq t_0$. Then we can easily see that ρ is a fuzzy equivalence relation in X.

(b) Let $X = \{a, b, c, d, e\}$ and let $\sigma = \triangle_X \cup \{B\}$ be the fuzzy relation in X defined as follows:

B	a	b	c	d	e
a	(r_1, r_1)	(r_1, r_2)	(r_1, r_3)	(r_1, r_4)	(r_1, r_5)
b	(r_2, r_1)	(r_2, r_2)	(r_2, r_3)	(r_2, r_4)	(r_2, r_5)
c	(r_3, r_1)	(r_3, r_2)	(r_3, r_3)	(r_3, r_4)	(r_3, r_5)
d	(r_4, r_1)	(r_4, r_2)	(r_4, r_3)	(r_4, r_4)	(r_4, r_5)
e	(r_5, r_1)	(r_5, r_2)	(r_5, r_3)	(r_5, r_4)	(r_5, r_5)

, where $r_1, r_2, r_3, r_4, r_5 \in I$ are fixed and $r_i \neq r_j$ for each $i \neq j$. Then clearly σ is a fuzzy equivalence relation in X.

(c) Let $X = \{a, b, c, d, e\}$ and let $\eta = \triangle_X \cup \{C_1, C_2, C_3, C_4, {C_4}^{-1}\}$ be the fuzzy relation in X defined as follows:

C_1	a	b	c	d	e
a	(r_1, r_1)	(r_1, r_1)	(r_1, r_2)	(r_1, r_3)	(r_1, r_4)
b	(r_1, r_1)	(r_2, r_2)	(r_2, r_2)	(r_2, r_3)	(r_2, r_4)
c	(r_2, r_1)	(r_2, r_2)	(r_3, r_3)	(r_1, r_1)	(r_3, r_4)
d	(r_3, r_1)	(r_3, r_2)	(r_1, r_1)	(r_3, r_3)	(r_3, r_4)
e	(r_4, r_1)	(r_4, r_2)	(r_4, r_3)	(r_4, r_3)	(r_4, r_4)

C_2	a	b	c	d	e
a	(r_1, r_1)	(r_1, r_1)	(r_1, r_2)	(r_1, r_3)	(r_1, r_4)
b	(r_1, r_1)	(r_1, r_1)	(r_1, r_2)	(r_1, r_3)	(r_1, r_4)
c	(r_2, r_1)	(r_2, r_1)	(r_2, r_2)	(r_2, r_3)	(r_2, r_4)
d	(r_3, r_1)	(r_3, r_1)	(r_3, r_2)	(r_3, r_3)	(r_3, r_4)
e	(r_4, r_1)	(r_4, r_1)	(r_4, r_2)	(r_4, r_3)	(r_4, r_4)

C_3	a	b	c	d	e
a	(r_1, r_1)	(r_1, r_2)	(r_1, r_3)	(r_1, r_3)	(r_1, r_4)
b	(r_2, r_1)	(r_2, r_2)	(r_2, r_3)	(r_2, r_3)	(r_2, r_4)
c	(r_3, r_1)	(r_3, r_2)	(r_1, r_1)	(r_3, r_3)	(r_3, r_4)
d	(r_3, r_1)	(r_3, r_2)	(r_3, r_3)	(r_3, r_3)	(r_3, r_4)
e	(r_4, r_1)	(r_4, r_2)	(r_4, r_3)	(r_4, r_3)	(r_1, r_1)

C_4	a	b	c	d	e
a	(r_1, r_1)	(r_1, r_2)	(r_1, r_3)	(r_1, r_3)	(r_1, r_4)
b	(r_2, r_1)	(r_2, r_1)	(r_1, r_3)	(r_1, r_3)	(r_1, r_4)
c	(r_3, r_1)	(r_3, r_1)	(r_3, r_2)	(r_3, r_3)	(r_3, r_4)
d	(r_3, r_1)	(r_3, r_1)	(r_3, r_3)	(r_3, r_3)	(r_3, r_4)
e	(r_4, r_1)	(r_4, r_2)	(r_4, r_3)	(r_4, r_3)	(r_1, r_1)

, where $r_1, r_2, r_3, r_4 \in I$ are fixed and $r_i \neq r_j$ for each $i \neq j$. Then we can easily see that η is a fuzzy equivalence relation in X. □

Result 2.A[2, Proposition in p.303]. Let ρ and σ be fuzzy relations in a nonempty set X. Then:

(1) If ρ is reflexive [resp. symmetric and transitive], then ρ^{-1} is reflexive [resp. symmetric and transitive].

(2) If ρ is reflexive [resp. symmetric and transitive], then $\rho \circ \rho$ is reflexive [resp. symmetric and transitive].

(3) If ρ is reflexive, then $\rho \circ \rho \supset \rho$.

(4) If ρ is symmetric, then $\rho \cup \rho^{-1}$, $\rho \cap \rho^{-1}$ are symmetric and $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho$.

(5) If ρ and σ are reflexive [resp. symmetric and transitive], then $\rho \cap \sigma$ is reflexive [resp. symmetric and transitive].

(6) If ρ and σ are symmetric, then $\rho \cup \sigma$ is symmetric.

¿From (1), (2) and (5), it is clear that if $\rho, \sigma \in \mathrm{F}Rel_E(X)$, then ρ^{-1}, $\rho \circ \rho$, $\rho \cap \sigma \in \mathrm{F}Rel_E(X)$.

Result 2.B[2, Theorem 1]. Let $\rho \in \mathrm{F}Rel_E(X)$. Then

(1) For each $x_0 \in X$, ρ induces an (ordinary) equivalence relation $\rho_I(x_0)$ in I defined by:

$$\rho_I(x_0) = \{(r,s) \in J : \exists A \in \rho \text{ s.t. } ((x_0,x_0),(r,s)) \in A\}.$$

(2) For each $r_0 \in I$, ρ induces an (ordinary) equivalence relation $\rho_X(r_0)$ in X defined by:

$$\rho_X(r_0) = \{(x,y) \in X \times X : \exists A \in \rho \text{ s.t. } ((x,y),(r_0,r_0)) \in A\}.$$

Example 2.B. Consider three fuzzy equivalence relations ρ, σ and η in Example 2.1.

