

**EIGHT-DIMENSIONAL EINSTEIN'S  
CONNECTION FOR THE FIRST CLASS  
I. THE RECURRENCE RELATIONS IN 8- $g$ -UFT**

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**Abstract.**

Lower dimensional cases of Einstein's connection were already investigated by many authors for  $n = 2, 3, 4, 5, 6, 7$ . This paper is the first part of the following series of two papers, in which we obtain a surveyable tensorial representation of 8-dimensional Einstein's connection in terms of the unified field tensor, with main emphasis on the derivation of powerful and useful recurrence relations which hold in 8-dimensional Einstein's unified field theory (i.e., 8- $g$ -UFT):

I. The recurrence relations in 8- $g$ -UFT

II. The Einstein's connection in 8- $g$ -UFT

All considerations in these papers are restricted to the first class only of the generalized 8-dimensional Riemannian manifold  $X_8$ .

## 1. INTRODUCTION.

In Appendix II to his last book Einstein ([14], 1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical,

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its exposition is mainly geometrical. Characterizing Einstein's unified field theory as a set of geometrical postulates in the space-time  $X_4$ , Hlavatý([15],1957) gave its mathematical foundation for the first time. Since then Hlavatý and number of mathematicians contributed for the development of this theory and obtained many geometrical consequences of these postulates.

Generalizing  $X_4$  to  $n$ -dimensional generalized Riemannian manifold  $X_n$ ,  $n$ -dimensional generalization of this theory, so called *Einstein's  $n$ -dimensional unified field theory* ( $n$ - $g$ -UFT hereafter), had been attempted by Wrede([18],1958) and Mishra([17],1959). On the other hand, corresponding to  $n$ - $g$ -UFT, Chung([1], 1963) introduced a new unified field theory, called *the Einstein's  $n$ -dimensional  $*g$ -unified field theory*( $n$ - $*g$ -UFT hereafter). This theory is more useful than  $n$ - $g$ -UFT in some physical aspects. Chung and et al obtained many results concerning this theory ([2],1969; [6],1981; [9],1988; [10]-[11],1998), particularly proving that  $n$ - $*g$ -UFT is equivalent to  $n$ - $g$ -UFT so far as the classes and indices of inertia are concerned ([8],1985). The case of the *third class*, which is the simplest case of both unified field theories, was completely studied for a general  $n$  by many authors([17], [18], [9], etc.). However, in the cases of the first and second class of both  $n$ -dimensional generalizations, it has been unable yet to represent the general  $n$ -dimensional Einstein's connection in a surveyable tensorial form in terms of the unified field tensor  $g_{\lambda\mu}$ . This is probably due to the complexity of the higher dimensions.

However, the lower dimensional cases of the Einstein's connection in  $n$ - $g$ -UFT were investigated by many authors: 2-dimensional case by Jakubowicz([16], 1969) and Chung et al([7], 1983), 3-dimensional case by Chung et al([3]-[5], 1979-1981), and 4-dimensional case by Hlavatý([15], 1957) and many other geometricians. Recently, Chung

et al also studied the Einstein's connection in 4- $g$ -UFT([1], 1963), in 3- and 5- $g$ -UFT([9], 1988), and in 6- $g$ -UFT([12]-[13], 1999), and obtained respective Einstein's connection in a surveyable tensorial form.

The purpose of the present paper, the first part of a series of two papers, is to derive powerful recurrence relations which hold in 8- $g$ -UFT. In the second part, we prove a necessary and sufficient condition for the existence and uniqueness of the Einstein's connection in 8- $g$ -UFT and establish a linear system of 110 equations for the solution of 8-dimensional Einstein's connection for the first class, employing the powerful recurrence relations obtained in Part I.

*All considerations in this and subsequent papers are dealt for the first class only.*

## 2. PRELIMINARIES.

This section is a brief collection of basic concepts, notations, and results, which are needed in our subsequent considerations. They are due to Chung([1], 1963; [9], 1988), Hlavatý([15], 1957) and Mishra([17], 1959). All considerations in this section are dealt for a general  $n > 1$ .

**2.1.  $n$ -dimensional  $g$ -unified field theory.** The Einstein's  $n$ -dimensional unified field theory, denoted by  $n$ - $g$ -UFT, is an  $n$ -dimensional generalization of the usual Einstein's 4-dimensional unified field theory in the space-time  $X_4$ . It is based on the following three principles as indicated by Hlavatý([15]).

*Principle A.* Let  $X_n$  be an  $n$ -dimensional generalized Riemannian manifold referred to a real coordinate system  $x^\nu$ , which obeys

the coordinate transformation  $x^\nu \rightarrow x^{\nu'}$  for which

$$(2.1) \quad \det\left(\frac{\partial x'}{\partial x}\right) \neq 0$$

In  $n$ - $g$ -UFT the manifold  $X_n$  is endowed with a real nonsymmetric tensor  $g_{\lambda\mu}$ , called *the unified field tensor of  $X_n$* . This tensor may be decomposed into its symmetric part  $h_{\lambda\mu}$  and skew-symmetric part  $k_{\lambda\mu}$ :

$$(2.2a) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

where

$$(2.2b) \quad \mathfrak{g} = \det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = \det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = \det(k_{\lambda\mu})$$

We may define a unique tensor  $h^{\lambda\nu} = h^{\nu\lambda}$  by

$$(2.3) \quad h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu$$

In  $n$ - $g$ -UFT the tensors  $h_{\lambda\mu}$  and  $h^{\lambda\nu}$  will serve for raising and/or lowering indices of tensors in  $X_n$  in the usual manner.

*Principle B.* The differential geometric structure on  $X_n$  is imposed by the tensor  $g_{\lambda\mu}$  by means of a connection  $\Gamma_{\lambda\mu}^\nu$  defined by a system of equations

$$(2.4) \quad D_\omega g_{\lambda\mu} = 2S_{\omega\mu}^\alpha g_{\lambda\alpha}$$

Here  $D_\omega$  denotes the symbolic vector of the covariant derivative with respect to  $\Gamma_{\lambda\mu}^\nu$  and  $S_{\lambda\mu}^\nu$  is the torsion tensor of  $\Gamma_{\lambda\mu}^\nu$ . The connection

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<sup>1</sup>Throughout the present paper, Greek indices are used for the holonomic components of tensors, while Roman indices are used for the nonholonomic components of a tensor in  $X_n$ . All indices take the values  $1, 2, \dots, n$ , and follow the summation convention with the exception of nonholonomic indices  $x, y, z, t$ .

$\Gamma_{\lambda\mu}^\nu$  satisfying (2.4) is called *the Einstein's connection*. Under certain conditions the system (2.4) admits a unique solution  $\Gamma_{\lambda\mu}^\nu$ .

*Principle C.* In order to obtain  $g_{\lambda\mu}$  involved in the solution for  $\Gamma_{\lambda\mu}^\nu$  certain conditions are imposed. These conditions may be condensed to

$$(2.5) \quad S_\lambda = S_{\lambda\alpha}^\alpha = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]}$$

where  $X_\lambda$  is an arbitrary non-zero vector, and  $R_{\omega\mu\lambda}^\nu$  and  $R_{\mu\lambda}$  are the curvature tensors of  $\Gamma_{\lambda\mu}^\nu$  defined by

$$(2.6) \quad R_{\omega\mu\lambda}^\nu = 2(\partial_{[\mu} \Gamma_{|\lambda|}^\nu \omega] + \Gamma_{\alpha}^\nu \Gamma_{[\mu} \Gamma_{|\lambda|}^\alpha \omega]), \quad R_{\mu\lambda} = R_{\alpha\mu\lambda}^\alpha$$

**2.2. Algebraic preliminaries.** In this subsection, notations, concepts, and several algebraic results in  $n$ -g-UFT are introduced.

(i) **Notations.** The following scalars, tensors, and notations are frequently used in our further considerations:

$$(2.7a) \quad g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}}$$

$$(2.7b) \quad K_p = k_{[\alpha_1}^{\alpha_1} k_{\alpha_2}^{\alpha_2} \dots k_{\alpha_p}^{\alpha_p]}, \quad (p = 0, 1, 2, \dots)$$

$$(2.7c) \quad {}^{(0)}k_\lambda^\nu = \delta_\lambda^\nu, \quad {}^{(1)}k_\lambda^\nu = k_\lambda^\nu, \quad {}^{(p)}k_\lambda^\nu = {}^{(p-1)}k_\lambda^\alpha k_\alpha^\nu, \quad (p = 1, 2, \dots)$$

$$(2.7d) \quad K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu}$$

$$(2.7e) \quad \sigma = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

where  $\nabla_\omega$  is the symbolic vector of covariant derivative with respect to the Christoffel symbols  $\{\overset{\nu}{\lambda\mu}\}$  defined by  $h_{\lambda\mu}$ . It has been shown that the scalars and tensors introduced in (2.7) satisfy

(2.8a)

$$K_0 = 1, \quad K_n = k \quad \text{if } n \text{ is even,} \quad \text{and} \quad K_p = 0 \quad \text{if } p \text{ is odd}$$

(2.8b)

$$g = 1 + K_2 + \cdots + K_{n-\sigma}$$

(2.8c)

$${}^{(p)}k_{\lambda\mu} = (-1)^p {}^{(p)}k_{\mu\lambda}, \quad {}^{(p)}k^{\lambda\nu} = (-1)^p {}^{(p)}k^{\nu\lambda}$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor  $T_{\omega\mu\lambda}$ , skew-symmetric in the first two indices, by  $T$ :

(2.9a)

$$\overset{pqr}{T} = \overset{pqr}{T}_{\omega\mu\lambda} = T_{\alpha\beta\gamma} {}^{(p)}k_\omega^\alpha {}^{(q)}k_\mu^\beta {}^{(r)}k_\lambda^\gamma$$

(2.9b)

$$T = T_{\omega\mu\lambda} = \overset{000}{T}$$

(2.9c)

$$2 \overset{pqr}{T}_{\omega[\lambda\mu]} = \overset{pqr}{T}_{\omega\lambda\mu} - \overset{pqr}{T}_{\omega\mu\lambda}, \quad 2 \overset{(pq)r}{T}_{\omega\lambda\mu} = \overset{pqr}{T}_{\omega\lambda\mu} + \overset{qpr}{T}_{\omega\lambda\mu}, \text{ etc}$$

We then have

(2.10)

$$\overset{pqr}{T}_{\omega\lambda\mu} = -\overset{qpr}{T}_{\lambda\omega\mu}$$

(ii) Classification, basic vectors, and basic scalars.

