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VARIOUS CONTINUITIES OF A MAP $f : (X, k, T^n_X) \to (Y, 2, T_Y)$ IN COMPUTER TOPOLOGY

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Abstract. For a set $X \subset \mathbb{Z}^n$, let (X, T^n_X) be the subspace of the Khalimsky n-dimensional space $(\mathbb{Z}^n, T^n)$, $n \in \mathbb{N}$. Considering a k-adjacency of (X, T^n_X), we use the notation (X, k, T^n_X). In this paper for a map $f : (X, k, T^n_X) \to (Y, 2, T_Y)$, we find the condition that weak $(k, 2)$-continuity of the map f implies strong $(k, 2)$-continuity of f.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}^n$ be the set of natural numbers, integers, and points in the Euclidean n-dimensional space with integer coordinates, respectively. Let $(\mathbb{Z}^n, T^n)$ be the Khalimsky n-dimensional space, $n \in \mathbb{N}$ [1, 2, 14]. For a set $X \subset \mathbb{Z}^n$, considering the subspace $(X, T^n_X) \subset (\mathbb{Z}^n, T^n)$ with k-adjacency, we use the notation (X, k, T^n_X). In [2, 7] for a map $f : (X, k_0, T^{n_0}_X) \to (Y, k_1, T^{n_0}_Y)$, the notions of (k_0, k_1)-continuity was developed [2, 7]. Meanwhile, for a discrete topological subspaces with k_0- and k_1-adjacency, denoted by (X, k_0) and (Y, k_1), digital (k_0, k_1)-continuity of a map $f : (X, k_0) \to (Y, k_1)$ was also developed [3, 4, 5, 6, 9, 10]. Further, the comparison between (k_0, k_1)-continuity of f and digital

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(k_0, k_1)-continuity of f were proceeded in [2]. Recently, the notion of weak (k_0, k_1)-continuity was also established [11].
In this paper for a map $f : (X, k, T_X^n) \to (Y, 2, T_Y)$, the notions of strong $(k, 2)$-continuity and weak $(k, 2)$-continuity are investigated and compared with each other. Further, in order to apply this result to discrete geometry, digital topology, and computer science we need to study various properties of these continuities of a map $f : (X, k, T_X^n) \to (Y, 2, T_Y)$.

2. Preliminaries

A set $X \subset \mathbb{Z}^n$ with k-adjacency, denoted by (X, k), has usually been considered in a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$, where $n \in \mathbb{N}$, k represents an adjacency relation for X, and $\bar{k}$ represents an adjacency relation for $\mathbb{Z}^n - X$ [2, 3, 4, 5, 6, 13]. However, in this paper we are not concerned with adjacencies among n-xels of $\mathbb{Z}^n - X$. Namely, we only consider a set $X \subset \mathbb{Z}^n$ with k-adjacency and Khalimsky product topology.

As the generalization of the commonly used 4- and 8-adjacency of $\mathbb{Z}^2$ and further, 6-, 18- and 26-adjacency of $\mathbb{Z}^3$ in [13], the k-adjacency relations of $\mathbb{Z}^n$ are established in [2, 3, 4, 5, 8] as follows:

$$k \in \{2n(n \geq 1), 3^n - 1(n \geq 2),$$
$$3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1(2 \leq r \leq n-1, n \geq 3)\}, (2-1)$$

where $C_t^n = n!/(n-t)!t!$.

For example, 8-, 32-, 64- and 80-adjacency relations of $\mathbb{Z}^4$ are considered and further, 10-, 50-, 130-, 210- and 242-adjacency relations of $\mathbb{Z}^5$ are used. Hereafter, each space $X \subset \mathbb{Z}^n$ is assumed with one of the k-adjacency relations of $\mathbb{Z}^n$.