(a) $\rho_I(a) = \{(r,r) \in J\} \cup \{(r_0,t_0),(t_0,r_0)\}$
$= \rho_I(b) = \rho_I(c) = \rho_I(d) = \rho_I(e)$

and

$$\rho_X(r_0) = X \times X = \rho_X(t_0).$$

(b) $\sigma_I(a) = \sigma_I(b) = \sigma_I(c) = \sigma_I(d) = \sigma_I(e) = \{(r,r) \in J\}$

and

$$\sigma_X(r_1) = \sigma_X(r_2) = \sigma_X(r_3) = \sigma_X(r_4) = \sigma_X(r_5)$$
$$= \{(a,a),(b,b),(c,c),(d,d),(e,e)\}.$$

(c)
$$\begin{aligned}\eta_I(a) &= \{(r,r) \in J\} = \eta_I(d) = \eta_I(e),\\ \eta_I(b) &= \{(r,r) \in J\} \cup \{(r_1,r_2),(r_2,r_1)\},\\ \eta_I(c) &= \{(r,r) \in J\} \cup \{(r_2,r_3),(r_3,r_2)\}\end{aligned}$$

and

$$\begin{aligned}\eta_X(r_1) &= \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(b,a),(c,d),(d,c)\},\\ \eta_X(r_2) &= \{(a,a),(b,b),(c,c),(d,d),(e,e),(b,c),(c,b)\},\\ \eta_X(r_3) &= \{(a,a),(b,b),(c,c),(d,d),(e,e),(c,d),(d,c)\},\\ \eta_X(r_4) &= \{(a,a),(b,b),(c,c),(d,d),(e,e)\}.\end{aligned}$$

□

Result 2.C[2, Corollary in p.304]. To each $\rho \in \mathrm{F}Rel_E(X)$ there are associated an ordinary equivalence relation ρ_I in I and an ordinary equivalence relation ρ_X in X. In fact,

$\rho_I = \bigcap_{x \in X} \rho_I(x)$ and $\rho_X = \bigcap_{r \in I} \rho_X(r)$.

In this case, ρ_I [resp. ρ_X] is called the *equivalence relation in I [resp. X] associated to the fuzzy equivalence relation ρ.*

The following is the immediate result of Definition 2.1 and Result 2.A(3).

Proposition 2.1. If $\rho \in \mathrm{F}Rel_E(X)$, then $\rho \circ \rho = \rho$.

Proposition 2.2. Let ρ be a fuzzy relation in X. If ρ is reflexive, then $\sigma \subset \rho \circ \sigma$ and $\sigma \subset \sigma \circ \rho$ for each $\sigma \subset X \underline{\times} X$.

Proof. Suppose ρ is reflexive and let $\sigma \subset X \overline{\underline{\times}} X$. Let $((x,y),(r_1,r_2)) \in A \in \sigma$. Since ρ is reflexive, there exists $B \in \rho$ such that $((y,y),(r_2,r_2)) \in B$. Thus there exists $C \in \rho \circ \sigma$ such that $((x,y),(r_1,r_2)) \in C$. So $\sigma \subset \rho \circ \sigma$. Similarly, we can see that $\sigma \subset \sigma \circ \rho$. □

Proposition 2.3. Let ρ and σ be fuzzy relations in X, ρ be fuzzy reflexive and let σ be fuzzy reflexive and transive. Then $\rho \subset \sigma$ if and only if $\rho \circ \sigma = \sigma$. In particular, this holds ρ and σ are fuzzy equivalence relations in X.

Proof. $(\Rightarrow)$: Suppose $\rho \subset \sigma$. Then, by Result 1.A(2), $\rho \circ \sigma \subset \sigma \circ \sigma$. Since σ is transitive, $\sigma \circ \sigma \subset \sigma$. Thus $\rho \circ \sigma \subset \sigma$. Since ρ is reflexive, by Proposition 2.2, $\sigma \subset \rho \circ \sigma$. Hence $\rho \circ \sigma = \sigma$.

$(\Leftarrow)$: Suppose $\rho \circ \sigma = \sigma$. Since σ is reflexive, by Proposition 2.2, $\rho \subset \rho \circ \sigma$. Hence, by the hypothesis, $\rho \subset \sigma$. □

Proposition 2.4. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$. Then $\rho \circ \sigma \in \mathrm{F}Rel_E(X)$ if and only if $\rho \circ \sigma = \sigma \circ \rho$.

Proof. $(\Rightarrow)$: Suppose $\rho \circ \sigma \in \mathrm{F}Rel_E(X)$. Then

$$\begin{aligned} \rho \circ \sigma &= (\rho \circ \sigma)^{-1} \text{ (Since } \rho \circ \sigma \text{ is symmetric)} \\ &= \sigma^{-1} \circ \rho^{-1} \text{ (By Result 1.A(6))} \\ &= \sigma \circ \rho. \text{ (Since } \sigma \text{ and } \rho \text{ are symmetric)} \end{aligned}$$

$(\Leftarrow)$: Suppose $\rho \circ \sigma = \sigma \circ \rho$. Since $\Delta_X \in \mathrm{F}Rel_E(X)$, ρ and σ are reflexive, by Result 2.A(3) and 1.B(2), $\Delta_X \subset \Delta_X \circ \Delta_X \subset \rho \circ \sigma$. Thus $\rho \circ \sigma$ is reflexive. On the other hand,

$$\begin{aligned} (\rho \circ \sigma)^{-1} &= \sigma^{-1} \circ \rho^{-1} \text{ (By Result 1.A(6))} \\ &= \sigma \circ \rho \text{ (Since } \sigma \text{ and } \rho \text{ are symmetric)} \\ &= \rho \circ \sigma. \text{ (By the hypothesis)} \end{aligned}$$

So $\rho \circ \sigma$ is symmetric. Moreover

$$\begin{aligned} (\rho \circ \sigma) \circ (\rho \circ \sigma) &= \rho \circ (\sigma \circ \rho) \circ \sigma \text{ (By Result 1.A(1))} \\ &= \rho \circ (\rho \circ \sigma) \circ \sigma \text{ (By the hypothesis)} \\ &= (\rho \circ \rho) \circ (\sigma \circ \sigma) \text{ (By Result 1.A(1))} \\ &\subset \rho \circ \sigma. \text{ (Since } \rho \text{ and } \sigma \text{ are transitive)} \end{aligned}$$

Thus $\rho \circ \sigma$ is transitive. Hence $\rho \circ \sigma \in \mathrm{F}Rel_E(X)$. □

Proposition 2.5. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$. Then $\rho \cup \sigma \in \mathrm{F}Rel_E(X)$ if and only if $\rho \circ \sigma \subset \rho \cup \sigma$ and $\sigma \circ \rho \subset \rho \cup \sigma$.