DEFINITION 2.1. The tensor  $g_{\lambda\mu}$  (or  $k_{\lambda\mu}$ ) is said to be :

1. of the first class, if  $K_{n-\sigma} \neq 0$
2. of the second class with the  $j$ th category ( $j \geq 1$ ), if

$$K_{2j} \neq 0, \quad K_{2j+2} = K_{2j+4} = \dots = K_{n-\sigma} = 0$$

3. of the third class, if  $K_2 = K_4 = \dots = K_{n-\sigma} = 0$

The solution of the system of equations (2.4) is most conveniently brought about in a nonholonomic frame of reference, which may be introduced by the projectivity

$$(2.11) \quad M A^\nu = k_\mu{}^\nu A^\mu, \quad (M \text{ a scalar})$$

DEFINITION 2.2. An eigenvector  $A^\nu$  of  $k_{\lambda\mu}$  that satisfies (2.11) is called a *basic vector* in  $X_n$ , and the corresponding eigenvalue  $M$  is termed a *basic scalar*.

It has been shown that the basic scalars  $M$  are solutions of the characteristic equation

$$(2.12) \quad M^\sigma (M^{n-\sigma} + K_2 M^{n-2-\sigma} + \dots + K_{n-2-\sigma} M^2 + K_{n-\sigma}) = 0$$

(iii) Nonholonomic frame of reference. In the first and second class, we have a set of  $n$  linearly independent basic vectors  $A^\nu$  ( $i = 1, \dots, n$ ) and a unique reciprocal set  $A_\lambda^i$  ( $i = 1, \dots, n$ ), satisfying

$$(2.13) \quad A_\lambda^j A_i^\lambda = \delta_i^j, \quad A_\lambda^i A^\nu = \delta_\lambda^\nu$$

With these two sets of vectors, we may construct a nonholonomic frame of reference as follows:

DEFINITION 2.3. If  $T_{\lambda \dots}^{\nu \dots}$  are holonomic components of a tensor, then its *nonholonomic components*  $T_{j \dots}^{i \dots}$  are defined by

$$(2.14a) \quad T_{j \dots}^{i \dots} = T_{\lambda \dots}^{\nu \dots} A_{\nu}^i \dots A_j^{\lambda} \dots$$

An easy inspection shows that

$$(2.14b) \quad T_{\lambda \dots}^{\nu \dots} = T_{j \dots}^{i \dots} A_i^{\nu} \dots A_{\lambda}^j \dots$$

Furthermore, if  $M_x$  is the basic scalar corresponding to  $A_x^{\nu}$ , then the nonholonomic components of  ${}^{(p)}k_{\lambda}^{\nu}$  are given by

$$(2.15) \quad {}^{(p)}k_x^i = M_x^p \delta_x^i, \quad {}^{(p)}k_{xi} = M_x^p h_{xi}, \quad {}^{(p)}k^{xi} = M_x^p h^{xi}$$

Without loss of generality we may choose the nonholonomic components of  $h_{\lambda\mu}$  as

$$(2.16) \quad h_{12} = h_{34} = \dots = h_{n-1-\sigma, n-\sigma} = 1 \\ \sigma h_{ni_0} = \delta_{\sigma}^1, \quad \text{the remaining } h_{ij} = 0$$

where the index  $i_0$  is taken so that  $\text{Det}(h_{ij}) \neq 0$  when  $n$  is odd.

**2.3. Differential geometric preliminaries.** In this subsection, we present several useful results involving Einstein's connection. These results are needed in our subsequent considerations for the solution of (2.4).



If the system (2.4) admits a solution  $\Gamma_{\lambda\mu}^\nu$ , it must be of the form

$$(2.17) \quad \Gamma_{\lambda\mu}^\nu = \{ \lambda\mu^\nu \} + S_{\lambda\mu}^\nu + U^\nu{}_{\lambda\mu}$$

where

$$(2.18) \quad U_{\nu\lambda\mu} = 2 S^{\quad 001}_{\nu(\lambda\mu)}$$

The above two relations show that *our problem of determining  $\Gamma_{\lambda\mu}^\nu$  in terms of  $g_{\lambda\mu}$  is reduced to that of studying the tensor  $S_{\lambda\mu}^\nu$* . On the other hand, it has been shown that the tensor  $S_{\lambda\mu}^\nu$  satisfies

$$(2.19) \quad S = B - 3 S^{(110)}$$

where

$$(2.20) \quad 2 B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3 K_{[\alpha\beta\nu]} k_\omega^\alpha k_\mu^\beta$$

Therefore, the Einstein's connection  $\Gamma_{\lambda\mu}^\nu$  satisfying (2.4) may be determined if the solution  $S_{\lambda\mu}^\nu$  of the system (2.19) is found. *The main purpose of the present paper is to find a device to solve the system (2.19) when  $n = 8$ .*

Furthermore, for the first class, the nonholonomic solution of (2.19) is given by

$$(2.21a) \quad M_{xyz} S_{xyz} = B_{xyz}$$

or equivalently

$$(2.21b) \quad 2 M_{xyz} S_{xyz} = K_{xyz} + 3 K_{[xyz]} M_x M_y$$

where

$$(2.22) \quad M_{xyz} = 1 + M_x M_y + M_y M_z + M_z M_x$$

Therefore, in virtue of (2.21), we see that *a necessary and sufficient nonholonomic conditions for the system (2.4) to have a unique solution in the first class is*

$$(2.23) \quad M_{xyz} \neq 0 \quad \text{for all } x, y, z$$

### 3. THE RECURRENCE RELATIONS OF THE FIRST KIND IN $n$ - $g$ -UFT.

This section is devoted to the derivation of the recurrence relations of the first kind and two other useful relations which hold in  $n$ - $g$ -UFT. *All considerations in this section are also dealt for a general  $n > 1$ .*

*The recurrence relations of the first kind in  $n$ - $g$ -UFT are those which are satisfied by the tensors  ${}^{(p)}k_\lambda{}^\nu$ . These relations will be proved in the following theorem.*

**THEOREM 3.1.** *(The recurrence relations of the first kind in  $n$ - $g$ -UFT). The tensors  ${}^{(p)}k_\lambda{}^\nu$  satisfy the following recurrence relations:*

**(For the first class).**

$$(3.1a) \quad \begin{aligned} &{}^{(n+p)}k_\lambda{}^\nu + K_2 {}^{(n+p-2)}k_\lambda{}^\nu + \dots + \\ &+ K_{n-\sigma-2} {}^{(p+\sigma+2)}k_\lambda{}^\nu + K_{n-\sigma} {}^{(p+\sigma)}k_\lambda{}^\nu = 0 \end{aligned}$$

*which may be condensed to*

$$(3.1b) \quad \sum_{f=0}^{n-\sigma} K_f {}^{(n+p-f)}k_\lambda{}^\nu = 0, \quad (p = 0, 1, 2, \dots)$$

*Proof.* Let  $M_x$  be a basic scalar. Then, in virtue of (2.12), we have

$$(3.2) \quad \sum_{f=0}^{n-\sigma} K_f M_x^{n-f} = 0$$

Multiplying  $\delta_x^i$  to both sides of (3.2) and making use of (2.15), we have

$$(3.3a) \quad \sum_{f=0}^{n-\sigma} K_f {}^{(n-f)}k_x^i = 0$$

whose holonomic form is

$$(3.3b) \quad \sum_{f=0}^{n-\sigma} K_f {}^{(n-f)}k_\lambda^\alpha = 0$$

The relation (3.1) immediately follows by multiplying  ${}^{(p)}k_\alpha^\nu$  to both sides of (3.3b). □

In the following two theorems we prove two useful relations.

**THEOREM 3.2. (For the first class).** *In the first class, a tensor  $T_{\omega\mu\nu}$ , skew-symmetric in the first two indices, satisfies*

$$(3.4a) \quad T^{(pq)r}{}_{\omega\mu\nu} = \sum_{x,y,z} T_{xyz} M_x^{(p} M_y^{q)} M_z^r A_\omega^x A_\mu^y A_\nu^z$$

$$(3.4b) \quad T^{r(pq)}{}_{\nu[\omega\mu]} = \sum_{x,y,z} T_{x[yz]} M_y^{(p} M_z^{q)} M_x^r A_\nu^x A_\omega^y A_\mu^z$$

*Proof.* Making use of (2.14b) and (2.16), the relation (3.4a) may be proved as in the following way:

$$\begin{aligned}
 {}^{(pq)r}T_{\omega\mu\nu} &= \sum_{x,y,z} {}^{(pq)r}T_{xyz} A_\omega^x A_\mu^y A_\nu^z \\
 &= \frac{1}{2} \sum_{x,y,z} T_{ijk} [({}^{(p)}k_x^i ({}^{(q)}k_y^j + ({}^{(q)}k_x^i ({}^{(p)}k_y^j)] ({}^{(r)}k_z^k A_\omega^x A_\mu^y A_\nu^z \\
 &= \frac{1}{2} \sum_{x,y,z} T_{xyz} (M_x^p M_y^q + M_x^q M_y^p) M_z^r A_\omega^x A_\mu^y A_\nu^z
 \end{aligned}$$

