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ξ-PARALLEL STRUCTURE JACOBI OPERATORS OF REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

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Abstract. Let M be a real hypersurface with almost contact metric structure (ϕ, ξ, η, g) in a nonflat complex space form $M_n(c)$. In this paper, we prove that if the structure Jacobi operator R_ξ is ξ-parallel and the Ricci tensor S commutes with the structure operator ϕ, then a real hypersurface in $M_n(c)$ is a Hopf hypersurface. Further, we characterize such Hopf hypersurface in $M_n(c)$.

0. Introduction

An n-dimensional complex space form $M_n(c)$ is a Kähler manifold of constant holomorphic sectional curvature c. As is well known, complete and simply connected complex space form is isometric to a complex projective space $\mathbb{P}_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$ or a complex hyperbolic space $\mathbb{H}_n\mathbb{C}$ according as $c > 0, c = 0$ or $c < 0$.

Let M be a real hypersurface of $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J and the Kähler metric of $M_n(c)$. The characteristic vector ξ is said to be *principal* if $A\xi = \alpha\xi$, where A is the shape operator in the direction of the unit normal N and $\alpha = \eta(A\xi)$. A real hypersurface is said to a *Hopf hypersurface* if the characteristic vector ξ of M is principal.

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Typical examples of Hopf hypersurfaces in $P_n\mathbb{C}$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary group $PU(n+1)$. Takagi [14] completely classified such hypersurfaces as six model spaces which are said to be A_1, A_2, B, C, D and E.

On the other hand, real hypersurfaces in $H_n\mathbb{C}$ have been investigated by Berndt [1], Montiel and Romero [9] and so on. Berndt [1] classified Hopf hypersurfaces in a complex hyperbolic space whose all principal curvatures are constant as four model spaces which are said to be A_0, A_1, A_2 and B.

Let M be a real hypersurface of type A_1 or A_2 in a complex projective space $P_n\mathbb{C}$, or *type* A_0, A_1 or A_2 in a complex hyperbolic space $H_n\mathbb{C}$. Then M is said to be *type* A for simplicity. For example, Okumura [10](resp. Montiel and Romero [9]) showed that a real hypersurface in $P_n\mathbb{C}$(resp. $H_n\mathbb{C}$) is locally congruent to one of real hypersurfaces of type A if and only if the structure operator ϕ commutes with the shape operator A.

We donote by ∇, S and R_ξ be the Riemannian connection, the Ricci tensor and the structure Jacobi operator with respect to the chracteristic vector ξ of a real hypersurface M in $M_n(c)$ respectively (for detail see section 1). Then the classification of M with the commutativity condition $S\phi = \phi S$ is still open and very important problem.

Recently, in [11] the authors proved that there exist no real hypersurfaces in $P_n\mathbb{C}$, $n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$.

In a continuing work [13] they consider a weaker condition, called D-parallelness, that is $\nabla_V R_\xi = 0$ for any vector field V orthogonal to ξ. But, it was proved further that there exist no real hypersurfaces in $P_n\mathbb{C}$, $n \geq 3$ with D-parallel structure Jacobi operator. In this situation, it is naturally leads us to consider another weaker condition ξ-parallelness, that is $\nabla_\xi R_\xi = 0$. Along this direction we introduce a theorem due to [5] as follows:

Theorem CK ([5]). *Let M be a connected real hypersurface of $M_n(c), c \neq 0$ whose shape operator A commutes R_ξ, that is $R_\xi A = AR_\xi$. Then M satisfies $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following:*

(1) *In case that $M_n(c) = \mathbb{P}_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

(A_2) *a tube of radius r over a totally goedesic $\mathbb{P}_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$*

and $r \neq \pi/4$.