Proof. $(\Rightarrow)$: Suppose $\rho \cup \sigma \in \mathrm{F}Rel_E(X)$. Then clearly $\rho \subset \rho \cup \sigma$ and $\sigma \subset \rho \cup \sigma$. Thus, by Result 1.A(2), $\rho \circ \sigma \subset (\rho \cup \sigma) \circ (\rho \cup \sigma)$. Since $\rho \cup \sigma$ is transitive, $(\rho \cup \sigma) \circ (\rho \cup \sigma) \subset \rho \cup \sigma$. So $\rho \circ \sigma \subset \rho \cup \sigma$. Similarly, we can see that $\sigma \circ \rho \subset \rho \cup \sigma$.

$(\Leftarrow)$: Suppose the necessary conditions hold. Since ρ and σ are reflexive, $\Delta_X \subset \rho$ and $\Delta_X \subset \sigma$. Then $\Delta_X \subset \rho \cup \sigma$. Thus $\rho \cup \sigma$ is reflexive. By Result 2.A(6), $\rho\cup\sigma$ is symmetric. On the other hand,

$$\begin{aligned}(\rho\cup\sigma)\circ(\rho\cup\sigma) &= \rho\circ(\rho\cup\sigma)\cup\sigma\circ(\rho\cup\sigma) \text{ (By Result 1.A(3))}\\ &= (\rho\circ\rho)\cup(\rho\circ\sigma)\cup(\sigma\circ\rho)\cup(\sigma\circ\sigma) \text{ (By Result 1.A(3))}\\ &\subset \rho\cup(\rho\circ\sigma)\cup(\sigma\circ\rho)\cup\sigma \text{ (Since } \rho \text{ and } \sigma \text{ are transitive)}\\ &\subset \rho\cup(\rho\cup\sigma)\cup(\rho\cup\sigma)\cup\sigma \text{ (By the hypothesis)}\\ &= \rho\cup\sigma.\end{aligned}$$

So $\rho\cup\sigma$ is transitive. Hence $\rho\cup\sigma\in \mathrm{F}Rel_E(X)$. □

Proposition 2.6. Let $\{\rho_\alpha\}_{\alpha\in\Gamma}$ be an indexed family of fuzzy equivalence relations in X. Then $\bigcap_{\alpha\in\Gamma}\rho_\alpha\in \mathrm{F}Rel_E(X)$.

Proof. Let $\rho=\bigcap_{\alpha\in\Gamma}\rho_\alpha$. Then we can easily see that ρ is reflexive and symmetric. Suppose $((x,y),(r_1,r_2))\in A\in\rho$ and $((y,z),(r_2,r_3))\in B\in\rho$. Since $\rho=\bigcap_{\alpha\in\Gamma}\rho_\alpha$, there exist $A_\alpha\in\rho_\alpha$ and $B_\alpha\in\rho_\alpha$ such that $((x,y),(r_{1_\alpha},r_{2_\alpha}))\in A_\alpha$ and $((y,z),(r_{2_\alpha},r_{3_\alpha}))\in B_\alpha$, where $A=\bigcap_{\alpha\in\Gamma}A_\alpha$ and $B=\bigcap_{\alpha\in\Gamma}B_\alpha$. Thus there exists $C_\alpha\in\rho_\alpha$ such that $((x,z),(r_{1_\alpha},r_{3_\alpha}))\in C_\alpha$. Let $C=\bigcap_{\alpha\in\Gamma}C_\alpha$. Then clearly $C\in\rho$ and $C(x,z)=\bigwedge_{\alpha\in\Gamma}(r_{1_\alpha},r_{3_\alpha})=(\bigwedge_{\alpha\in\Gamma}r_{1_\alpha},\bigwedge_{\alpha\in\Gamma}r_{3_\alpha})=(r_1,r_3)$. Thus $((x,z),(r_1,r_3))\in C$. So ρ is transitive. Hence $\bigcap_{\alpha\in\Gamma}\rho_\alpha\in \mathrm{F}Rel_E(X)$. □

3. Fuzzy partitions

Definition 3.1[2]. Let X be a nonempty set. Then H is called a *fuzzy cutting on* X if H is a function of X into the power set of I, i.e., $H: X\to P(I)$. In this case, we will denote H as $\{(x,H_x): x\in X\}$ or simply $\{(x,H_x)\}$, where $H_x=H(x)$ is the subset of I corresponding to $x\in X$.

A fuzzy cutting on X is said to be *empty* and is denoted by $\emptyset$ if $\emptyset_x=\emptyset$ for each $x\in X$. A fuzzy cutting on X is said to be *universal* and is denoted by $\cup$ if $\cup_x = I$ for each $x \in X$, i.e., $\cup = \{(x, I)\}$.

Example 3.1. Let $X = \{a, b, c\}$ and let $H : X \to P(I)$ be the mapping defined as follows:

$$H_a = \left(0, \frac{1}{2}\right), \ H_b = \left[\frac{1}{3}, \frac{2}{3}\right) \text{ and } H_c = \left(\frac{1}{2}, 1\right].$$

Then clearly H is a fuzzy cutting of X. □

A fuzzy set $A \in I^X$ is said to *be contained in* the fuzzy cutting H of X, symbolically $A \subset H$, if $A(x) \in H_x$ for each $x \in X$ for which $H_x \neq \emptyset$ and $A(x) = 0$ whenever $H_x = \emptyset$. Note that $\emptyset$ is the fuzzy set in X contained in the empty fuzzy cutting, and that $\cup$ contains all fuzzy sets in X.

Let $H = \{(x, H_x)\}$ and $H' = \{(x, H_x')\}$ be fuzzy cuttings on X. Then H is said to *be contained in* H', symbolically $H \subset H'$, if $H_x \subset H_x'$ for each $x \in X$. Clearly, $\emptyset \subset H \subset \cup$ for each fuzzy cutting H of X. The union $H \cup H'$ of H and H' is the fuzzy cutting defined by $H \cup H' = \{(x, H_x \cup H_x')\}$. The operations of intersection, complements, etc. on fuzzy cuttings are similarly defined. The fuzzy cuttings H and H' are said to be *disjoint* if $H_x \cap H_x' = \emptyset$ for each $x \in X$. A collection of fuzzy cuttings is said to be *disjoint* if each pair of this collection is disjoint.