The second relation can be proved similarly. □

**THEOREM 3.3.** (For all classes). *The tensor  $B_{\omega\mu\nu}$ , given by (2.20), satisfies*

$$(3.5) \quad {}^{(pq)r}B = {}^{(pq)r}S + {}^{(p'q')r}S + {}^{(p'q)r'}S + {}^{(pq')r'}S$$

$$(3.6) \quad 2 {}^{(pq)r}B_{\omega\mu\nu} = {}^{(pq)r}K_{\omega\mu\nu} + {}^{(p'q')r}K_{\omega\mu\nu} + {}^{(p'q)r'}K_{\nu[\omega\mu]} + {}^{(pq')r'}K_{\nu[\omega\mu]}$$

where

$$(3.7) \quad p' = p + 1, \quad q' = q + 1, \quad r' = r + 1$$

*Proof.* In virtue of (2.9) and (2.19), the relation (3.5) may be shown as in the following way:

$$\begin{aligned}
 {}^{(pq)r}B &= {}^{(pq)r}B_{\omega\mu\nu} = \frac{1}{2} B_{\alpha\beta\gamma} [({}^{(p)}k_\omega^\alpha ({}^{(q)}k_\mu^\beta + ({}^{(q)}k_\omega^\alpha ({}^{(p)}k_\mu^\beta)] ({}^{(r)}k_\nu^\gamma \\
 &= \frac{1}{2} [S_{\alpha\beta\gamma} + S_{\epsilon\eta\gamma} k_\alpha^\epsilon k_\beta^\eta + S_{\epsilon\beta\eta} k_\alpha^\epsilon k_\gamma^\eta + S_{\alpha\epsilon\eta} k_\beta^\epsilon k_\gamma^\eta] \times \\
 &\quad \times [({}^{(p)}k_\omega^\alpha ({}^{(q)}k_\mu^\beta + ({}^{(q)}k_\omega^\alpha ({}^{(p)}k_\mu^\beta)] ({}^{(r)}k_\nu^\gamma
 \end{aligned}$$

After a lengthy calculation, we note that the right-hand side of the above equation is equal to (3.5). The relation (3.6) may be proved similarly.  $\square$

#### 4. THE RECURRENCE RELATIONS OF THE SECOND KIND IN 8-g-UFT.

This section is particularly concerned with the 8-dimensional case; that is with 8-g-UFT. In this section, we first investigate the basic scalars and some relations satisfied by them. In order to obtain a tensorial representation of the 8-dimensional Einstein's connection  $\Gamma_{\lambda\mu}^{\nu}$  in terms of  $g_{\lambda\mu}$ , we need powerful *recurrence relations of the second kind* which are satisfied by an arbitrary tensor  $T_{\omega\lambda\mu}$ , skew-symmetric in the first two indices. Therefore, we finally derive these relations, after introducing the *recurrence relations of the second kind* which are satisfied by the basic scalars. *All considerations in this section are restricted to  $n = 8$ .*

In 8-g-UFT there are five cases; that is, the unified field tensor  $g_{\lambda\mu}$  belongs to

- (1) *the first class*, if  $K_8 \neq 0$
- (2) *the second class with the first category*, if  $K_2 \neq 0$ ,  
 $K_4 = K_6 = K_8 = 0$
- (3) *the second class with the second category*, if  $K_4 \neq 0$ ,  
 $K_6 = K_8 = 0$
- (4) *the second class with the third category*, if  $K_6 \neq 0$ ,  
 $K_8 = 0$
- (5) *the third class*, if  $K_2 = K_4 = K_6 = K_8 = 0$

In this section we investigate the first case.

Before we start investigations about the basic scalars, we first note that in 8-g-UFT the relation (2.8b) is reduced to

$$(4.1) \quad g = 1 + K_2 + K_4 + K_6 + K_8$$

and formally state in the following theorem the recurrence relations of the first kind when  $n = 8$ , which are direct consequences of (3.1):

**THEOREM 4.1.** (The recurrence relations of the first kind in 8-g-UFT). *The tensors  ${}^{(p)}k_{\lambda}{}^{\nu}$  satisfy the following recurrence relations in 8-g-UFT for  $p = 0, 1, 2, \dots$ :*

	class and category	Recurrence relation of the first kind in 8-g-UFT
(4.2)	The first class	${}^{(p+8)}k_{\lambda}{}^{\nu} = -K_2 {}^{(p+6)}k_{\lambda}{}^{\nu} - K_4 {}^{(p+4)}k_{\lambda}{}^{\nu} - K_6 {}^{(p+2)}k_{\lambda}{}^{\nu} - K_8 {}^{(p)}k_{\lambda}{}^{\nu}$

*Proof.* The relations (4.2) are direct consequences of (3.1).□

**THEOREM 4.2.** *The basic scalars in 8-g-UFT are given by*

	class and category	The basic scalars $M_x$
(4.3)	The first class	$M_1 = -M_2 = \sqrt{-\frac{K_2}{4} + m + A}$ $M_3 = -M_4 = \sqrt{-\frac{K_2}{4} + m - A}$ $M_5 = -M_6 = \sqrt{-\frac{K_2}{4} - m + B}$ $M_7 = -M_8 = \sqrt{-\frac{K_2}{4} - m - B}$

where  $z_1$  is a zero of

$$(4.4a) \quad 8z^3 + 20pz^2 + 8(2p^2 - r)z + 4p(p^2 - r) - q^2 = 0$$

and

(4.4b)

$$p = K_4 - \frac{3}{8}(K_2)^2, \quad q = -\frac{1}{8}(K_2)^3 + \frac{1}{2}K_2 K_4 - K_6$$

(4.4c)

$$r = -\frac{3}{256}(K_2)^4 + \frac{1}{16}(K_2)^2 K_4 - \frac{1}{4}K_2 K_6 + K_8$$

(4.4d)

$$m = \frac{1}{2}(2z_1 + p)^{\frac{1}{2}}$$

(4.4e)

$$A = (m^2 - p - z_1 + \frac{q}{4m})^{\frac{1}{2}}, \quad B = (m^2 - p - z_1 - \frac{q}{4m})^{\frac{1}{2}}$$

*Proof.* Since the characteristic equation (2.12) for the first class in 8-g-UFT is reduced to

$$(4.5) \quad (M^2)^4 + K_2(M^2)^3 + K_4(M^2)^2 + K_6 M^2 + K_8 = 0$$

(4.3) follows by the method of Cardano, using the notations given by (4.4a)-(4.4e). □

**THEOREM 4.3.** *The basic scalars  $M_x$  in 8-g-UFT satisfy the following relations:*

	class and category	The basic scalars $M_x$
(4.6)	The first class	$M_1 + M_2 = M_3 + M_4 = M_5 + M_6 = M_7 + M_8 = 0$ $M_a^2 + M_b^2 + M_c^2 + M_d^2 = -K_2$ $M_a^2 M_b^2 + M_a^2 M_c^2 + M_a^2 M_d^2 + M_b^2 M_c^2 + M_b^2 M_d^2 + M_c^2 M_d^2 = K_4$ $M_a^2 M_b^2 M_c^2 + M_a^2 M_b^2 M_d^2 + M_a^2 M_c^2 M_d^2 + M_b^2 M_c^2 M_d^2 = -K_6$ $M_a^2 M_b^2 M_c^2 M_d^2 = K_8$

Here, the indices  $a, b, c, d$  are assumed to take values as  $a = 1, 2$ ;  $b = 3, 4$ ;  $c = 5, 6$ ;  $d = 7, 8$ .

*Proof.* The relations (4.6) follow from (4.3). In the proof of (4.6), use of the following relations are made:

$$m^2 = \frac{1}{4}(K_4 - \frac{3}{8}(K_2)^2 + 2z_1)$$

$$A^2 + B^2 = -\frac{3}{2}K_4 + \frac{9}{16}(K_2)^2 - z_1$$

$$m(B^2 - A^2) = \frac{1}{16}(K_2)^3 - \frac{1}{4}K_2 K_4 + \frac{1}{2}K_6$$

$$\begin{aligned} A^2 B^2 = & -\frac{7}{16}(K_4)^2 - \frac{75}{1024}(K_2)^4 - \frac{3}{4}(z_1)^2 + \frac{25}{64}(K_2)^2 K_4 + \\ & + \frac{15}{32}(K_2)^2 z_1 - \frac{5}{4}K_4 z_1 - \frac{1}{4}K_2 K_6 + K_8 \quad \square \end{aligned}$$

Using the relations given in Theorem (4.3), we may prove the recurrence relations of the second kind in the following theorem.