(2) *In case that $M_n(c) = \mathbb{H}_n\mathbb{C}$*

(A_0) *a horosphere;*

(A_1) *a geodesic hypersphere or a tube over complex hyperbolic hyperplane $\mathbb{H}_{n-1}\mathbb{C}$;*

(A_2) *a tube over a totally goedesic $\mathbb{H}_k\mathbb{C}$ $(1 \leq k \leq n-2)$.*

In this paper, we study a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $S\phi = \phi S$.

The main purpose of the present paper is to prove

Theorem . *Let M be a real hypersurface in a nonflat complex space form which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $\phi S = S\phi$. Then M is a Hopf hypersurface. Further, M is locally congruent to one of the following hypersurfaces:*

(1) *In case that $M_n(c) = \mathbb{P}_n\mathbb{C}$*

(A_1) *a tube of radius r over a hyperplane $\mathbb{P}_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{P}_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*

(T) *a tube of radius $\frac{\pi}{4}$ over a certain complex submanifold in $\mathbb{P}_n\mathbb{C}$,*

(2) *In case $M_n(c) = \mathbb{H}_n\mathbb{C}$*

(A_0) *a horosphere in $\mathbb{H}_n\mathbb{C}$, i.e., a Montiel tube,*

(A_1) *a geodesic hypersphere, or a tube of a hyperplane* $\mathbb{H}_{n-1}\mathbb{C}$,

(A_2) *a tube over a totally geodesic* $\mathbb{H}_k\mathbb{C}$ $(1 \leq k \leq n-2)$.

All manifolds in the present paper are assumed to be connected and of class $\mathcal{C}^\infty$ and the real hypersurfaces are supposed to be orientable.

1. Preliminaries

In this section elemental factors of a real hypersurface are recalled. Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X) N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where g denoted the Riemannian metric tensor of M induced from $\tilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X) N, \quad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we may see that M induces an almost contact metric structure (ϕ, ξ, η, g) that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M.

Since J is parallel, we verify from the Gauss and Weingarten formulas the following:

$$\nabla_X \xi = \phi AX, \tag{1.1}$$

$$(\nabla_X \phi)\, Y = \eta(Y)AX - g(AX,Y)\xi. \tag{1.2}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations respectively:

(1.3)
$$R\,(X,Y)\,Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y$$
$$-2g(\phi X,Y)\phi Z\} + g\,(AY,Z)\,AX - g(AX,Z)AY,$$

$$(\nabla_X A)\,Y - (\nabla_Y A)\,X = \frac{c}{4}\{\eta\,(X)\,\phi Y - \eta\,(Y)\,\phi X - 2g\,(\phi X,Y)\,\xi\} \tag{1.4}$$

for any vector fields X, Y and Z on M, where R denotes Riemannian curvature tensor of M.

In what follows, to write our formulas in convention forms, we denote by $\alpha = \eta\,(A\xi)\,, \beta = \eta\,(A^2\xi)\,, \mu^2 = \beta - \alpha^2\,,\ h = \mathrm{Tr}\ A$, and ∇f the gradient vector field of a function f defined on M. In the following, we use the same terminology and notation as above unless otherwise stated. We shall denote the Ricci tensor of type (1,1) by S. Then it follows from (1.3) that

$$S = \frac{c}{4}\{(2n+1)I - 3\eta\otimes\xi\} + hA - A^2, \tag{1.5}$$

where I is an identity map, which implies

$$S\xi = \frac{c}{2}(n-1)\xi + hA\xi - A^2\xi. \tag{1.6}$$

If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector field ξ. Using (1.1) we see that

$$\phi U = -A\xi + \alpha\xi, \tag{1.7}$$

which shows that $g(U,U) = \beta - \alpha^2$. By definition of U and (1.1) we verify that

$$g\left(\nabla_X \xi, U\right) = g\left(A^2\xi, X\right) - \alpha g(A\xi, X). \tag{1.8}$$

Now, differentiating (1.7) covariantly along M and making use of (1.1) and (1.2) we find

$$\begin{aligned} &\eta\left(X\right) g(AU + \nabla\alpha, Y) + g\left(\phi X, \nabla_Y U\right) \\ &= g\left(\left(\nabla_Y A\right) X, \xi\right) - g\left(A\phi AX, Y\right) + \alpha g(A\phi X, Y), \end{aligned} \tag{1.9}$$

which together with (1.4) gives

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{1.10}$$

From (1.9) we also have

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha, \tag{1.11}$$

where we have used (1.1) and (1.8).