Let $\rho \in \mathrm{F}Rel_E(X)$. Then, by Result 2.B(1), ρ induces an equivalence relation $\rho_I(x)$ in I, for each $x \in X$. Let the equivalence class of $r \in I$ with respect to $\rho_I(x)$ be denoted by $[r]_x$ or $r/\rho_I(x)$. In fact,

$$\begin{aligned} [r]_x &= \{s \in I : (r, s) \in \rho_I(x)\} \\ &= \{s \in I : \exists A \in \rho \text{ s.t. } ((x, x), (r, s)) \in A\}. \end{aligned}$$

For each $x_0 \in X$ and $r_0 \in I$ we define a fuzzy cutting $H(x_0, r_0)$ on X as follows: for each $y \in X$, if there exists $r \in I$ such that $((x_0, y), (r_0, r)) \in$

A for some $A \in \rho$, we set $H(x_0, r_0)(y) = [r]_y$ and it such r does not exist, we set $H(x_0, r_0)(y) = \emptyset$. It is clear that the function $H(x_0, r_0) : X \to P(I)$ is well-defined (see [2]).

Result 3.A[2, Proposition in p.305]. Let $\rho \in \mathrm{F}Rel_E(X)$. For every $x, x_1, x_2 \in X$ and let $r, r_1, r_2 \in I$, we have:

(1) $(x, [r]_x) \in H(x, r)$. Hence $H(x, r) \neq \emptyset$.

(2) $((x_1, x_2), (r_1, r_2)) \in A \in \rho$ if and only if $H(x_1, r_1) = H(x_2, r_2)$.

(3) $(r, r_1) \in \rho_I(x)$ if and only if $H(x, r) = H(x, r_1)$.

(4) If $H(x_1, r_1) \cap H(x_2, r_2) \neq \emptyset$, then $H(x_1, r_1) = H(x_2, r_2)$.

(5) $\cup_{x \in X, r \in I} H(x, r) = \cup$.

Definition 3.2[2]. Let $\rho \in \mathrm{F}Rel_E(X)$, let $x \in X$ and let $r \in I$. Then the fuzzy cutting $H(x, r)$ is called the *fuzzy equivalence class of* (x, r) *with respect to* ρ (or the ρ*-fuzzy equivalence class of* (x, r)) and is denoted by $[(x, r)]_\rho$ or simply $[(x, r)]$ if there is no ambiguity.

Example 3.2. Consider three fuzzy equivalence relations in Example 2.1.

(a) For any $x, y \in X$ and each $r \in I$,

$$[(x, r)]_\rho(y) = \begin{cases} \{r_0, t_0\} & \text{if } r = r_0 \text{ or } r = t_0, \\ \{r\} & \text{if } x = y,\ r \neq r_0 \text{ and } r \neq t_0, \\ \emptyset & \text{if } x \neq y,\ r \neq r_0 \text{ and } r \neq t_0. \end{cases}$$

(b) For each $r \in I$,

$$[(a, r)]_\sigma(a) = \{r\},$$

$$[(a, r)]_\sigma(b) = \begin{cases} \{r_2\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1, \end{cases}$$

$$[(a, r)]_\sigma(c) = \begin{cases} \{r_3\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1, \end{cases}$$

$$[(a,r)]_\sigma(d) = \begin{cases} \{r_4\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1, \end{cases}$$

$$[(a,r)]_\sigma(e) = \begin{cases} \{r_5\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1, \end{cases}$$

$$[(a,r)]_\sigma(a) = \begin{cases} \{r_1\} & \text{if } r = r_2, \\ \emptyset & \text{if } r \neq r_2, \end{cases}$$

$$[(b,r)]_\sigma(b) = \{r\},$$

$$[(b,r)]_\sigma(c) = \begin{cases} \{r_3\} & \text{if } r = r_2, \\ \emptyset & \text{if } r \neq r_2, \end{cases}$$

$$[(b,r)]_\sigma(d) = \begin{cases} \{r_4\} & \text{if } r = r_2, \\ \emptyset & \text{if } r \neq r_2, \end{cases}$$

$$[(b,r)]_\sigma(e) = \begin{cases} \{r_5\} & \text{if } r = r_2, \\ \emptyset & \text{if } r \neq r_3, \end{cases}$$

$$[(c,r)]_\sigma(a) = \begin{cases} \{r_1\} & \text{if } r = r_3, \\ \emptyset & \text{if } r \neq r_3, \end{cases}$$

$$[(c,r)]_\sigma(b) = \begin{cases} \{r_2\} & \text{if } r = r_3, \\ \emptyset & \text{if } r \neq r_3, \end{cases}$$

$$[(c,r)]_\sigma(c) = \{r\},$$

$$[(c,r)]_\sigma(d) = \begin{cases} \{r_4\} & \text{if } r = r_3, \\ \emptyset & \text{if } r \neq r_3, \end{cases}$$

$$[(c,r)]_\sigma(e) = \begin{cases} \{r_5\} & \text{if } r = r_3, \\ \emptyset & \text{if } r \neq r_3, \end{cases}$$

$$[(d,r)]_\sigma(a) = \begin{cases} \{r_1\} & \text{if } r = r_4, \\ \emptyset & \text{if } r \neq r_4, \end{cases}$$

$$[(d,r)]_\sigma(b) = \begin{cases} \{r_2\} & \text{if } r = r_4, \\ \emptyset & \text{if } r \neq r_4, \end{cases}$$

$$[(d,r)]_\sigma(c) = \begin{cases} \{r_3\} & \text{if } r = r_4, \\ \emptyset & \text{if } r \neq r_4, \end{cases}$$

$$[(d,r)]_\sigma(d) = \{r\},$$

$$[(d,r)]_\sigma(e) = \begin{cases} \{r_5\} & \text{if } r = r_4, \\ \emptyset & \text{if } r \neq r_4, \end{cases}$$

$$[(e,r)]_\sigma(a) = \begin{cases} \{r_1\} & \text{if } r = r_5, \\ \emptyset & \text{if } r \neq r_5, \end{cases}$$

$$[(e,r)]_\sigma(b) = \begin{cases} \{r_2\} & \text{if } r = r_5, \\ \emptyset & \text{if } r \neq r_5, \end{cases}$$

$$[(e,r)]_\sigma(c) = \begin{cases} \{r_3\} & \text{if } r = r_5, \\ \emptyset & \text{if } r \neq r_5, \end{cases}$$

$$[(e,r)]_\sigma(d) = \begin{cases} \{r_4\} & \text{if } r = r_5, \\ \emptyset & \text{if } r \neq r_5, \end{cases}$$