**THEOREM 4.4.** (The recurrence relations of the second kind in 8-g-UFT). In 8-g-UFT the basic scalars  $M$  satisfy the following recurrence relations which hold for all values of  $x$  and  $y$  when  $x \neq y$ :

(For the first class)

(4.7a)

$$\begin{aligned} M_{x \ y}^{(7 \ M^0)} = & -M_{x \ y}^{(6 \ M^1)} - M_{x \ y}^{(5 \ M^2)} - K_2 M_{x \ y}^{(5 \ M^0)} \\ & - M_{x \ y}^{(4 \ M^3)} - K_2 M_{x \ y}^{(4 \ M^1)} - K_2 M_{x \ y}^{(3 \ M^2)} \\ & - K_4 M_{x \ y}^{(3 \ M^0)} - K_4 M_{x \ y}^{(2 \ M^1)} - K_6 M_{x \ y}^{(1 \ M^0)} \end{aligned}$$



(4.7b)

$$2M_x^{(7)} M_y^{(1)} = -2M_x^{(5)} M_y^{(3)} - 2K_2 M_x^{(5)} M_y^{(1)} - K_6 M_x M_y \\ - 2K_4 M_x^{(3)} M_y^{(1)} - K_2 M_x^{(3)} M_y^{(3)}$$

(4.7c)

$$M_x^{(7)} M_y^{(2)} = -M_x^{(6)} M_y^{(3)} - M_x^{(5)} M_y^{(4)} - K_2 M_x^{(5)} M_y^{(2)} \\ - K_2 M_x^{(4)} M_y^{(3)} - K_4 M_x^{(3)} M_y^{(2)} + K_8 M_x^{(1)} M_y^{(0)}$$

(4.7d)

$$2M_x^{(7)} M_y^{(3)} = -2M_x^{(6)} M_y^{(4)} - M_x^{(5)} M_y^{(5)} - 2K_2 M_x^{(5)} M_y^{(3)} \\ - K_2 M_x^{(4)} M_y^{(4)} - K_4 M_x^{(3)} M_y^{(3)} + K_8 M_x M_y \\ + 2K_8 M_x^{(2)} M_y^{(0)} + K_6 M_x^{(2)} M_y^{(2)}$$

(4.7e)

$$M_x^{(7)} M_y^{(4)} = -M_x^{(6)} M_y^{(5)} - K_2 M_x^{(5)} M_y^{(4)} + K_6 M_x^{(3)} M_y^{(2)} \\ + K_8 M_x^{(3)} M_y^{(0)} + K_8 M_x^{(2)} M_y^{(1)}$$

(4.7f)

$$2M_x^{(7)} M_y^{(5)} = -M_x^{(6)} M_y^{(6)} - K_2 M_x^{(5)} M_y^{(5)} + 2K_8 M_x^{(4)} M_y^{(0)} \\ + 2K_6 M_x^{(4)} M_y^{(2)} + K_6 M_x^{(3)} M_y^{(3)} + K_4 M_x^{(4)} M_y^{(4)} \\ + 2K_8 M_x^{(3)} M_y^{(1)} + K_8 M_x^{(2)} M_y^{(2)}$$

(4.7g)

$$M_x^{(7)} M_y^{(6)} = K_4 M_x^{(5)} M_y^{(4)} + K_6 M_x^{(5)} M_y^{(2)} + K_8 M_x^{(5)} M_y^{(0)} \\ + K_6 M_x^{(4)} M_y^{(3)} + K_8 M_x^{(4)} M_y^{(1)} + K_8 M_x^{(3)} M_y^{(2)}$$

(4.7h)

$$\begin{aligned}
M_x^7 M_y^7 &= K_2 M_x^6 M_y^6 + 2K_4 M_x^6 M_y^4 + 2K_6 M_x^6 M_y^2 \\
&+ 2K_8 M_x^6 M_y^0 + K_4 M_x^5 M_y^5 + 2K_6 M_x^5 M_y^3 \\
&+ 2K_8 M_x^5 M_y^1 + K_6 M_x^4 M_y^4 + 2K_8 M_x^4 M_y^2 \\
&\quad + K_8 M_x^3 M_y^3
\end{aligned}$$

Furthermore, we also have

(4.8a)

$$\begin{aligned}
K_2 M_x^7 M_y^0 - M_x^7 M_y^2 &= M_x^6 M_y^3 - K_2 M_x^6 M_y^1 + M_x^5 M_y^4 \\
&- (K_2)^2 (M_x^5 M_y^0 + M_x^4 M_y^1) + K_4 M_x^3 M_y^2 \\
&- (K_2)^2 M_x^3 M_y^2 - K_2 K_4 (M_x^3 M_y^0 + M_x^2 M_y^1) \\
&\quad - (K_2 K_6 + K_8) M_x^1 M_y^0
\end{aligned}$$

(4.8b)

$$\begin{aligned}
K_6 M_x^7 M_y^0 + M_x^7 M_y^6 &= -K_6 M_x^6 M_y^1 + K_4 M_x^5 M_y^4 \\
&+ (K_8 - K_2 K_6) (M_x^5 M_y^0 + M_x^4 M_y^1 + M_x^3 M_y^2) \\
&- K_4 K_6 (M_x^3 M_y^0 + M_x^2 M_y^1) - (K_6)^2 M_x^1 M_y^0
\end{aligned}$$

(4.8c)

$$\begin{aligned}
K_8 M_x^7 M_y^0 + K_2 M_x^7 M_y^6 &= -K_8 M_x^6 M_y^1 + K_2 K_4 M_x^5 M_y^4 \\
&+ (K_2 K_6 - K_8) (M_x^5 M_y^2 + M_x^4 M_y^3) \\
&- K_4 K_8 (M_x^3 M_y^0 + M_x^2 M_y^1) - K_6 K_8 M_x^1 M_y^0
\end{aligned}$$

(4.8d)

$$\begin{aligned}
K_4 M_x^{(7)} M_y^{(0)} - K_2 M_x^{(7)} M_y^{(2)} &= K_2 (M_x^{(6)} M_y^{(3)} + M_x^{(5)} M_y^{(4)}) \\
&- K_4 M_x^{(6)} M_y^{(1)} + ((K_2)^2 - K_4) (M_x^{(5)} M_y^{(2)} + M_x^{(4)} M_y^{(3)}) \\
&- K_2 K_4 (M_x^{(5)} M_y^{(0)} + M_x^{(4)} M_y^{(1)}) - (K_4)^2 (M_x^{(3)} M_y^{(0)} \\
&+ M_x^{(2)} M_y^{(1)}) - (K_4 K_6 + K_2 K_8) M_x^{(1)} M_y^{(0)}
\end{aligned}$$

(4.8e)

$$\begin{aligned}
K_6 M_x^{(7)} M_y^{(0)} + K_2 M_x^{(7)} M_y^{(4)} &= -K_2 M_x^{(6)} M_y^{(5)} - K_6 M_x^{(6)} M_y^{(1)} \\
&- (K_2)^2 M_x^{(5)} M_y^{(4)} - K_6 M_x^{(5)} M_y^{(2)} - K_2 K_6 (M_x^{(5)} M_y^{(0)} \\
&+ M_x^{(4)} M_y^{(1)}) - K_6 M_x^{(4)} M_y^{(3)} - (K_6)^2 M_x^{(1)} M_y^{(0)} \\
&+ (K_2 K_8 - K_4 K_6) (M_x^{(3)} M_y^{(0)} + M_x^{(2)} M_y^{(1)})
\end{aligned}$$

(4.8f)

$$\begin{aligned}
K_8 M_x^{(7)} M_y^{(0)} + K_4 M_x^{(7)} M_y^{(4)} &= -K_4 M_x^{(6)} M_y^{(5)} - K_2 K_4 M_x^{(5)} M_y^{(4)} \\
&- K_8 (M_x^{(6)} M_y^{(1)} + M_x^{(5)} M_y^{(2)} + M_x^{(4)} M_y^{(3)}) \\
&- K_2 K_8 (M_x^{(5)} M_y^{(0)} + M_x^{(4)} M_y^{(1)}) - K_6 K_8 M_x^{(1)} M_y^{(0)} \\
&+ (K_4 K_6 - K_2 K_8) M_x^{(3)} M_y^{(2)}
\end{aligned}$$

(4.8g)

$$\begin{aligned}
K_8 M_x^{(7)} M_y^{(0)} + K_6 M_x^{(7)} M_y^{(2)} &= -K_6 (M_x^{(6)} M_y^{(3)} + M_x^{(5)} M_y^{(4)}) \\
&- K_8 M_x^{(6)} M_y^{(1)} - (K_2 K_6 + K_8) (M_x^{(5)} M_y^{(2)} + M_x^{(4)} M_y^{(3)}) \\
&- K_2 K_8 (M_x^{(5)} M_y^{(0)} + M_x^{(4)} M_y^{(1)}) - K_4 K_8 M_x^{(3)} M_y^{(0)} \\
&- (K_2 K_8 + K_4 K_6) M_x^{(3)} M_y^{(2)} - K_4 K_8 M_x^{(2)} M_y^{(1)}
\end{aligned}$$

(4.8h)

$$\begin{aligned}
2K_2M_x^{(7)}M_y^{(1)} - 2M_x^{(7)}M_y^{(3)} &= 2M_x^{(6)}M_y^{(4)} + M_x^{(5)}M_y^{(5)} - 2(K_2)^2M_x^{(5)}M_y^{(1)} \\
&+ K_2M_x^{(4)}M_y^{(4)} + (K_4 - (K_2)^2)M_x^{(3)}M_y^{(3)} - 2K_2K_4M_x^{(3)}M_y^{(1)} \\
&- K_6M_x^{(2)}M_y^{(2)} - 2K_8M_x^{(2)}M_y^{(0)} - (K_8 + K_2K_6)MM_{xy}
\end{aligned}$$

(4.8i)

$$\begin{aligned}
2K_6M_x^{(7)}M_y^{(1)} + M_x^{(7)}M_y^{(7)} &= K_2M_x^{(6)}M_y^{(6)} + 2K_4M_x^{(6)}M_y^{(4)} + K_6(2M_x^{(6)}M_y^{(2)} \\
&+ M_x^{(4)}M_y^{(4)}) + 2K_8(M_x^{(6)}M_y^{(0)} + M_x^{(4)}M_y^{(2)}) + K_4M_x^{(5)}M_y^{(5)} \\
&+ (K_8 - K_2K_6)(2M_x^{(5)}M_y^{(1)} + M_x^{(3)}M_y^{(3)}) - (K_6)^2MM_{xy} \\
&- 2K_4K_6M_x^{(3)}M_y^{(1)}
\end{aligned}$$

(4.8j)

$$\begin{aligned}
2K_8M_x^{(7)}M_y^{(1)} + K_2M_x^{(7)}M_y^{(7)} &= (K_2)^2M_x^{(6)}M_y^{(6)} + 2(K_2K_6 - K_8)M_x^{(5)}M_y^{(3)} \\
&+ K_2K_4(2M_x^{(6)}M_y^{(4)} + M_x^{(5)}M_y^{(5)}) + 2K_2K_8(2M_x^{(6)}M_y^{(0)} \\
&+ M_x^{(4)}M_y^{(2)}) - 2K_4K_8M_x^{(3)}M_y^{(1)} - 2K_6K_8MM_{xy}
\end{aligned}$$