If $A\xi - \eta(A\xi)\xi \neq 0$, we can put

$$A\xi = \alpha\xi + \mu W, \tag{1.12}$$

where W is a unit vector field orthogonal to ξ. Then from (1.7) it is clear that $U = \mu\phi W$ and hence $g(U,U) = \mu^2$, and W is also orthogonal to U. Using (1.1) we see that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{1.13}$$

From Gauss equation (1.3) we know that the structure Jacobi operator R_ξ is given by

$$(1.14)\qquad R_\xi X = R(X,\xi)\xi = \frac{c}{4}\{(X - \eta(X)\xi\} + \alpha AX - \eta\,(AX)\,A\xi$$

for any vector field X on M.

In what follows we assume that $\mu \neq 0$ on M, that is, ξ is not a principal curvature vector field and we put $\Omega = \{p \in M \mid \mu(p) \neq 0\}$. Suppose that Ω is not empty. Then Ω is an open subset of M, and from now on we discuss our arguments on Ω.

2. ξ-parallel structure Jacobi operator

Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ satisfying $\nabla_\xi R_\xi = 0$, which means that the structure Jacobi operator is ξ-parallel.

Differentiating (1.14) covariantly, we find

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) = &-\tfrac{c}{4}\{\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z) + (X\alpha)g(AY, Z)\} \\ &+ \alpha g((\nabla_X A)Y, Z) - g(A\xi, Z)\{g((\nabla_X A)\xi, Y) - g(A\phi AY, X)\} \\ &- g(A\xi, Y)\{g((\nabla_X A)\xi, Z) - g(A\phi AZ, X)\}. \end{aligned}$$

Putting $X = \xi$ in this and using (1.1) and (1.10), we get

$$\begin{aligned} g((\nabla_\xi R_\xi)Y, Z) = &-\tfrac{c}{4}\{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha)g(AY, Z) \\ &+ \alpha g((\nabla_\xi A)Y, Z) - g(A\xi, Z)\{3g(AU, Y) + Y\alpha\} \\ &- g(A\xi, Y)\{3g(AU, Z) + Z\alpha\}, \end{aligned}$$

where u is a 1-form by $u(X) = g(U, X)$ for any vector field X.

From the last equation and $\nabla_\xi R_\xi = 0$ we see that

$$\begin{aligned} (2.1)\quad \alpha(\nabla_\xi A)X + (\xi\alpha)AX = &-\frac{c}{4}\{u(X)\xi + \eta(X)U\} + \eta(AX)\{3AU + \nabla\alpha\} \\ &+ \{3g(AU, X) + X\alpha\}A\xi. \end{aligned}$$

Putting $X = \xi$ in this and using (1.10) we obtain

$$\alpha AU + \frac{c}{4}U = 0. \tag{2.2}$$

which tells us that $\alpha \neq 0$ on Ω.