$$[(e,r)]_\sigma(e) = \{r\}.$$

(c) For each $r \in I$,

$$[(a,r)]_\eta(a) = \{r\},$$

$$[(a,r)]_\eta(b) = \begin{cases} \{r_1, r_2\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1, \end{cases}$$

$$[(a,r)]_\eta(c) = \begin{cases} \{r_2, r_3\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1, \end{cases}$$

$$[(a,r)]_\eta(d) = \begin{cases} \{r_3\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1, \end{cases}$$

$$[(a,r)]_\eta(e) = \begin{cases} \{r_4\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1, \end{cases}$$

$$[(b,r)]_\eta(a) = \begin{cases} \{r_1\} & \text{if } r = r_1 \text{ or } r = r_2, \\ \emptyset & \text{if } r \neq r_1 \text{ and } r \neq r_2, \end{cases}$$

$$[(b,r)]_\eta(b) = \begin{cases} \{r_1, r_2\} & \text{if } r = r_1 \text{ or } r = r_2, \\ \{r\} & \text{if } r \neq r_1 \text{ and } r \neq r_2, \end{cases}$$

$$[(b,r)]_\eta(c) = \begin{cases} \{r_2, r_3\} & \text{if } r = r_1 \text{ or } r = r_2, \\ \emptyset & \text{if } r \neq r_1 \text{ and } r \neq r_2, \end{cases}$$

$$[(b,r)]_\eta(d) = \begin{cases} \{r_3\} & \text{if } r = r_1 \text{ or } r = r_2, \\ \emptyset & \text{if } r \neq r_1 \text{ and } r \neq r_2, \end{cases}$$

$$[(b,r)]_\eta(e) = \begin{cases} \{r_4\} & \text{if } r = r_1 \text{ or } r = r_2, \\ \emptyset & \text{if } r \neq r_1 \text{ and } r \neq r_2, \end{cases}$$

$$[(c,r)]_\eta(a) = \begin{cases} \{r_1\} & \text{if } r = r_2 \text{ or } r = r_3, \\ \emptyset & \text{if } r \neq r_2 \text{ and } r \neq r_3, \end{cases}$$

$$[(c,r)]_\eta(b) = \begin{cases} \{r_1, r_2\} & \text{if } r = r_2 \text{ or } r = r_3, \\ \emptyset & \text{if } r \neq r_2 \text{ and } r \neq r_3, \end{cases}$$

$$[(c,r)]_\eta(c) = \begin{cases} \{r_2, r_3\} & \text{if } r = r_2 \text{ or } r = r_3, \\ \{r\} & \text{if } r \neq r_2 \text{ and } r \neq r_3, \end{cases}$$

$$[(c,r)]_\eta(d) = \begin{cases} \{r_3\} & \text{if } r = r_2 \text{ or } r = r_3, \\ \emptyset & \text{if } r \neq r_2 \text{ and } r \neq r_3, \end{cases}$$

$$[(c,r)]_\eta(e) = \begin{cases} \{r_4\} & \text{if } r = r_2 \text{ or } r = r_3, \\ \emptyset & \text{if } r \neq r_2 \text{ and } r \neq r_3, \end{cases}$$

$$[(d,r)]_\eta(a) = \begin{cases} \{r_1\} & \text{if } r = r_3, \\ \emptyset & \text{if } r \neq r_3, \end{cases}$$

$$[(d,r)]_\eta(b) = \begin{cases} \{r_2, r_3\} & \text{if } r = r_3, \\ \emptyset & \text{if } r \neq r_3, \end{cases}$$

$$[(d,r)]_\eta(c) = \begin{cases} \{r_2, r_3\} & \text{if } r = r_3, \\ \{r_1\} & \text{if } r = r_1, \\ \emptyset & \text{if } r \neq r_1 \text{ and } r \neq r_3, \end{cases}$$

$$[(d,r)]_\eta(d) = \{r\},$$

$$[(d,r)]_\eta(e) = \begin{cases} \{r_4\} & \text{if } r = r_3, \\ \emptyset & \text{if } r \neq r_3, \end{cases}$$

$$[(e,r)]_\eta(a) = \begin{cases} \{r_1\} & \text{if } r = r_4, \\ \emptyset & \text{if } r \neq r_4, \end{cases}$$

$$[(e,r)]_\eta(b) = \begin{cases} \{r_1, r_2\} & \text{if } r = r_4, \\ \emptyset & \text{if } r \neq r_4, \end{cases}$$

$$[(e,r)]_\eta(c) = \begin{cases} \{r_2, r_3\} & \text{if } r = r_4, \\ \emptyset & \text{if } r \neq r_4, \end{cases}$$

$$[(e,r)]_\eta(d) = \begin{cases} \{r_3\} & \text{if } r = r_4, \\ \emptyset & \text{if } r \neq r_4, \end{cases}$$

$$[(e,r)]_\eta(e) = \{r\}.$$

□

Definition 3.3[2]. Let $\{H_\alpha\}_{\alpha\in\Gamma}$ (Γ is an index set) be a family of fuzzy cuttings H_α on X. Then $\{H_\alpha\}_{\alpha\in\Gamma}$ is called a *fuzzy partition* of X if it satisfies the following conditions:

(i) $H_\alpha \neq \emptyset$ for each $\alpha \in \Gamma$.

(ii) $H_\alpha \cap H_\beta = \emptyset$ for $\alpha \neq \beta$; $\alpha, \beta \in \Gamma$.

(iii) $\bigcup_{\alpha\in\Gamma} H_\alpha = \mathsf{U}$.

Example 3.3. Let $X = \{a, b, c, d, e\}$, let $r_0 \neq t_0 \in I$ and let $\{H_r\}_{r\in I}$ be the family of fuzzy cuttings on X defined as follows: for each $x \in X$ and $r \in I$,

$$H_r(x) = \begin{cases} \{r_0, t_0\} & \text{if } r = r_0 \text{ or } r = t_0, \\ \{r\} & \text{if } r \neq r_0 \text{ and } r \neq t_0. \end{cases}$$

Then clearly $\{H_r\}_{r\in I}$ is a fuzzy partition of X.

Definition 3.4[2]. Let $A \in X \underline{\times} Y$. Then A is called the *J-fuzzy singleton with support* $(x_0, y_0) \in X \times Y$ and *value* $(r_0, t_0) \in J$ if

$$A(x,y) = \begin{cases} (r_0, t_0) & \text{if } \ (x,y) = (x_0, y_0), \\ (0,0) & \text{if } \ (x,y) \neq (x_0, y_0), \end{cases}$$

for each $(x, y) \in X \times Y$. In this case, we write $A = [(x_0, y_0), (r_0, t_0)]$ or $(x_0, y_0)_{(r_0,t_0)}$.