Indeed, *Khalimsky line topology* on $\mathbb{Z}$ is induced from the subbasis $\{[2n-1, 2n+1]_{\mathbb{Z}} | n \in \mathbb{Z}\}$ [2, 9, 14] and is denoted by $(\mathbb{Z}, T)$. Namely, the family of the subset $\{\{2n+1\}, [2m-1, 2m+1]_{\mathbb{Z}} | m, n \in \mathbb{Z}\}$, which induces open sets for $(\mathbb{Z}, T)$, is a basis of the Khalimsky line topology

on $\mathbf{Z}$ [1, 2, 9, 14].
If the set $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b : n \in \mathbf{Z}\}$ with 2-adjacency is considered with the discrete topology, then it is called a *digital interval* and further, if the set $[a, b]_{\mathbf{Z}}$ is considered as a subspace of $(\mathbf{Z}, T)$ with the Khalimsky line topology, then it is called a *Khalimsky interval.* For a digital image (X, k) in $\mathbf{Z}^n$, we use the notation $N_k^*(p) := N_k(p) \cup \{p\}$, where $N_k(p) := \{x \in X | p \quad \text{is} \quad k-\text{adjacent to} \quad x\}$ which is called the *k-neighbors* of p [13].

Now consider the *product topology* on $\mathbf{Z}^n$ derived from the Khalimsky line topology on $\mathbf{Z}, n \geq 2$. Then the topology on $\mathbf{Z}^n$ is called the *product Khalimsky topology* on $\mathbf{Z}^n$, and we use the notation $(\mathbf{Z}^n, T^n)$. Let us examine the structure of the Khalimsky n-dimensional space. A point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ is *open* if all coordinates are odd, and *closed* if each of the coordinates is even [1, 9, 12, 13, 14]. These points are called *pure* and the other points in $\mathbf{Z}^n$ is called *mixed.* In all subspaces of $(\mathbf{Z}^n, T^n), n \in \mathrm{N}$, of Fig.1, 2, and 3, the symbol ■, jumbo dot, and ◉ mean a pure closed point, a pure open point, and a mixed point, respectively.

For a set $X \subset \mathbf{Z}^n$, consider the subspace (X, T_X^n) induced from the Khalimsky n-dimensional space $(\mathbf{Z}^n, T^n)$. Further, considering a topological space (X, T_X^n) with one of the k-adjacency relations of $\mathbf{Z}^n$ in (2-1), we call it a (computer topological) *space with k-adjacency* and denote it (X, k, T_X^n) [2, 7, 11].

Meanwhile, a set $X \subset \mathbf{Z}^n$ has often been studied with discrete topology with a k-adjaency [2, 3, 4, 5, 6, 7, 8, 13]. For a set with k-adjacency (X, k) in $\mathbf{Z}^n$, two distinct points $x, y \in X$ are called *k-connected* if there is a sequence $(x_i)_{i \in [0,m]_{\mathbf{Z}}} \subset X$ such that $x_0 = x$, $x_m = y$ and further, x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbf{Z}}, m \geq 1$. The number m is called the *length* of this k-path [13]. For an adjacency relation k, a *simple k-path* in X is the sequence $(x_i)_{i\in[0,m]_{\mathbb{Z}}} \subset X$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$ [2, 3, 4, 5, 6].

By computer topology is now meant the mathematical recognition of discrete space in $\mathbb{Z}^n$, *e.g.,* a development of tools implementing topological concepts for use in computer science and information technology. Computer topology plays a significant role in computer graphics, image synthesis, image analysis and so forth. It grew out of discrete geometry expanded into applications where significant topological issues arise. Computer topology may be of interest both for computer scientist who try to apply topological knowledge for investigating discrete spaces and for mathematicians who want to use computers to solve complicated topological problems. We can see some difference between *computer topology* and *digital topology.* Precisely, while computer topology needs some reasonable topological structure for the research of spaces $X \subset \mathbb{Z}^n$, digital topology requires the discrete topology for the study of them.

3. Various continuities in computer topology

For a discrete space with k-adjacency (X, k) in $\mathbb{Z}^n$, let us recall the digital k-neighborhood of a point $x \in X$ as follows. The *k-neighborhood* of $x_0 \in X$ with radius ε is defined to be the set [2, 3, 4, 5, 6, 7, 8]

$$N_k(x_0, \varepsilon) := \{x \in X| \quad l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \qquad (3-1)$$

where $l_k(x_0, x)$ is the length of a *shortest simple k-path* from x_0 to x in X. Meanwhile, for a space (X, k, T_X^n) and $x \in X$, by the *neighborhood* V of the point x is typically meant the existence of some open set $O_x \in T_X^n$ such that $x \in O_x \subset V$.