Putting $X = \alpha U$ in (2.1) and making use of (2.2), we find

$$\alpha^2(\nabla_\xi A)U - \frac{c}{4}(\xi\alpha)U = \frac{c}{4}\alpha\mu^2\xi + \{\alpha(U\alpha) - \frac{c}{4}c\mu^2\}A\xi. \tag{2.3}$$

Because of (2.2), the equation (1.11) is reduced to

$$\alpha\nabla_\xi U = \frac{3}{4}c\mu W + \alpha^2 A\xi - \alpha\beta\xi + \alpha\phi\nabla\alpha. \tag{2.4}$$

Differentiating (2.2) covariantly along Ω, we find

$$(X\alpha)AU + \alpha(\nabla_X A)U + \alpha A(\nabla_X U) + \frac{c}{4}\nabla_X U = 0. \tag{2.5}$$

If we replace $X = \alpha\xi$ in this equation and take account of (2.2) and (2.3), we can obtain

$$\frac{c}{4}\alpha\mu^2\xi + \{\alpha(U\alpha) - \frac{3}{4}c\mu^2\}A\xi + \alpha^2 A(\nabla_\xi U) + \frac{c}{4}\alpha\nabla_\xi U = 0.$$

which together with (2.4) implies that

$$\alpha A\phi\nabla\alpha + \frac{c}{4}\phi\nabla\alpha + (U\alpha)A\xi + \mu(\alpha^2 + \frac{3}{4}c)\{AW - \mu\xi - \frac{1}{\alpha}(\mu^2 - \frac{c}{4})W\} = 0, \tag{2.6}$$

where we have used (1.12).

Using (1.4) and (1.7), we verify from (2.5) that

$$\begin{aligned} &\frac{c}{4}\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + \frac{c}{4}\alpha^2\mu\{\eta(X)w(Y) - \eta(Y)w(X)\} \\ &+ \alpha^2\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + \frac{c}{4}\alpha du(X,Y) = 0, \end{aligned} \tag{2.7}$$

where w is a 1-form defind by $w(X) = g(W, X)$, and the exterior derivative du of 1-form u is given by

$$du(X,Y)=\frac{1}{2}\{Yu(X)-Xu(Y)-u([X,Y])\}.$$

If we replace X by U in (2.7), then obtain

$$\frac{c}{4}(\mu^2\nabla\alpha-(U\alpha)U)+\alpha^2A\nabla_UU+\frac{c}{4}\alpha\nabla_UU=0, \tag{2.8}$$

because U and W are mutually orthogonal.

Combining (1.9) to (2.1) and using the Codazzi equation (1.4) and (2.2), we obtain

$$\begin{aligned}\alpha^2\phi\nabla_XU=&-\alpha^2(X\alpha)\xi+\tfrac{c}{4}\alpha u(X)\xi-\alpha(\xi\alpha)AX-\tfrac{c}{4}\alpha^2\phi X\\&+g(A\xi,X)(\alpha\nabla\alpha-\tfrac{3}{4}cU)+(\alpha(X\alpha)-\tfrac{3}{4}cu(X))A\xi\\&+\tfrac{c}{4}(u(X)\xi+\eta(X)U)-\alpha^2A\phi AX+\alpha^3A\phi AX.\end{aligned}$$

Applying this by ϕ and making use of (1.8) and (1.12), we have

$$\begin{aligned}&\alpha^2\nabla_XU+\alpha^2g(AW,X)\xi-\alpha g(A\xi,X)\phi\nabla\alpha\\&=\alpha(\xi\alpha)\phi AX+\tfrac{c}{4}\alpha^2(X-\eta(X)\xi)+\tfrac{3}{4}c\mu g(A\xi,X)W+\alpha(X\alpha)U\\&-\tfrac{3}{4}cu(X)U+\alpha^3AX-\tfrac{3}{4}\alpha\mu\eta(X)W-\alpha^3\eta(X)A\xi-\alpha^2\phi A\phi AX.\end{aligned} \tag{2.9}$$

Putting $X=U$ in (2.9) and taking account of (1.7), (1.12) and (2.2), we verify that

$$\alpha^2\nabla_UU=-\frac{c}{4}\mu(\xi\alpha)W+\{\alpha(U\alpha)-\frac{3}{4}c\mu^2\}U+\frac{c}{4}\mu\alpha\phi AW. \tag{2.10}$$