Let ρ be a fuzzy equivalence relation in X. Then we will denote the set of all ρ-fuzzy equivalence classes as X/ρ and call it the *fuzzy quotient set of X by ρ*.

Result 3.B[2, Theorem 2]. Let X be a nonempty set. If $\rho \in \mathrm{F}Rel_E(X)$, then X/ρ is a fuzzy partition of X. Conversely, if $\mathcal{P}$ is a fuzzy partition of X, then $\mathcal{P}$ defines a fuzzy equivalence relation in X. In fact, we define the fuzzy equivalence relation in X, in terms of $\mathcal{P}$, as follows: $[(x_1, x_2), (r_1, r_2)] \in \rho$ if and only if there exists $H \in \mathcal{P}$ such that $r_1 \in H_{x_1}$ and $r_2 \in H_{x_2}$.

We will denote the fuzzy equivalence relation in X determined a fuzzy partition $\mathcal{P}$ of X as $\rho_{\mathcal{P}}$.

Example 3.B. Let ρ be the fuzzy equivalence relation in $X = \{a, b, c, d, e\}$ defined in Example 2.1. Then clearly $X/\rho = \{[(a, r_0)]_\rho\} \cup \{[(x, r)]_\rho : x \in X,\ r \neq r_0 \text{ and } r \neq t_0\}$ is a fuzzy partition of X. Conversely, let $\mathcal{P} = \{H_r\}_{r \in I}$ be the fuzzy partition of X defined in Example 3.3. Then we can easily see that $\rho_{\mathcal{P}} = \rho$, where ρ is the fuzzy equivalence relation in X defined in Example 2.1. □

Result 3.C[2, Theorem 3]. Let X be a nonempty set. If $\rho \in \mathrm{F}Rel_E(X)$, then $\rho_{X/\rho} = \rho$. Conversely, if $\mathcal{P}$ is a fuzzy partition of X, then $X/\rho_{\mathcal{P}} = \mathcal{P}$.

Definition 3.1. Let $f : X \to Y$ be a mapping, let H be a fuzzy cutting on X and let K be a fuzzy cutting on Y. Then:

(1) The *inverse image of* K *under* f, denoted by $f^{-1}(K)$, is a fuzzy cutting on X defined as follows: for each $x \in X$,

$$f^{-1}(K)(x) = (K \circ f)(x) = K_{f(x)}.$$

(2) The *image of* H *under* f, denoted by $f(H)$, is a fuzzy cutting on Y defined as follows: for each $y \in Y$,

$$f(H)(y) = \begin{cases} \bigcup_{x \in f^{-1}(y)} H_x & \text{if } f^{-1}(y) \neq \emptyset, \\ \emptyset & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Definition 3.2. Let $f : X \to Y$ be a mapping, let $\{H_\alpha\}_{\alpha\in\Gamma}$ be a family of fuzzy cuttings on X and let $\{K_\beta\}_{\beta\in\Lambda}$ be a family of fuzzy cuttings on Y.

(1) $f(\emptyset) = \emptyset$.

(2) $f(\cup_X) = \cup_Y$ if f is surjective, where $\cup_X$ [resp. $\cup_Y$] denote the universal fuzzy cutting on X [resp. Y].

(3) If $H_\alpha \subset H_\beta$, then $f(H_\alpha) \subset f(H_\beta)$.

(4) $f^{-1}(\cup_Y) = \cup_X$.

(5) If $K_\alpha \subset K_\beta$, then $f^{-1}(K_\alpha) \subset f^{-1}(K_\beta)$.

(6) $f^{-1}(f(H_\alpha)) \supset H_\alpha$ (equality holds if f is bijective).

(7) $f(\bigcup_{\alpha\in\Gamma} H_\alpha) = \bigcup_{\alpha\in\Gamma} f(H_\alpha)$.

(8) $f(\bigcap_{\alpha\in\Gamma} H_\alpha) \subset \bigcap_{\alpha\in\Gamma} f(H_\alpha)$(equality holds of f is injective).

(9) $f^{-1}(\bigcup_{\beta\in\Lambda} K_\beta) = \bigcup_{\beta\in\Lambda} f^{-1}(K_\beta)$.

(10) $f^{-1}(\bigcap_{\beta\in\Lambda} K_\beta) = \bigcap_{\beta\in\Lambda} f^{-1}(K_\beta)$.

(11) $f(f^{-1}(K_\beta)) \subset K_\beta$ (equality holds if f is surjective).

(12) If f is surjective, then $f({H_\alpha}^c) \supset (f(H_\alpha))^c$ (equality holds if f is bijective).

(13) $f^{-1}({K_\beta}^c) = (f^{-1}(K_\beta))^c$.

Proof. The proofs of (1), (2), (3), (4) and (5) are obvious.

(6) Let $x \in X$. Then

$$\begin{aligned} f^{-1}(f(H_\alpha))(x) &= f(H_\alpha)(f(x)) \\ &= \textstyle\bigcup_{x'\in f^{-1}(f(x))} H_{\alpha,x'} \\ &\supset H_{\alpha,x}\,. \ (*) \end{aligned}$$

So $f^{-1}(f(H_\alpha))(x) \supset H_{\alpha,x}$. Suppose f is injective. Then clearly $f^{-1}(f(x)) = x$. Thus, by $(*)$, $f^{-1}(f(H_\alpha))(x) = H_{\alpha,x}$. Hence $f^{-1}(f(H_\alpha)) = H_\alpha$.

(7) Let $y \in Y$. Suppose $f^{-1}(y) \neq \emptyset$. Then

$$\begin{aligned} f(\textstyle\bigcup_{\alpha\in\Gamma} H_\alpha)(y) &= \textstyle\bigcup_{x\in f^{-1}(y)}(\bigcup_{\alpha\in\Gamma} H_\alpha)(x) \\ &= \textstyle\bigcup_{x\in f^{-1}(y)}(\bigcup_{\alpha\in\Gamma} H_{\alpha,x}) \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\alpha\in\Gamma}(\bigcup_{x\in f^{-1}(y)}H_{\alpha,x}) \\
&= \bigcup_{\alpha\in\Gamma}[f(H_\alpha)(y)] \\
&= [\bigcup_{\alpha\in\Gamma}f(H_\alpha)](y).
\end{aligned}
$$

Suppose $f^{-1}(y) = \emptyset$. Then clearly $f(\bigcup_{\alpha\in\Gamma} H_\alpha)(y) = \emptyset = [\bigcup_{\alpha\in\Gamma} f(H_\alpha)](y)$. Hence $f(\bigcup_{\alpha\in\Gamma} H_\alpha) = \bigcup_{\alpha\in\Gamma} f(H_\alpha)$.