Further, if the set $N_k(x_0, \varepsilon)$ in (3-1) is a *topological neighborhood* of x_0 in (X, T_X^n), then this set is called a *(computer topological) k-neighborhood of x_0 with radius* $\varepsilon \in \mathbb{N}$ and is denoted by

$$N_k^*(x_0, \varepsilon) := \{x \in X| l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}. \qquad (3-2)$$

With the terminologies and definitions above at hand we have the following in terms of (3-1).

Definition 1 (Digital (k_0, k_1)-continuity). [2, 3, 4, 5, 6, 8, 9, 10] *For two discrete topological spaces with k_i-adjacency, $i \in \{0,1\}$, (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a function $f : (X, k_0) \to (Y, k_1)$ is said to be digitally (k_0, k_1)-continuous at a point $x \in X$ if for any $N_{k_1}(f(x), \varepsilon) \subset Y$, there is $N_{k_0}(x, \delta) \subset X$ such that$f(N_{k_0}(x, \delta)) \subset N_{k_1}(f(x), \varepsilon)$.*
Further, we say that a map $f : (X, k_0) \to (Y, k_1)$ is digitally (k_0, k_1)-continuous if the map f is digitally (k_0, k_1)-continuous at every point $x \in X$.

Indeed, digital (k_0, k_1)-continuity from (X, k_0) to (Y, k_1) implies the preservation of the k_0-connectedness of (X, k_0) into the k_1-connectedness of (Y, k_1). Further, digital (k_0, k_1)-continuity has the *uniform* (k_0, k_1)-*continuity* in [6].

Definition 2 (Khalimsky-continuity). [1, 2, 7, 14] *For two spaces $(X, T_X^{n_0}) \subset (\mathbf{Z}^n, T^n)$ and $(Y, T_Y) \subset (\mathbf{Z}, T)$, a function $f : X \to Y$ is said to be Khalimsky continuous at a point $x \in X$ if f is continuous at the point x with the Khalimsky product topology.*
Further, we say that a map $f : X \to Y$ is Khalimsky continuous if the map f is Khalimsky continuous at every point $x \in X$.

Definition 3 (Weak $(k, 2)$-continuity). [11] *For computer topological spaces (X, k, T_X^n) and $(Y, 2, T_Y)$, a function $f : X \to Y$ is said to be weakly $(k, 2)$-continuous at a point $x \in X$ if*
(1) $f : X \to Y$ is Khalimsky continuous at the point $x \in X$, and
(2) f is digitally $(k, 2)$-continuous at a point $x \in X$.
Further, we say that a map $f : X \to Y$ is weakly $(k, 2)$-continuous if the map f is weakly $(k, 2)$-continuous at every point $x \in X$.

Example 3.1. *For a Khalimsky continuous map $f : \mathbf{Z} \to \mathbf{Z}$, consider the new map $g(x) = f(x) + (2n + 1), n \in \mathbf{Z}$. Then the map g can* *not be a weakly $(2,2)$-continuous map. Meanwhile, if we take $h(x) = f(x) + 2n, n \in \mathbb{Z}$, then the map h is a weakly $(2,2)$-continuous map.*

Now we have the following in terms of (3-2).

Definition 4 ((k_0, k_1)-continuity). [2, 7] *For two spaces $(X, k_0, T_X^{n_0})$ and$(Y, k_1, T_Y^{n_1})$, a function $f : X \to Y$ is said to be (k_0, k_1)-continuous at a point $x \in X$ if for any $N_{k_1}^*(f(x), \varepsilon) \subset Y$, there is $N_{k_0}^*(x, \delta) \subset X$ such that*

$$f(N_{k_0}^*(x, \delta)) \subset N_{k_1}^*(f(x), \varepsilon).$$

Further, we say that a map $f : X \to Y$ is (k_0, k_1)-continuous if the map f is (k_0, k_1)-continuous at every point $x \in X$.