Substituting (2.10) into (2.8) and taking account of (2.2), we verify that

$$\alpha\mu^2\nabla\alpha-\alpha(U\alpha)U=\mu(\xi\alpha)(\alpha AW+\frac{c}{4}W)-\alpha\mu\{\alpha A\phi AW+\frac{c}{4}\phi AW\}. \tag{2.11}$$

Using (2.2), the equation (2.1) can be rewritten as

$$\begin{aligned}\alpha^2(\nabla_\xi A)X=&-\alpha(\xi\alpha)AX+\frac{c}{4}\alpha\{u(X)\xi+\eta(X)U\}\\&+\{\alpha(X\alpha)-\tfrac{3}{4}cu(X)\}A\xi+(\alpha\nabla\alpha-\tfrac{3}{4}cU)g(A\xi,X).\end{aligned} \tag{2.12}$$

3. Real hypersurfaces satisfying $S\phi=\phi S$

Let M be a real hypersurface in $M_n(c)$, $c \neq 0$ satisfying $S\phi = \phi S$. Then from (1.5) we have

$$A^2\phi - \phi A^2 = h(A\phi - \phi A), \tag{3.1}$$

which enables us to obtain $\phi(A^2\xi - hA\xi) = 0$. Because of properties of the almost contact metric structure, it follows from this that

$$A^2\xi = hA\xi + (\beta - h\alpha)\xi. \tag{3.2}$$

From (1.12) and (3.2), we see that

$$AW = \mu\xi + (h - \alpha)W \tag{3.3}$$

and hence

$$A^2W = hAW + (\beta - h\alpha)W \tag{3.4}$$

because of $\mu \neq 0$.

Now, differentiating (3.3) covariantly along Ω, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(h-\alpha)W + (h-\alpha)\nabla_X W. \tag{3.5}$$

If we take a inner product with W in the last equation, then we find

$$g((\nabla_X A)W, W) = -2g(AU, X) + Xh - X\alpha \tag{3.6}$$

since W is a unit vector field orthogonal to ξ. We also obtain by applying ξ to this,

$$\mu g((\nabla_X A)W, \xi) = (h - 2\alpha)(AU, X) + \mu(X\mu),$$

where we have used (1.8) and (3.2), which together with (1.4) implies that

$$\mu(\nabla_\xi A)W = (h - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu. \tag{3.7}$$

Replacing X by ξ in (3.5) and making use of (3.7), we find

$$\begin{aligned}(h - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu + \mu\{A\nabla_\xi W - (h - \alpha)\nabla_\xi W\} \\ = \mu(\xi\mu)\xi + \mu^2 U + \mu(\xi h - \xi\alpha)W.\end{aligned} \tag{3.8}$$

On the other hand, from the fact that $\phi U = -\mu W$ we see that

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W.$$

If we replace X by ξ in this and take account of (1.11) and (1.12), then we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W. \tag{3.9}$$

Combining this to (3.8), we verify that

$$\begin{aligned} 3A^2U - 2hAU + A\nabla\alpha + \frac{1}{2}\nabla\beta - h\nabla\alpha + (\alpha h - \beta - \frac{c}{4})U \\ = 2\mu(W\alpha)\xi + \mu(\xi h)W - (h - 2\alpha)(\xi\alpha)\xi, \end{aligned} \tag{3.10}$$

which shows that

$$\xi\beta = 2\alpha(\xi\alpha) + 2\mu(W\alpha). \tag{3.11}$$

From (3.6) and (3.7) it is seen that

$$W\mu = \xi h - \xi\alpha. \tag{3.12}$$

Differentiating (3.2) covariantly and using (1.1), we find

$$\begin{aligned} &(\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX - hA\phi AX \\ &= (Xh)A\xi + h(\nabla_X A)\xi + X(\beta - h\alpha)\xi + (\beta - h\alpha)\phi AX, \end{aligned}$$

which together with the Codazzi equation (1.4) yields

$$\begin{aligned} &\tfrac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y)\} + \tfrac{c}{2}(h-\alpha)g(\phi Y, X) - g(A^2\phi AX, Y) \\ &+ g(A^2\phi AY, X) + 2hg(\phi AX, AY) - (\beta - h\alpha)\{g(\phi AY, X) - g(\phi AX, Y)\} \\ &= g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + (Yh)\eta(AX) - (Xh)\eta(AY) \\ &+ Y(\beta - h\alpha)\eta(X) - X(\beta - h\alpha)\eta(Y). \end{aligned}$$