(8) Let $y \in Y$. Suppose $f^{-1}(y) \neq \emptyset$. Then

$$
\begin{aligned}
f(\bigcap_{\alpha\in\Gamma}H_\alpha)(y) &= \bigcup_{x\in f^{-1}(y)}(\bigcap_{\alpha\in\Gamma}H_\alpha)(x) \\
&= \bigcup_{x\in f^{-1}(y)}(\bigcap_{\alpha\in\Gamma}H_{\alpha,x}) \ (**) \\
&\subset \bigcap_{\alpha\in\Gamma}(\bigcup_{x\in f^{-1}(y)}H_{\alpha,x}) \\
&= \bigcap_{\alpha\in\Gamma}[f(H_\alpha)(y)] \\
&= [\bigcap_{\alpha\in\Gamma}f(H_\alpha)](y).
\end{aligned}
$$

Suppose $f^{-1}(y) = \emptyset$. Then clearly $f(\bigcap_{\alpha\in\Gamma} H_\alpha)(y) = \emptyset = \bigcap_{\alpha\in\Gamma} f(H_\alpha)(y)$. Hence $f(\bigcap_{\alpha\in\Gamma} H_\alpha) = \bigcap_{\alpha\in\Gamma} f(H_\alpha)$. But if f is injective, then, since $x = f^{-}1(y)$, by (**), $f(\bigcap_{\alpha\in\Gamma} H_\alpha)(y) = [\bigcap_{\alpha\in\Gamma} f(H_\alpha)](y)$. So $f(\bigcap_{\alpha\in\Gamma} H_\alpha) = \bigcap_{\alpha\in\Gamma} f(H_\alpha)$

(9) Let $x \in X$. Then

$$
\begin{aligned}
f^{-1}(\bigcup_{\beta\in\Lambda}K_\beta)(x) &= (\bigcup_{\beta\in\Lambda}K_\beta)(f(x)) \\
&= \bigcup_{\beta\in\Lambda}K_{\beta,f(x)} \\
&= \bigcup_{\beta\in\Lambda}(f^{-1}(K_\beta))(x) \\
&= [\bigcup_{\beta\in\Lambda}f^{-1}(K_\beta)](x).
\end{aligned}
$$

So $f^{-1}(\bigcup_{\beta\in\Lambda} K_\beta) = \bigcup_{\beta\in\Lambda} f^{-1}(K_\beta)$.

(10) Let $x \in X$. Then

$$
\begin{aligned}
f^{-1}(\bigcap_{\beta\in\Lambda}K_\beta)(x) &= (\bigcap_{\beta\in\Lambda}K_\beta)(f(x)) \\
&= \bigcap_{\beta\in\Lambda}K_{\beta,f(x)} \\
&= \bigcap_{\beta\in\Lambda}f^{-1}(K_\beta)(x) \\
&= [\bigcap_{\beta\in\Lambda}f^{-1}(K_\beta)](x).
\end{aligned}
$$

So $f^{-1}(\bigcap_{\beta\in\Lambda} K_\beta) = \bigcap_{\beta\in\Lambda} f^{-1}(K_\beta)$.

(11) Let $y \in Y$. Suppose $f^{-1}(y) \neq 0$. Then

$$
\begin{aligned}
[f(f^{-1}(K_\beta))](y) &= \bigcup_{x\in f^{-1}(y)}f^{-1}(K_\beta)(x) \\
&= \bigcup_{x\in f^{-1}(y)}K_{\beta,f(x)} \\
&= K_{\beta,y}
\end{aligned}
$$

$$= K_\beta(y).$$

Suppose $f^{-1}(y) = \emptyset$. Then clearly $[f(f^{-1}(K_\beta))](y) = \emptyset$. But $K_\beta(y) = \emptyset$ or $K_\beta(y) \neq \emptyset$. So $[f(f^{-1}(K_\beta))](y) \subset K_\beta(y)$. Hence $f(f^{-1}(K_\beta)) \subset K_\beta$. Now suppose f is surjective. Then clearly $f(f^{-1}(K_\beta)) = K_\beta$.

(12) Let $y \in Y$. Since f is surjective, $f^{-1}(y) \neq \emptyset$. Then

$$\begin{aligned} f(H_\alpha{}^c)(y) &= \bigcup_{x \in f^{-1}(y)} (H_\alpha{}^c)(x) \\ &= \bigcup_{x \in f^{-1}(y)} (I \backslash H_{\alpha,x}) \\ &\supset I \backslash (\bigcup_{x \in f^{-1}(y)} H_{\alpha,x}) \\ &= I \backslash f(H_\alpha)(y) \\ &= (f(H_\alpha))^c(y). \end{aligned}$$

Thus $f(H_\alpha{}^c) \supset (f(H_\alpha))^c$. Now suppose f is bijective. Then clearly $f(H_\alpha{}^c) = (f(H_\alpha))^c$.

(13) Let $x \in X$. Then

$$\begin{aligned} f^{-1}(K_\beta{}^c)(x) &= (K_\beta{}^c)(f(x)) \\ &= I \backslash K_{\beta, f(x)} \\ &= (I \backslash f^{-1}(K_\beta))(x) \\ &= [f^{-1}(K_\beta)]^c(x). \end{aligned}$$

Hence $f^{-1}(K_\beta{}^c) = (f^{-1}(K_\beta))^c$. □

Proposition 3.3. Let $f : X \to Y$ be a mapping. If $\{K_\beta\}_{\beta \in \Lambda}$ is a fuzzy partition of Y, then $\{f^{-1}(K_\beta)\}_{\beta \in \Lambda}$ is a fuzzy partition of X.