Remark 3.2. *The $(k, 2)$-continuity of a map $f : (X, k, T_X^n) \to (Y, 2, T_Y)$ in Definition 4 is equivalent to the following: for any point $x \in X$,*

$$f(N_k^*(x, r)) \subset N_2^*(f(x), 1),$$

where the number r is the least element of $\mathbb{N}$ such that $N_k^*(x, r)$ contains an open set including the point x.

Indeed, the current (k_0, k_1)-continuity can be used to develop essential tools in computer topology such as (k_0, k_1)-homotopy, (k_0, k_1)-homotopy equivalence, (k_0, k_1)-covering theory and so forth [7].

By Definitions 2 and 4 we obtain the following.

Definition 5 (Strong (k_0, k_1)-continuity). [11] *For two spaces $(X, k_0, T_X^{n_0})$ and $(Y, k_1, T_Y^{n_1})$, a function $f : X \to Y$ is said to be strongly (k_0, k_1)-continuous at a point $x \in X$ if*
(1) f is (Khalimsky) continuous at the point x; and
(2) f is (k_0, k_1)-continuous at the point x.
Further, we say that a map $f : X \to Y$ is strongly (k_0, k_1)-continuous if the map f is strongly (k_0, k_1)-continuous at every point $x \in X$.

Theorem 3.3. *For a map $f : (X, k, T_X^n) \to (Y, 2, T_Y)$, $(k, 2)$-continuity of f is irrelevant to weak $(k, 2)$-continuity of f.*

Proof: Consider the two spaces (X, k, T_X^n) and $(Y, 2, T_Y)$, where $X =: \{x_1, x_2\}$ in which x_1 is a mixed point and $x_2 \in N_k(x_1)$ is a pure open point, and $Y := \{y_1, y_2\}$, where y_1 is an even number and y_2 is an odd number such that $y_2 \in N_2(y_1)$. Then consider the map $f : X \to Y$ such that $f(x_1) = y_2$ and $f(x_2) = y_1$. Then we see that the map f is a $(k, 2)$-continuous map but it can not be weakly $(k, 2)$-continuous at the point x_1 because the map f can not be Khalimsky continuous at the point x_1.
Conversely, in case that there is no $N_{k_0}^*(x, r)$ in (X, k, T_X^n), the assertion that weak $(k, 2)$-continuity of f leads to $(k, 2)$-continuity of f can not be successful. More precisely, consider the map
$g : Z := \{a_i | i \in [0, 11]_{\mathbb{Z}}\} \to Y := \{0, 1, 2, 3\}$ in Fig.2(b) given by
$g(\{a_0, a_6, a_7, a_8, a_9, a_{10}, a_{11}\}) = \{0\}$,
$g(\{a_1, a_2\}) = \{1\}$, and $g(\{a_3, a_4, a_5\}) = \{2\}$.
Then, while the map g is weakly $(4, 2)$-continuous, it can not be $(4, 2)$-continuous at the point a_0 because there is only the 4-neighborhood of a_0, $N_4^*(a_0, 6) = Z$, which can not lead to $(4, 2)$-continuity of the map g at the point a_0. □

Indeed, strong $(k, 2)$-continuity of f leads to weak $(k, 2)$-continuity of f. Now, we investigate some relations between weak $(k, 2)$- and strong $(k, 2)$-continuities of a map $f : (X, k, T_X^n) \to (Y, 2, T_Y)$, $n \in \mathbb{N}$ (see Theorem 3.4 and Corollary 3.5).

Theorem 3.4. *Assume that (X, k, T_X^n) and $(Y, 2, T_Y)$ are k- and 2-connected, respectively. Let $f : (X, k, T_X^n) \to (Y, 2, T_Y)$ be a map, $n \in \mathbb{N}$. Then we obtain the following.*

(1) Weak $(k, 2)$-continuity of f implies strong $(k, 2)$-continuity of f if $k = 3^n - 1$.