(4.8k)

$$\begin{aligned}
2K_4M_x^{(7)}M_y^{(1)} - 2K_2M_x^{(7)}M_y^{(3)} &= 2K_2M_x^{(6)}M_y^{(4)} + (K_2)^2M_x^{(4)}M_y^{(4)} \\
&+ K_2M_x^{(5)}M_y^{(5)} + 2((K_2)^2 - K_4)M_x^{(5)}M_y^{(3)} - K_2K_6M_x^{(2)}M_y^{(2)} \\
&- 2(K_4)^2M_x^{(3)}M_y^{(1)} - 2K_2K_8M_x^{(2)}M_y^{(0)} - 2K_2K_4M_x^{(5)}M_y^{(1)} \\
&- (K_4K_6 + K_2K_8)MM_{xy}
\end{aligned}$$

(4.8l)

$$\begin{aligned}
2K_6M_x^{(7)}M_y^{(1)} + 2K_2M_x^{(7)}M_y^{(5)} &= -(K_2)^2M_x^{(5)}M_y^{(5)} - 2K_6M_x^{(5)}M_y^{(3)} \\
&- 2K_2K_6M_x^{(5)}M_y^{(1)} - (K_6)^2MM_{xy} + 2(K_2K_8 \\
&- K_4K_6)M_x^{(3)}M_y^{(1)}
\end{aligned}$$

(4.8m)

$$\begin{aligned}
2K_8 M_x^{(7} M_y^{1)} + 2K_4 M_x^{(7} M_y^{5)} &= -K_2 K_4 M_x^5 M_y^5 - 2K_8 M_x^{(5} M_y^{3)} \\
&\quad - 2K_2 K_8 M_x^{(5} M_y^{1)} - K_6 K_8 M M_x M_y \\
&\quad + (K_4 K_6 - K_2 K_8) M_x^3 M_y^3
\end{aligned}$$

(4.8n)

$$\begin{aligned}
2K_8 M_x^{(7} M_y^{1)} + 2K_6 M_x^{(7} M_y^{3)} &= -2K_6 M_x^{(6} M_y^{4)} - K_6 M_x^5 M_y^5 \\
&\quad - 2(K_2 K_6 + K_8) M_x^{(5} M_y^{3)} - 2K_2 K_8 M_x^{(5} M_y^{1)} \\
&\quad - K_2 K_6 M_x^4 M_y^4 - (K_2 K_8 + K_4 K_6) M_x^3 M_y^3 \\
&\quad - 2K_4 K_8 M_x^{(3} M_y^{1)} + (K_6)^2 M_x^2 M_y^2 + 2K_6 K_8 M_x^{(2} M_y^{0)}
\end{aligned}$$

(4.8o)

$$\begin{aligned}
K_2 M_x^{(7} M_y^{2)} - M_x^{(7} M_y^{4)} &= M_x^{(6} M_y^{5)} - K_2 M_x^{(6} M_y^{3)} - (K_2)^2 M_x^{(5} M_y^{2)} \\
&\quad - (K_2)^2 M_x^{(4} M_y^{3)} - (K_2 K_4 + K_6) M_x^{(3} M_y^{2)} \\
&\quad - K_8 (M_x^{(3} M_y^{0)} - 2K_4 K_8 (2M_x^{(3} M_y^{0)} + M_x^{(2} M_y^{1)}) \\
&\quad + K_2 K_8 M_x^{(1} M_y^{0)}
\end{aligned}$$

(4.8p)

$$\begin{aligned}
K_4 M_x^{(7} M_y^{2)} + M_x^{(7} M_y^{6)} &= -K_4 M_x^{(6} M_y^{3)} + (K_6 - K_2 K_4) M_x^{(5} M_y^{2)} \\
&\quad + K_8 M_x^{(5} M_y^{0)} + (K_6 - K_2 K_4) M_x^{(4} M_y^{3)} + K_8 M_x^{(4} M_y^{1)} \\
&\quad + (K_8 - (K_4)^2) M_x^{(3} M_y^{2)} + K_4 K_8 M_x^{(1} M_y^{0)}
\end{aligned}$$

(4.8q)

$$\begin{aligned}
K_6 M_x^{(7} M_y^{2)} + K_2 M_x^{(7} M_y^{6)} &= -K_6 M_x^{(6} M_y^{3)} + (K_2 K_4 - K_6) M_x^{(5} M_y^{4)} \\
&\quad + K_2 K_8 (M_x^{(5} M_y^{0)} + M_x^{(4} M_y^{1)}) + (K_2 K_8 \\
&\quad - K_4 K_6) M_x^{(3} M_y^{2)} + K_6 K_8 M_x^{(1} M_y^{0)}
\end{aligned}$$

(4.8r)

$$\begin{aligned}
K_6 M_x^{(7)} M_y^{(2)} + K_4 M_x^{(7)} M_y^{(4)} &= -K_4 M_x^{(6)} M_y^{(5)} - K_6 M_x^{(6)} M_y^{(3)} \\
&\quad - (K_6 + K_2 K_4) M_x^{(5)} M_y^{(4)} - K_2 K_6 (M_x^{(5)} M_y^{(2)} + M_x^{(4)} M_y^{(3)}) \\
&\quad + K_4 K_8 (M_x^{(3)} M_y^{(0)} + M_x^{(2)} M_y^{(1)}) + K_6 K_8 M_x^{(1)} M_y^{(0)}
\end{aligned}$$

(4.8s)

$$\begin{aligned}
K_8 M_x^{(7)} M_y^{(2)} + K_4 M_x^{(7)} M_y^{(6)} &= -K_8 M_x^{(6)} M_y^{(3)} + ((K_4)^2 - K_8) M_x^{(5)} M_y^{(4)} \\
&\quad + (K_4 K_6 - K_2 K_8) M_x^{(5)} M_y^{(2)} + K_4 K_8 M_x^{(5)} M_y^{(0)} + (K_4 K_6 \\
&\quad - K_2 K_8) M_x^{(4)} M_y^{(3)} + K_4 K_8 M_x^{(4)} M_y^{(1)} + (K_8)^2 M_x^{(1)} M_y^{(0)}
\end{aligned}$$

(4.8t)

$$\begin{aligned}
2K_4 M_x^{(7)} M_y^{(3)} + M_x^7 M_y^7 &= K_2 M_x^6 M_y^6 + 2K_6 M_x^{(6)} M_y^{(2)} + 2K_8 M_x^{(6)} M_y^{(0)} \\
&\quad + 2(K_6 - K_2 K_4) M_x^{(5)} M_y^{(3)} + 2K_8 M_x^{(5)} M_y^{(1)} + (K_6 \\
&\quad - K_2 K_4) M_x^4 M_y^4 + 2K_8 M_x^{(4)} M_y^{(2)} + (K_8 - (K_4)^2) M_x^3 M_y^3 \\
&\quad + K_4 K_6 M_x^2 M_y^2 + 2K_4 K_8 M_x^{(2)} M_y^{(0)} + K_4 K_8 M_x M_y
\end{aligned}$$

(4.8u)

$$\begin{aligned}
2K_2 M_x^{(7)} M_y^{(3)} - 2M_x^{(7)} M_y^{(5)} &= -2K_2 M_x^{(6)} M_y^{(4)} - 2(K_2)^2 M_x^{(5)} M_y^{(3)} \\
&\quad - (K_2)^2 M_x^4 M_y^4 - (K_2 K_4 + K_6) M_x^3 M_y^3 + K_2 K_6 M_x^2 M_y^2 \\
&\quad + 2K_2 K_8 M_x^{(2)} M_y^{(0)} - 2K_8 M_x^{(3)} M_y^{(1)} + K_2 K_8 M_x M_y
\end{aligned}$$

(4.8v)

$$\begin{aligned}
2K_6M_x^{(7)}M_y^{(3)} + K_2M_x^{(7)}M_y^{(7)} &= (K_2)^2M_x^{(6)}M_y^{(6)} + 2(K_2K_4 - K_6)M_x^{(6)}M_y^{(4)} \\
&+ 2K_2K_6M_x^{(6)}M_y^{(2)} + 2K_2K_8M_x^{(6)}M_y^{(0)} + (K_2K_4 \\
&- K_6)M_x^{(5)}M_y^{(5)} + 2K_2K_8M_x^{(5)}M_y^{(1)} + 2K_2K_8M_x^{(4)}M_y^{(2)} \\
&+ (K_2K_8 - K_4K_6)M_x^{(3)}M_y^{(3)} + (K_6)^2M_x^{(2)}M_y^{(2)} \\
&+ K_6K_8(2M_x^{(2)}M_y^{(0)} + MM)
\end{aligned}$$

(4.8w)

$$\begin{aligned}
2K_6M_x^{(7)}M_y^{(3)} + 2K_4M_x^{(7)}M_y^{(5)} &= -2K_6M_x^{(6)}M_y^{(4)} - (K_2K_4 + K_6)M_x^{(5)}M_y^{(5)} \\
&- 2K_2K_6M_x^{(5)}M_y^{(3)} - K_2K_6M_x^{(4)}M_y^{(4)} + 2K_4K_8M_x^{(3)}M_y^{(1)} \\
&+ (K_6)^2M_x^{(2)}M_y^{(2)} + 2K_6K_8M_x^{(2)}M_y^{(0)} + K_6K_8MM
\end{aligned}$$

(4.8x)

$$\begin{aligned}
2K_8M_x^{(7)}M_y^{(3)} + K_4M_x^{(7)}M_y^{(7)} &= K_2K_4M_x^{(6)}M_y^{(6)} + 2K_4K_6M_x^{(6)}M_y^{(2)} \\
&+ ((K_4)^2 - K_8)(2M_x^{(6)}M_y^{(4)} + M_x^{(5)}M_y^{(5)}) \\
&+ 2K_4K_8(M_x^{(6)}M_y^{(0)} + M_x^{(5)}M_y^{(1)} + M_x^{(4)}M_y^{(2)}) \\
&+ (K_4K_6 - K_2K_8)(2M_x^{(5)}M_y^{(3)} + M_x^{(4)}M_y^{(4)}) \\
&+ K_6K_8M_x^{(2)}M_y^{(2)} + (K_8)^2(2M_x^{(2)}M_y^{(0)} + MM)
\end{aligned}$$