Replacing X by μW to the both sides of the last equation and using (1.4), (1.10), (3.3), (3.4) and (3.7), we obtain (for detail, see [6])

$$\begin{aligned} (3\alpha - 2h)A^2U + 2(h^2 + \beta - 2h\alpha + \frac{c}{4})AU + (h-\alpha)(\beta - h\alpha - \frac{c}{2})U \\ = \mu A\nabla\mu + (\alpha h - \beta)\nabla\alpha - \tfrac{1}{2}(h-\alpha)\nabla\beta + \mu^2\nabla h \\ - \mu(Wh)A\xi - \mu W(\beta - h\alpha)\xi. \end{aligned} \tag{3.13}$$

4. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$ and $S\phi = \phi S$

We will continue our arguments under the assumptions $\nabla_\xi R_\xi = 0$ and at the same time $S\phi = \phi S$ on real hypersurfaces in $M_n(c)$, $c \neq 0$. Then (2.6) and (2.11) turns out respectively to be

$$(4.1)\quad \alpha A\phi\nabla\alpha + \frac{c}{4}\phi\nabla\alpha + (U\alpha)A\xi + \frac{1}{\alpha}\mu(\alpha^2 + \frac{3}{4}c)(h\alpha + \frac{c}{4} - \beta)W = 0,$$

$$(4.2)\qquad \alpha\mu\nabla\alpha = \frac{\alpha}{\mu}(U\alpha)U + \alpha\mu(\xi\alpha)\xi + (h\alpha - \alpha^2 + \frac{c}{4})(\xi\alpha)W$$

by virtue of (2.2) and (2.3). If we take a inner product (4.1) and (4.2) with W and make use of (3.3), then we have respectively

$$(4.3)\qquad (\beta - h\alpha - \frac{c}{4})\{\alpha(U\alpha) - \mu^2(\alpha^2 + \frac{3}{4}c)\} = 0,$$

$$(4.4)\qquad \mu\alpha(W\alpha) = (h\alpha - \alpha^2 + \frac{c}{4})\xi\alpha.$$

Now, taking a inner product $\alpha^2 U$ to (3.1) and using (2.2) and (3.4), we find

$$(4.5)\qquad \alpha^2\sigma = \frac{c}{4}(\beta - \sigma + \frac{c}{4}),$$

where we have put

$$(4.6)\qquad \sigma = \beta - h\alpha.$$

Combining (4.3) to the last two equations, we verify that

$$(4.7)\qquad \alpha(U\alpha) = \mu^2(\alpha^2 + \frac{3}{4}).$$

Using (4.4) and (4.7), the equation (4.2) turns out to be

$$(4.8)\qquad \alpha\nabla\alpha = \alpha(\xi\alpha)\xi + \alpha(W\alpha)W + (\alpha^2 + \frac{3}{4}c)U.$$

Putting $X = \mu W$ in (2.12) and taking account of (2.2) and (3.7)

$$\alpha\{\frac{1}{2}\alpha\nabla\beta - \beta\nabla\alpha\} + \frac{c}{4}(3\beta - 2\alpha^2 - h\alpha)U = -\mu\alpha(\xi\alpha)AW + \mu\alpha(W\alpha)A\xi,$$

where we have used $\mu^2 = \beta - \alpha^2$, or using (1.12), (3.3), (4.4) and (4.6),

$$\alpha^2 \nabla\beta - \beta\nabla\alpha^2 + \frac{c}{2}(2\mu^2 + \sigma)U = (\xi\alpha)\{\frac{c}{2}A\xi - 2\alpha\sigma\xi\}, \tag{4.9}$$

which together with (4.7) gives

$$\frac{1}{2}\alpha(U\beta) = \{\beta\alpha + \frac{c}{4}(h + 2\alpha)\}\mu^2 \tag{4.10}$$