(2) Weak $(k, 2)$-continuity of f implies strong $(k, 2)$-continuity of f with some hypothesis if $k \neq 3^n - 1$.

Proof: (1) If $n = 1$, then weak $(2, 2)$-continuity of f is easily equivalent to strong $(2, 2)$-continuity of f because for any point $x \in X$ and $f(x) \in Y$ we see that $N_2(x, 1) = N_2^*(x, 1)$ and further, $N_2(f(x), 1) = N_2^*(f(x), 1)$.

(2) In case that $n \geq 2$, in order to prove (1), (2), we need to consider a pair of distinct points $x_1, x_2 \in X$ such that x_1 and x_2 are k-adjacent, we investigate the following six cases according to the locations of both $x_1 \in X$ and $f(x_1) \in Y$ (see Fig.1).

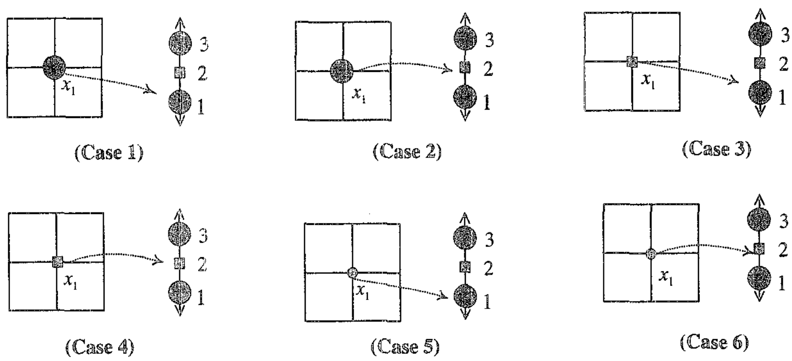

FIGURE 1

(Case 2-1) Assume that the point $x_1 \in X$ is a *pure open point* and $f(x_1) \in Y$ is *odd* in Y. Since both $\{x_1\}$ and $\{f(x_1)\}$ are the smallest open sets containing the points x_1 and $f(x_1)$, respectively, by the weak $(k, 2)$-continuity of f, we see that

$$f(N_k^*(x_1, 1)) = f(N_k(x_1, 1)) \subset N_2(f(x_1), 1) = N_2^*(f(x_1), 1),$$

which means that f is a strongly $(k, 2)$-continuous map at the point x_1.

(Case 2-2) Assume that x_1 is a *pure open point* and $f(x_1)$ is an *even* integer. By the weak $(k, 2)$-continuity of f at the point x_1 and Remark 3.2, take $N_2^*(f(x_1), 1) \subset (Y, 2, T_Y)$. Then, since $\{x_1\}$ is an open set, there is $N_k^*(x_1, 1) = N_k(x_1, 1) \subset X$ such that

$$f(N_k^*(x_1, 1)) = f(N_k(x_1, 1)) \subset N_2(f(x_1), 1) = N_2^*(f(x_1), 1),$$

which means that the map f is strongly $(k, 2)$-continuous at the point x_1.

(Case 2-3) Assume that x_1 is a *pure closed point* and $f(x_1)$ is *odd* in Y.

In case that $k = 3^n - 1$, the smallest open set containing the point x_1 is $N^*_{3^n-1}(x_1, 1) = N_{3^n-1}(x_1, 1)$ and further, $N^*_2(f(x_1), 1) = N_2(f(x_1), 1)$ because $\{f(x_1)\}$ is an open set. By the weak $(k, 2)$-continuity of f at the point x_1, we see that $f(N_{3^n-1}(x_1, 1)) = \{f(x_1)\}$. Consequently,

$$f(N^*_{3^n-1}(x_1, 1)) = f(N_{3^n-1}(x_1, 1)) \subset N_2(f(x_1), 1) = N^*_2(f(x_1), 1),$$

which implies that f is strongly $(k, 2)$-continuous at the point x_1.

In case that $k \neq 3^n - 1$, the assertion need not be true.