(4.8y)

$$\begin{aligned}
K_4M_x^{(7)}M_y^{(4)} + K_2M_x^{(7)}M_y^{(6)} &= -K_4M_x^{(6)}M_y^{(5)} + K_2K_6(M_x^{(5)}M_y^{(2)} \\
&+ M_x^{(4)}M_y^{(3)}) + K_2K_8(M_x^{(5)}M_y^{(0)} + M_x^{(4)}M_y^{(1)}) + (K_4K_6 \\
&+ K_2K_8)M_x^{(3)}M_y^{(2)} + K_4K_8(M_x^{(3)}M_y^{(0)} + M_x^{(2)}M_y^{(1)})
\end{aligned}$$

(4.8z)

$$\begin{aligned}
K_8 M_x^{(7)} M_y^{(4)} - K_6 M_x^{(7)} M_y^{(6)} &= -K_8 M_x^{(6)} M_y^{(5)} - (K_2 K_8 \\
&+ K_4 K_6) (M_x^{(5)} M_y^{(4)} - (K_6)^2 (M_x^{(5)} M_y^{(2)} + M_x^{(4)} M_y^{(3)}) \\
&- K_6 K_8 (M_x^{(5)} M_y^{(0)} + M_x^{(4)} M_y^{(1)}) \\
&+ (K_8)^2 (M_x^{(3)} M_y^{(0)} + M_x^{(2)} M_y^{(1)})
\end{aligned}$$

(4.9a)

$$\begin{aligned}
2K_4 M_x^{(7)} M_y^{(5)} + K_2 M_x^{(7)} M_y^{(7)} &= (K_2)^2 M_x^{(6)} M_y^{(6)} + 2K_2 K_4 M_x^{(6)} M_y^{(4)} \\
&+ K_2 K_6 (2M_x^{(6)} M_y^{(2)} + 2M_x^{(5)} M_y^{(3)} + M_x^{(4)} M_y^{(4)}) \\
&+ 2K_2 K_8 (M_x^{(6)} M_y^{(0)} + M_x^{(5)} M_y^{(1)} + M_x^{(4)} M_y^{(2)}) \\
&+ (K_4 K_6 + K_2 K_8) M_x^{(3)} M_y^{(3)} + 2K_4 K_8 M_x^{(3)} M_y^{(1)}
\end{aligned}$$

(4.9b)

$$\begin{aligned}
2K_8 M_x^{(7)} M_y^{(5)} - K_6 M_x^{(7)} M_y^{(7)} &= -K_2 K_6 M_x^{(6)} M_y^{(6)} - 2K_4 K_6 M_x^{(6)} M_y^{(4)} \\
&- (K_6)^2 (2M_x^{(6)} M_y^{(2)} + 2M_x^{(5)} M_y^{(3)} + M_x^{(4)} M_y^{(4)}) \\
&- 2K_6 K_8 (M_x^{(6)} M_y^{(0)} + M_x^{(5)} M_y^{(1)} + M_x^{(4)} M_y^{(2)}) \\
&- (K_2 K_8 + K_4 K_6) M_x^{(5)} M_y^{(5)} + 2(K_8)^2 M_x^{(3)} M_y^{(1)}
\end{aligned}$$

*Proof.* The proof of the relations in (4.7). The values of index  $x$  belong to any one of the following four groups:

$$x = a = 1, 2; \quad x = b = 3, 4; \quad x = c = 5, 6; \quad x = d = 7, 8$$

For the values of two different indices  $x$  and  $y$ , we have two cases. The first case is that  $x$  and  $y$  belong to the same group. The second case is that each of  $x$  and  $y$  belongs to two different groups. In the first



case, it may be easily shown that the relations in (4.7) are identities  $0 = 0$ , because in the proof given below each of  $A, B, C, D, E, F, G$ , and  $H$  contains the factor  $\frac{M}{x} + \frac{M}{y} = 0$ . Therefore, in the proof of the relations (4.7) it suffices to show the validity of the relations for the second case only. Now let  $x, y, z$ , and  $w$  be indices each of which belongs to mutually different groups.

In the proof of each of the recurrence relations given in (4.7), we use the relations (4.3) and (4.6) whenever they are necessary. We prove the relations in (4.7) in order, and we also use the previously proved relations in (4.7) whenever necessary.

For example, we demonstrate in the following the proof of (4.7e). In order to prove (4.7e), we consider

$$E = M \frac{M}{x} (M + \frac{M}{y}) M^2 \frac{M^2}{x} \frac{M^2}{y} \frac{M^2}{z} \frac{M^2}{w}$$

The relation (4.6) gives

$$(4.10a) \quad E = 2K_8 \frac{M^{(2)}}{x} \frac{M^{(1)}}{y}$$

On the other hand, making use of (4.6), (4.3) in order, we have

$$(4.10b) \quad \begin{aligned} E &= 2M \frac{M^{(4)}}{x} \frac{M^{(3)}}{y} \frac{M^2}{z} \frac{M^2}{w} \\ &= 2M \frac{M^{(4)}}{x} \frac{M^{(3)}}{y} [K_4 + 2K_2 \frac{M^{(2)}}{x} \frac{M^{(0)}}{y} + 2M \frac{M^{(4)}}{x} \frac{M^{(0)}}{y} + \frac{M^2}{x} \frac{M^2}{y}] \\ &= 2K_4 \frac{M^{(4)}}{x} \frac{M^{(3)}}{y} + 2K_2 \frac{M^{(6)}}{x} \frac{M^{(3)}}{y} + 2K_2 \frac{M^{(5)}}{x} \frac{M^{(4)}}{y} \\ &\quad + 2M \frac{M^{(8)}}{x} \frac{M^{(3)}}{y} + 2M \frac{M^{(7)}}{x} \frac{M^{(4)}}{y} + 2M \frac{M^{(6)}}{x} \frac{M^{(5)}}{y} \\ &= 2M \frac{M^{(7)}}{x} \frac{M^{(4)}}{y} + 2M \frac{M^{(6)}}{x} \frac{M^{(5)}}{y} + 2K_2 \frac{M^{(5)}}{x} \frac{M^{(4)}}{y} - 2K_6 \frac{M^{(3)}}{x} \frac{M^{(2)}}{y} \\ &\quad - 2K_8 \frac{M^{(3)}}{x} \frac{M^{(0)}}{y} \end{aligned}$$

The relation (4.7e) follows immediately from (4.10a) and (4.10b).

The remaining relations in (4.7) may be proved similarly by con-

sidering

$$A = \left( \begin{matrix} M & M \\ x & y \end{matrix} \right) \left[ \left( \begin{matrix} M^2 & M^2 \\ x & y \end{matrix} \right) \begin{matrix} M^2 & M^2 \\ z & w \end{matrix} \right. \\ \left. + \left( \begin{matrix} M^2 & M^2 \\ z & w \end{matrix} \right) \begin{matrix} M^2 & M^2 \\ x & y \end{matrix} \right] \quad \text{for the proof of (4.7a)}$$

$$B = \left( \begin{matrix} M & M \\ x & y \end{matrix} \right) \begin{matrix} M^{(7)} & M^0 \\ x & y \end{matrix} \quad \text{for the proof of (4.7b)}$$

$$C = \begin{matrix} M^2 & M^2 \\ x & y \end{matrix} \left( \begin{matrix} M & M \\ x & y \end{matrix} \right) \left[ \begin{matrix} M^2 & (M^2 + M^2 + M^2) \\ x & y \end{matrix} \right. \\ \left. + \begin{matrix} M^2 & (M^2 + M^2) \\ z & w \end{matrix} + \begin{matrix} M^2 & M^2 \\ z & w \end{matrix} \right] \quad \text{for the proof of (4.7c)}$$

$$D = \left( \begin{matrix} M & M \\ x & y \end{matrix} \right) \begin{matrix} M^{(7)} & M^2 \\ x & y \end{matrix} \quad \text{for the proof of (4.7d)}$$

$$F = 2 \begin{matrix} M & M \\ x & y \end{matrix} \left( \begin{matrix} M & M \\ x & y \end{matrix} \right) \begin{matrix} M^{(7)} & M^2 \\ x & y \end{matrix} \quad \text{for the proof of (4.7f)}$$

$$G = \left( \begin{matrix} M & M \\ x & y \end{matrix} \right) \begin{matrix} M^{(7)} & M^5 \\ x & y \end{matrix} \quad \text{for the proof of (4.7g)}$$

$$H = \begin{matrix} M & M \\ x & y \end{matrix} \left( \begin{matrix} M & M \\ x & y \end{matrix} \right) \begin{matrix} M^{(7)} & M^4 \\ x & y \end{matrix} \quad \text{for the proof of (4.7h)}$$

The relations (4.8) and (4.9) may be proved similarly. □

Now, we are ready to prove the recurrence relations of the third kind in the following theorem. These relations are very important for the solution of I(2.4) or I(2.19) in 8-*g*-UFT. We use these relations in our subsequent paper to establish a linear system equivalent to I(2.4) and to find a precise and surveyable tensorial representation of 8-dimensional Einstein's connection in terms of the unified field tensor  $g_{\lambda\mu}$ .

**THEOREM 4.5.** (The recurrence relations of the third kind in 8-*g*-UFT). *If  $T \equiv T_{\omega\lambda\mu}$  is a skew-symmetric tensor in the first two indices, the following recurrence relations hold in 8-*g*-UFT:*

(For the first class).