Proof. It is clear that $f^{-1}(K_\beta) \neq \emptyset$ for each $\beta \in \Lambda$. Let $\alpha \neq \beta \in \Lambda$. Then clearly $K_\alpha \cap K_\beta = \emptyset$. Thus $f^{-1}(K_\alpha \cap K_\beta) = \emptyset$. But, by Proposition 3.6(10), $f^{-1}(K_\alpha \cap K_\beta) = f^{-1}(K_\alpha) \cap f^{-1}(K_\beta)$. So $f^{-1}(K_\alpha) \cap f^{-1}(K_\beta) = \emptyset$. On the other hand,

$$\begin{aligned} \bigcup_{\beta \in \Lambda} f^{-1}(K_\beta) &= f^{-1}(\bigcup_{\beta \in \Lambda} K_\beta) \text{ (By Proposition 3.6(9))} \\ &= f^{-1}(\cup_Y) \quad \text{(Since } \{K_\beta\}_{\beta \in \Lambda} \text{ is a partition of } Y) \\ &= \cup_X. \text{ (By Proposition 3.6(4))} \end{aligned}$$

Hence $\{f^{-1}(K_\beta)\}_{\beta \in \Lambda}$ is a fuzzy partition of X. □

Proposition 3.4. Let $f : X \to Y$ be a surjection. If $\{H_\alpha\}_{\alpha\in\Gamma}$ is a fuzzy partition of X, then $\{f(H_\alpha)\}_{\alpha\in\Gamma}$ is a fuzzy partition of Y.

Proof. It is clear that $f(H_\alpha) \neq \emptyset$ for each $\alpha \in \Gamma$. Let $\alpha \neq \beta \in \Gamma$. Then $H_\alpha \cap H_\beta = \emptyset$. Thus, by Proposition 3.6(1), $f(H_\alpha \cap H_\beta) = \emptyset$. But, by Proposition 3.6(8), $f(H_\alpha \cap H_\beta) = f(H_\alpha) \cap f(H_\beta)$. So $f(H_\alpha) \cap f(H_\beta) = \emptyset$. On the other hand,

$$\begin{aligned} \bigcup_{\alpha\in\Gamma} f(H_\alpha) &= f(\bigcup_{\alpha\in\Gamma} H_\alpha) \quad \text{(By Proposition 3.6(7))} \\ &= f(\cup_X) \quad \text{(Since } \{H_\alpha\}_{\alpha\in\Gamma} \text{ is a fuzzy partition of } X) \\ &= \cup_Y. \quad \text{(By Proposition 3.6(2))} \end{aligned}$$

Hence $\{f(H_\alpha)\}_{\alpha\in\Gamma}$ is a fuzzy partition of X. □

The following is the immediate result of Result 3.A(2) and Definition 3.2.

Proposition 3.5. Let $\rho \in \mathrm{FRel_E}(X)$, let $x, y \in X$ and let $r, s \in I$. Then $[(x,r)]_\rho = [(y,s)]_\rho$ if and only if $[r]_x = [s]_y$.

Proposition 3.6. Let ρ and σ be fuzzy equivalence relations in X. Then

$$[(x,r)]_\rho \cap [(x,r)]_\sigma = [(x,r)]_{\rho\cap\sigma},$$

for each $x \in X$ and $r \in I$.

Proof. Let $y \in X$. Suppose there exists $s \in I$ such that $((x,y),(r,s)) \in A$ and $((x,y),(r,s)) \in B$ for some $A \in \rho$ and $B \in \sigma$. Then

$$[(x,r)]_\rho(y) = s/\rho_I(y) \text{ and } [(x,r)]_\sigma(y) = s/\sigma_I(y).$$

Let $t \in ([(x,r)]_\rho \cap [(x,r)]_\sigma)(y)$. Then there exists $A' \in \rho$ and $B' \in \sigma$ such that $((y,y),(s,t)) \in A'$ and $((y,y),(s,t)) \in B'$. Thus $((y,y),(s,t)) \in A' \cap B' \in \rho\cap\sigma$. So $t \in s/(\rho\cap\sigma)_I(y) = [(x,r)]_{\rho\cap\sigma}(y)$ and hence $([(x,r)]_\rho \cap [(x,r)]_\sigma)(y) \subset [(x,r)]_{\rho\cap\sigma}(y)$. Similarly, we can see that $[(x,r)]_{\rho\cap\sigma}(y) \subset$

$([(x,r)]_\rho \cap [(x,r)]_\sigma)(y)$. If there is no $s \in I$ such that $((x,y),(r,s)) \in A$ or $((x,y),(r,s)) \in B$ for each $A \in \rho$ and $B \in \sigma$, then $([(x,r)]_\rho \cap [(x,r)]_\sigma)(y) = \emptyset = [(x,r)]_{\rho\cap\sigma}(y)$. Hence $[(x,r)]_\rho \cap [(x,r)]_\sigma = [(x,r)]_{\rho\cap\sigma}$.

□

Proposition 3.7. Let $\rho, \sigma \in \mathrm{FRel_E}(X)$. If $\rho \cup \sigma \in \mathrm{FRel_E}(X)$, then

$$[(x,r)]_\rho \cup [(x,r)]_\sigma = [(x,r)]_{\rho\cup\sigma}$$

for each $x \in X$ and $r \in I$.

Proof. Let $y \in X$. Suppose there exists $s \in I$ such that $((x,y),(r,s)) \in A$ or $((x,y),(r,s)) \in B$ for some $A \in \rho$ and $B \in \sigma$. Then

$$[(x,r)]_\rho(y) = s/\rho_I(y) \text{ or } [(x,r)]_\sigma(y) = s/\sigma_I(y).$$

Let $t \in [(x,r)]_\rho \cup [(x,r)]_\sigma$. Then there exists $A' \in \rho$ or $B' \in \sigma$ such that $((y,y),(s,t)) \in A'$ or $((y,y),(s,t)) \in B'$. Thus $((y,y),(s,t)) \in A' \cup B' \in A \cup B$. So $t \in s/(\rho\cup\sigma)_I(y) = [(x,r)]_{\rho\cup\sigma}(y)$ and hence $([(x,r)]_\rho \cup [(x,r)]_\sigma)(y) \subset [(x,r)]_{\rho\cup\sigma}(y)$. Similarly, we can easily see that $[(x,r)]_{\rho\cup\sigma}(y) \subset ([(x,r)]_\rho \cup [(x,r)]_\sigma)(y)$. If there is no $s \in I$ such that $((x,y),(r,s)) \in A$ and $((x,y),(r,s)) \in B$ for each $A \in \rho$ and $B \in \sigma$, then $([(x,r)]_\rho \cup [(x,r)]_\sigma)(y) = \emptyset = [(x,r)]_{\rho\cup\sigma}(y)$. This completes the proof.

□

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