As a counterexample, consider the map $f : X \to Y$ in Fig.2(a) given by $f(\{a_0, a_1, a_2, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}\}) = \{1\}$, $f(\{a_4, a_8\}) = \{2\}$, and $f(\{a_5, a_6, a_7\}) = \{3\}$.

Then, while the map f is weakly $(4, 2)$-continuous at the point a_0, it can not be strongly $(4, 2)$-continuous at the point a_0. If not, suppose that the map f is a strongly $(4, 2)$-continuous at the point a_0. Then the minimal 4-neighborhood of of a_0 is the set $N^*_4(a_0, 6) = X - \{a_7, a_8, a_9\}$, so that $f(N^*_4(a_0, 6)) = \{0, 1, 2\} = N^*_2(f(a_0), 1)$, which contradicts to the weak $(4, 2)$-continuity of the map f at the point a_0.

(Case 2-4) Consider the case that x_1 is a *pure closed point* and $f(x_1)$ is an *even* integer.

In case that $k = 3^n - 1$, the smallest open set containing the point x_1 is $N^*_{3^n-1}(x_1, 1) = N_{3^n-1}(x_1, 1)$ and further, $N^*_2(f(x_1), 1) = N_2(f(x_1), 1)$. Due to the weak $(k, 2)$-continuity of f at the point x_1, we obtain the following.

$$f(N^*_{3^n-1}(x_1, 1)) = f(N_{3^n-1}(x_1, 1)) \subset N_2(f(x_1), 1) = N^*_2(f(x_1), 1),$$

which implies that f is strongly $(k, 2)$-continuous at the point x_1.

In case that $k \neq 3^n - 1$, the assertion need not be held.

As a counterexample, consider the map $g : Z \to Y := \{0, 1, 2, 3\}$ given by

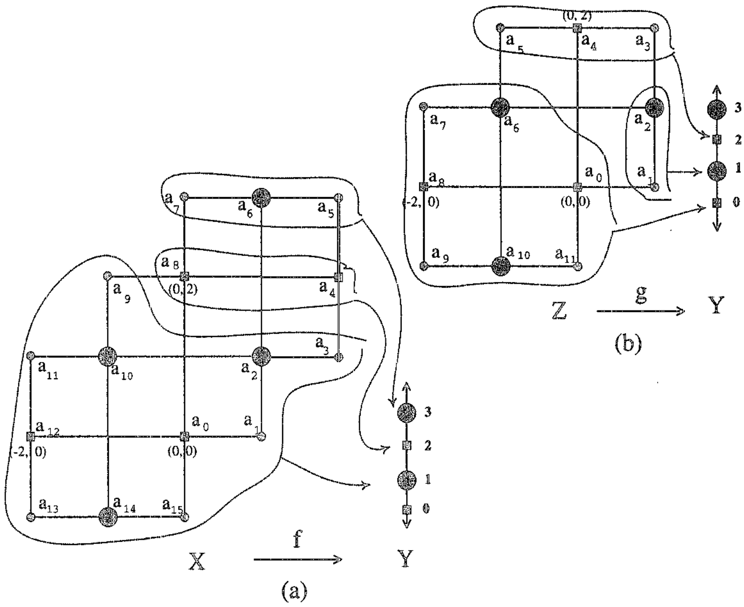

FIGURE 2

$g(\{a_0, a_6, a_7, a_8, a_9, a_{10}, a_{11}\}) = \{0\}$, $g(\{a_1, a_2\} = \{1\})$, and $g(\{a_3, a_4, a_5\}) = \{2\}$,where $Z := \{a_i | i \in [0, 11]_{\mathbb{Z}}\}$ and $Y := \{0, 1, 2, 3\}$.
Then, while the map g is weakly $(4, 2)$-continuous, g can not be strongly $(4, 2)$-continuous at the point a_0 because the minimal 4-neighborhood of a_0 is the set $X = N_4^*(a_0, 6)$, which contradicts to the weak $(4, 2)$-continuity of g at the point a_0.