(4.11a)

$$\begin{aligned} {}^{(70)r}T = & - {}^{(61)r}T - {}^{(52)r}T - K_2 {}^{(50)r}T - {}^{(43)r}T - K_2 {}^{(41)r}T \\ & - K_2 {}^{(32)r}T - K_4 {}^{(30)r}T - K_4 {}^{(21)r}T - K_6 {}^{(10)r}T \end{aligned}$$

(4.11b)

$$2 {}^{(71)r}T = -2 {}^{(53)r}T - 2K_2 {}^{(51)r}T - K_2 {}^{33r}T - 2K_4 {}^{(31)r}T - K_6 {}^{11r}T$$

(4.11c)

$${}^{(72)r}T = - {}^{(63)r}T - {}^{(54)r}T - K_2 {}^{(52)r}T - K_2 {}^{(43)r}T - K_4 {}^{(32)r}T + K_8 {}^{(10)r}T$$

(4.11d)

$$\begin{aligned} 2 {}^{(73)r}T = & -2 {}^{(64)r}T - {}^{55r}T - 2K_2 {}^{(53)r}T - K_2 {}^{44r}T - K_4 {}^{33r}T \\ & + K_6 {}^{22r}T + 2K_8 {}^{(20)r}T + K_8 {}^{11r}T \end{aligned}$$

(4.11e)

$${}^{(74)r}T = - {}^{(65)r}T - K_2 {}^{(54)r}T + K_6 {}^{(32)r}T + K_8 {}^{(30)r}T + K_8 {}^{(21)r}T$$

(4.11f)

$$2 {}^{(75)r}T = -K_2 {}^{55r}T + K_6 {}^{33r}T + 2K_8 {}^{(31)r}T$$

(4.11g)

$$\begin{aligned} {}^{(76)r}T = & K_4 {}^{(54)r}T + K_6 {}^{(52)r}T + K_8 {}^{(50)r}T + K_6 {}^{(43)r}T \\ & + K_8 {}^{(41)r}T + K_8 {}^{(32)r}T \end{aligned}$$

(4.11h)

$$\begin{aligned} {}^{77r}T = & K_2 {}^{66r}T + 2K_4 {}^{(64)r}T + 2K_6 {}^{(62)r}T + 2K_8 {}^{(60)r}T + K_4 {}^{55r}T \\ & + 2K_6 {}^{(53)r}T + 2K_8 {}^{(51)r}T + K_6 {}^{44r}T + 2K_8 {}^{(42)r}T + K_8 {}^{33r}T \end{aligned}$$

Furthermore, the following identities also hold in the first class:

(4.12a)

$$\begin{aligned} K_2 \overset{(70)r}{T} - \overset{(72)r}{T} &= \overset{(63)r}{T} - K_2 \overset{(61)r}{T} + \overset{(54)r}{T} - (K_2)^2 (\overset{(50)r}{T} + \overset{(41)r}{T}) \\ &\quad + (K_4 - (K_2)^2) \overset{(32)r}{T} - K_2 K_4 (\overset{(30)r}{T} + \overset{(21)r}{T}) \\ &\quad - (K_2 K_6 + K_8) \overset{(10)r}{T} \end{aligned}$$

(4.12b)

$$\begin{aligned} K_6 \overset{(70)r}{T} + \overset{(76)r}{T} &= -K_6 \overset{(61)r}{T} + K_4 \overset{(54)r}{T} + (K_8 - K_2 K_6) (\overset{(50)r}{T} \\ &\quad + \overset{(41)r}{T} + \overset{(32)r}{T}) - K_4 K_6 (\overset{(30)r}{T} + \overset{(21)r}{T}) - (K_6)^2 \overset{(10)r}{T} \end{aligned}$$

(4.12c)

$$\begin{aligned} K_8 \overset{(70)r}{T} + K_2 \overset{(76)r}{T} &= -K_8 \overset{(61)r}{T} + K_2 K_4 \overset{(54)r}{T} + (K_2 K_6 - K_8) (\overset{(52)r}{T} \\ &\quad + \overset{(43)r}{T}) - K_4 K_8 (\overset{(30)r}{T} + \overset{(21)r}{T}) - K_6 K_8 \overset{(10)r}{T} \end{aligned}$$

(4.12d)

$$\begin{aligned} K_4 \overset{(70)r}{T} - K_2 \overset{(72)r}{T} &= K_2 \overset{(63)r}{T} - K_4 \overset{(61)r}{T} + K_2 \overset{(54)r}{T} - K_2 K_4 \overset{(50)r}{T} \\ &\quad + ((K_2)^2 - K_4) (\overset{(52)r}{T} + \overset{(43)r}{T}) - K_2 K_4 \overset{(41)r}{T} \\ &\quad - (K_4)^2 (\overset{(30)r}{T} + \overset{(21)r}{T}) - (K_4 K_6 + K_2 K_8) \overset{(10)r}{T} \end{aligned}$$

(4.12e)

$$\begin{aligned} K_6 \overset{(70)r}{T} + K_2 \overset{(74)r}{T} &= -K_2 \overset{(65)r}{T} - K_6 \overset{(61)r}{T} - (K_2)^2 \overset{(54)r}{T} - K_6 \overset{(52)r}{T} \\ &\quad - K_2 K_6 \overset{(50)r}{T} - K_6 \overset{(43)r}{T} - (K_6)^2 \overset{(10)r}{T} \\ &\quad + (K_2 K_8 - K_4 K_6) (\overset{(30)r}{T} + \overset{(21)r}{T}) - K_2 K_6 \overset{(41)r}{T} \end{aligned}$$

(4.12f)

$$\begin{aligned}
K_8 {}^{(70)r}T + K_4 {}^{(74)r}T &= -K_4 {}^{(65)r}T - K_8 ({}^{(61)r}T + {}^{(52)r}T + {}^{(43)r}T) \\
&\quad - K_2 K_4 {}^{(54)r}T - K_2 K_8 ({}^{(50)r}T + {}^{(41)r}T) + (K_4 K_6 \\
&\quad - K_2 K_8) {}^{(32)r}T - K_6 K_8 {}^{(10)r}T
\end{aligned}$$

(4.12g)

$$\begin{aligned}
K_8 {}^{(70)r}T + K_6 {}^{(72)r}T &= -K_6 {}^{(63)r}T - K_8 {}^{(61)r}T - K_6 {}^{(54)r}T - (K_2 K_6 \\
&\quad + K_8) {}^{(52)r}T - K_2 K_8 {}^{(50)r}T - (K_2 K_6 + K_8) {}^{(43)r}T \\
&\quad - K_2 K_8 {}^{(41)r}T - (K_2 K_8 + K_4 K_6) {}^{(32)r}T - K_4 K_8 ({}^{(30)r}T + {}^{(21)r}T)
\end{aligned}$$

(4.12h)

$$\begin{aligned}
2K_2 {}^{(71)r}T - 2 {}^{(73)r}T &= 2 {}^{(64)r}T + {}^{(55)r}T - 2(K_2)^2 {}^{(51)r}T + K_2 {}^{(44)r}T \\
&\quad + (K_4 - (K_2)^2) {}^{(33)r}T - 2K_2 K_4 {}^{(31)r}T - K_6 {}^{(22)r}T - 2K_8 {}^{(20)r}T \\
&\quad - (K_8 + K_2 K_6) {}^{(11)r}T
\end{aligned}$$

(4.12i)

$$\begin{aligned}
2K_6 {}^{(71)r}T + {}^{(77)r}T &= K_2 {}^{(66)r}T + 2K_4 {}^{(64)r}T + 2K_6 {}^{(62)r}T + 2K_8 {}^{(60)r}T + K_4 {}^{(55)r}T \\
&\quad + 2(K_8 - K_2 K_6) {}^{(51)r}T + K_6 {}^{(44)r}T + 2K_8 {}^{(42)r}T \\
&\quad + (K_8 - K_2 K_6) {}^{(33)r}T - 2K_4 K_6 {}^{(31)r}T - (K_6)^2 {}^{(11)r}T
\end{aligned}$$

(4.12j)

$$\begin{aligned}
2K_8 {}^{(71)r}T + K_2 {}^{(77)r}T &= (K_2)^2 {}^{(66)r}T + K_2 K_4 (2 {}^{(64)r}T + {}^{(55)r}T) \\
&\quad + K_2 K_6 (2 {}^{(62)r}T + {}^{(44)r}T) - 2K_4 K_8 {}^{(31)r}T + 2K_2 K_8 ({}^{(60)r}T + {}^{(42)r}T) \\
&\quad - K_6 K_8 {}^{(11)r}T + 2(K_2 K_6 - K_8) {}^{(53)r}T
\end{aligned}$$

(4.12k)

$$\begin{aligned}
2K_4 \overset{(71)r}{T} - 2K_2 \overset{(73)r}{T} &= K_2(2 \overset{(64)r}{T} + \overset{55r}{T}) - 2K_2K_4 \overset{(51)r}{T} \\
&+ 2((K_2)^2 - K_4) \overset{(53)r}{T} + (K_2)^2 \overset{44r}{T} - 2(K_4)^2 \overset{(31)r}{T} \\
&- K_2K_6 \overset{22r}{T} - 2K_2K_8 \overset{(20)r}{T} - (K_4K_6 + K_2K_8) \overset{11r}{T}
\end{aligned}$$

(4.12l)

$$\begin{aligned}
2K_6 \overset{(71)r}{T} + 2K_2 \overset{(75)r}{T} &= -(K_2)^2 \overset{55r}{T} - 2K_6 \overset{(53)r}{T} - 2K_2K_6 \overset{(51)r}{T} \\
&- (K_6)^2 \overset{11r}{T} + 2(K_2K_8 - K_4K_6) \overset{(31)r}{T}
\end{aligned}$$