(Case 2-5) Assume that x_1 is a *mixed point* and $f(x_1)$ is *odd* in Y. In case that $k = 3^n - 1$, the smallest open set containing the point x_1 is $N_{3^n-1}^*(x_1, 1) = N_{3^n-1}(x_1, 1)$ and further, $N_2^*(f(x_1), 1) = N_2(f(x_1), 1)$ because $\{f(x_1)\}$ is an open set. By the weak $(k, 2)$-continuity of f at the point x_1, we obtain the following

$$g(N_{3^n-1}^*(x_1, 1)) = g(N_{3^n-1}(x_1, 1)) \subset N_2(g(x_1), 1) = N_2^*(g(x_1), 1),$$

which implies that f is strongly $(k, 2)$-continuous at the point x_1.

In case that $k \notin \{3^n - 1, 3^n - 2^n - 1\}$, the assertion need not be true. As a counterexample, consider the map $f : X \to Y$ in Fig.3(a) given by

$$f(\{x_1, x_2\}) = \{3\}, f(\{x_3, x_4\}) = \{2\}.$$

Then, while the map f is weakly $(6, 2)$-continuous at the point x_1, it can not be strongly $(6, 2)$-continuous at the point x_1 because $T_X^3 = \{X, \phi, \{x_1, x_2\}, \{x_4\}\}$ so that there is no $N_6^*(x_1, \varepsilon) \subset X, \varepsilon \in \mathrm{N}$ for the strong $(6, 2)$-continuity of f.

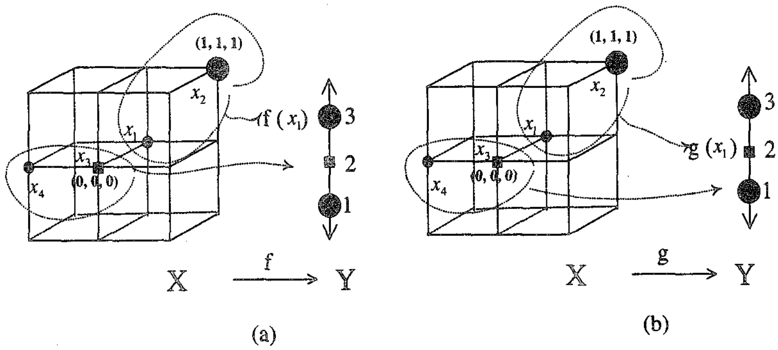

FIGURE 3

(Case 2-6) Assume that x_1 is a *mixed point* and $f(x_1)$ is an *even* integer.

In case that $k = 3^n - 1$, by the same method as the above and the weak $(k, 2)$-continuity of f we obtain the following.

$$f(N_{3^n-1}^*(x_1, 1)) = f(N_{3^n-1}(x_1, 1)) \subset N_2(f(x_1), 1) = N_2^*(f(x_1), 1),$$

which implies that f is strongly $(k, 2)$-continuous at the point x_1.

In case that $k \notin \{3^n - 1, 3^n - 2^n - 1\}$, the assertion need not be true. As an counterexample, consider the map $g : X \to Y$ in Fig.3(b) given by

$$g(\{x_1, x_2\}) = \{2\}, g(\{x_3, x_4\}) = \{1\}.$$

Then, while the map g is weakly $(6, 2)$-continuous at the point x_1, it can not be strongly $(6, 2)$-continuous at the point x_1 because there is no $N_6^*(x_1, \varepsilon) \subset X$ for any $\varepsilon \in \mathrm{N}$. □

By Theorem 3.4 we obtain the following because for a map $f : (X, k, T_X^n) \to (Y, 2, T_Y)$ if $k = 3^n - 1$, then strong $(k, 2)$-continuity of f implies weak $(k, 2)$-continuity of f.

Corollary 3.5. *For a map $f : (X, k, T_X^n) \to (Y, 2, T_Y)$ if $k = 3^n - 1$, then weak $(k, 2)$-continuity of f is equivalent to strong $(k, 2)$-continuity of f.*

4. Concluding remark

Several continuities have been studied in computer and digital topology, which are used in computer topological morphology, image processing, discrete geometry, digital topology, and computer science.

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