(4.12m)

$$\begin{aligned}
2K_8 \overset{(71)r}{T} + 2K_4 \overset{(75)r}{T} &= -K_2K_4 \overset{55r}{T} - 2K_8 \overset{(53)r}{T} - 2K_2K_8 \overset{(51)r}{T} \\
&- K_6K_8 \overset{11r}{T} + (K_4K_6 - K_2K_8) \overset{33r}{T}
\end{aligned}$$

(4.12n)

$$\begin{aligned}
2K_8 \overset{(71)r}{T} + 2K_6 \overset{(73)r}{T} &= -2K_6 \overset{(64)r}{T} - K_6 \overset{55r}{T} - 2(K_2K_6 + K_8) \overset{(53)r}{T} \\
&- 2K_2K_8 \overset{(51)r}{T} - K_2K_6 \overset{44r}{T} - (K_2K_8 + K_4K_6) \overset{33r}{T} \\
&- 2K_4K_8 \overset{(31)r}{T} + (K_6)^2 \overset{22r}{T} + 2K_6K_8 \overset{(20)r}{T}
\end{aligned}$$

(4.12o)

$$\begin{aligned}
K_2 \overset{(72)r}{T} - \overset{(74)r}{T} &= \overset{(65)r}{T} - K_2 \overset{(63)r}{T} - (K_2)^2 (\overset{(52)r}{T} + \overset{(43)r}{T}) \\
&- (K_2K_4 + K_6) \overset{(32)r}{T} - K_8 (\overset{(30)r}{T} + \overset{(21)r}{T}) + K_2K_8 \overset{(10)r}{T}
\end{aligned}$$

(4.12p)

$$\begin{aligned}
K_4 \overset{(72)r}{T} + \overset{(76)r}{T} &= -K_4 \overset{(63)r}{T} + (K_6 - K_2K_4) (\overset{(52)r}{T} + \overset{(43)r}{T}) \\
&+ K_8 (\overset{(50)r}{T} + \overset{(41)r}{T}) + (K_8 - (K_4)^2) \overset{(32)r}{T} + K_4K_8 \overset{(10)r}{T}
\end{aligned}$$

(4.12q)

$$K_6^{(72)r} T + K_2^{(76)r} T = -K_6^{(63)r} T + (K_2 K_4 - K_6)^{(54)r} T \\ + K_2 K_8^{(50)r} (T + T^{(41)r}) + (K_2 K_8 - K_4 K_6)^{(32)r} T + K_6 K_8^{(10)r} T$$

(4.12r)

$$K_6^{(72)r} T + K_4^{(74)r} T = -K_4^{(65)r} T - K_6^{(63)r} T - (K_6 + K_2 K_4)^{(54)r} T \\ - K_2 K_6^{(52)r} (T + T^{(43)r}) + K_4 K_8^{(30)r} (T + T^{(21)r}) + K_6 K_8^{(10)r} T$$

(4.12s)

$$K_8^{(72)r} T + K_4^{(76)r} T = -K_8^{(63)r} T + ((K_4)^2 - K_8)^{(54)r} T \\ + (K_4 K_6 - K_2 K_8)^{(52)r} (T + T^{(43)r}) \\ + K_4 K_8^{(50)r} T + K_4 K_8^{(41)r} T + (K_8)^2^{(10)r} T$$

(4.12t)

$$2K_4^{(73)r} T + T^{77r} = K_2^{66r} T + 2K_6^{(62)r} T + 2K_8^{(60)r} T + (2K_6 \\ - K_2 K_4)^{(53)r} T + 2K_8^{(51)r} T + (K_6 - K_2 K_4)^{44r} T + 2K_8^{(42)r} T \\ + (K_8 - (K_4)^2)^{33r} T + K_4 K_6^{22r} T + K_4 K_8 (2 T^{(20)r} + T^{11r})$$

(4.12u)

$$2K_2^{(73)r} T - 2 T^{(75)r} = -2K_2^{(64)r} T - (K_2)^2 (2 T^{(53)r} + T^{44r}) - (K_2 K_4 \\ + K_6)^{33r} T - 2K_8^{(31)r} T + K_2 K_6^{22r} T + K_2 K_8 (2 T^{(20)r} + T^{11r})$$

(4.12v)

$$2K_6^{(73)r} T + K_2^{77r} T = (K_2)^2^{66r} T + (K_2 K_4 - K_6) (2 T^{(64)r} + T^{55r}) \\ + 2K_2 K_6^{(62)r} T + 2K_2 K_8 (T^{(60)r} + T^{(51)r} + T^{(42)r}) \\ + (K_2 K_8 - K_4 K_6)^{33r} T + (K_6)^2^{22r} T + K_6 K_8 (2 T^{(20)r} + T^{11r})$$

(4.12w)

$$\begin{aligned}
2K_6^{(73)r} T + 2K_4^{(75)r} T &= -2K_6^{(64)r} T - (K_2K_4 + K_6)^{55r} T \\
&\quad - K_2K_6(2 T^{(53)r} + T^{44r}) + 2K_4K_8^{(31)r} T + (K_6)^2 T^{22r} \\
&\quad + K_6K_8(2 T^{(20)r} + T^{11r})
\end{aligned}$$

(4.12x)

$$\begin{aligned}
2K_8^{(73)r} T + K_4^{77r} T &= K_2K_4^{66r} T + 2K_4K_6^{(62)r} T + 2((K_4)^2 - K_8)^{(64)r} T \\
&\quad + 2K_4K_8(T^{(60)r} + T^{(51)r} + T^{(42)r}) + ((K_4)^2 - K_8)^{55r} T + K_6K_8^{22r} T \\
&\quad + (K_4K_6 - K_2K_8)(2 T^{(53)r} + T^{44r}) + (K_8)^2(2 T^{(20)r} + T^{11r})
\end{aligned}$$

(4.12y)

$$\begin{aligned}
K_4^{(74)r} T + K_2^{(76)r} T &= -K_4^{(65)r} T + K_2K_6(T^{(52)r} + T^{(43)r}) + K_2K_8(T^{(50)r} \\
&\quad + T^{(41)r}) + (K_4K_6 + K_2K_8)(T^{(32)r} + K_4K_8(T^{(30)r} + T^{(21)r}))
\end{aligned}$$

(4.12z)

$$\begin{aligned}
K_8^{(74)r} T - K_6^{(76)r} T &= -K_8^{(65)r} T - (K_2K_8 + K_4K_6)^{(54)r} T \\
&\quad - (K_6)^2(T^{(52)r} + T^{(43)r}) - K_6K_8(T^{(50)r} + T^{(41)r}) \\
&\quad + (K_8)^2(T^{(30)r} + T^{(21)r})
\end{aligned}$$

(4.13a)

$$\begin{aligned}
2K_4^{(75)r} T + K_2^{77r} T &= (K_2)^2 T^{66r} + 2K_2K_4^{(64)r} T + K_2K_6(2 T^{(62)r} \\
&\quad + 2 T^{(53)r} + T^{44r}) + 2K_2K_8(T^{(60)r} + T^{(51)r} + T^{(42)r}) \\
&\quad + (K_4K_6 + K_2K_8)^{33r} T + 2K_4K_8^{(31)r} T
\end{aligned}$$



(4.13b)

$$\begin{aligned}
 2K_8 \binom{75}{r} T - K_6 \binom{77}{r} T &= -K_2 K_6 \binom{66}{r} T - (K_6)^2 (2 \binom{62}{r} T + 2 \binom{53}{r} T + \binom{44}{r} T) \\
 &\quad - 2K_4 K_6 \binom{64}{r} T - 2K_6 K_8 (\binom{60}{r} T + \binom{51}{r} T + \binom{42}{r} T) \\
 &\quad - (K_2 K_8 + K_4 K_6) \binom{55}{r} T + 2(K_8)^2 \binom{31}{r} T
 \end{aligned}$$

*Proof.* We first note that the terms in the right-hand side of (3.4a) vanishes identically when  $x = y$ . Therefore, whenever we use (3.4a), it suffices to consider the terms corresponding to the cases  $x \neq y$  only. The proof of the above relations follow from (3.4a), using (4.7) for the proof of (4.11). For example, the relation (4.11a) may be proved as in the following way:

$$\begin{aligned}
 \binom{70}{r} T &= \binom{70}{r} T_{\omega\mu\nu} \\
 &= \sum_{x,y,z} T_{xyz} M_x^{(7)} M_y^{(0)} M_z^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\
 &= \sum_{x,y,z} T_{xyz} [M_x^{(6)} M_y^{(1)} - M_x^{(5)} M_y^{(2)} - K_2 M_x^{(5)} M_y^{(0)} - M_x^{(4)} M_y^{(3)} \\
 &\quad - K_2 M_x^{(4)} M_y^{(1)} - K_2 M_x^{(3)} M_y^{(2)} - K_4 M_x^{(3)} M_y^{(0)} \\
 &\quad - K_4 M_x^{(2)} M_y^{(1)} - K_6 M_x^{(1)} M_y^{(0)}] M_z^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\
 &= - \binom{61}{r} T_{\omega\mu\nu} - \binom{52}{r} T_{\omega\mu\nu} - K_2 \binom{50}{r} T_{\omega\mu\nu} - \binom{43}{r} T_{\omega\mu\nu} - K_2 \binom{41}{r} T_{\omega\mu\nu} \\
 &\quad - K_2 \binom{32}{r} T_{\omega\mu\nu} - K_4 \binom{30}{r} T_{\omega\mu\nu} - K_4 \binom{21}{r} T_{\omega\mu\nu} - K_6 \binom{10}{r} T_{\omega\mu\nu} \\
 &= - \binom{61}{r} T - \binom{52}{r} T - K_2 \binom{50}{r} T - \binom{43}{r} T - K_2 \binom{41}{r} T \\
 &\quad - K_2 \binom{32}{r} T - K_4 \binom{30}{r} T - K_4 \binom{21}{r} T - K_6 \binom{10}{r} T
 \end{aligned}$$

The relations (4.12) and (4.13) can be proved similarly, using (4.8) and (4.9). □

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