On the other hand, we have from (2.12)

$$\alpha^2(\xi h) = -\alpha h(\xi\alpha) + 2\alpha g(A\xi, \nabla\alpha),$$

which together with (1.12) and (4.4) yields

$$\alpha^2(\xi h) = (h\alpha + \frac{c}{2})\xi\alpha. \tag{4.11}$$

From (3.11) and (4.4) we also have

$$\frac{1}{2}\alpha(\xi\beta) = (h\alpha + \frac{c}{4})\xi\alpha. \tag{4.12}$$

Combining the last two equations, and using (4.6), we see that

$$\xi\sigma = 0. \tag{4.13}$$

If we differentiate (4.5), then we have

$$(\alpha^2 + \frac{c}{4})\nabla\sigma = \frac{c}{4}\nabla\beta - 2\alpha\sigma\nabla\alpha,$$

which together with (4.5), (4.8) and (4.9) implies that

$$\alpha^2(\alpha^2 + \frac{c}{4})\nabla\sigma = \frac{c}{2}\{\alpha(\sigma - \frac{c}{4})W\alpha + \frac{c}{4}\mu(\xi\alpha)\}W + \frac{c^2}{8}(\sigma - \mu^2 - \frac{c}{2})U. \tag{4.14}$$

Thus, it follows that

$$\alpha^2(\alpha^2 + \frac{c}{4})U\sigma = \frac{c^2}{8}(\sigma - \mu^2 - \frac{c}{2})\mu^2. \tag{4.15}$$

Because of (4.7) and (4.10), it is seen that

$$\alpha\mu(U\mu) = \{\alpha\mu^2 + \frac{c}{4}(h - \alpha)\}\mu^2, \tag{4.16}$$

which tells us that

$$\alpha^2 g(\mu A\nabla\mu, U) = -\frac{c}{4}\alpha\mu^2 + \frac{c}{4}(h - \alpha).$$

If we take a inner product $\alpha^2 U$ to (3.13) and making use of (2.2), (4.5), (4.7), (4.10) and the last equation, we find

$$\alpha(Uh) = \beta\alpha(h-\alpha) + \frac{c}{4}(2\beta - \alpha^2). \tag{4.17}$$

First of all we prove

Lemma 1. $\xi\alpha = 0$ and $W\alpha = 0$ on Ω.

Proof. From (4.6) we obtain

$$\alpha^2(U\sigma) = \alpha^2(U\beta) - h\alpha^2(U\alpha) - \alpha^3(Uh).$$

Substituting (4.7), (4.10) and (4.17) into this, we get

$$\alpha^2(U\sigma) = \frac{c}{4}\{(2\alpha^2 + \frac{c}{4})\mu^2 + \beta(\beta - \sigma + \frac{c}{4}) - \alpha^4\},$$

where we have used (4.5) and $\mu^2 = \beta - \alpha^2$. From this and (4.15) we see that

$$(\alpha^2 + \frac{c}{4})\{(2\alpha^2 + \frac{c}{4})\mu^2 + \beta(\beta - \sigma + \frac{c}{4}) - \alpha^4\} = \frac{c}{2}(\sigma - \mu^2 - \frac{c}{2}),$$

or, using (4.5),

$$\alpha^2(\beta + 3\alpha^2) + c\alpha^2 + (\frac{c}{4})^2 + \frac{c}{2} = 0.$$

Differentiating this and taking account of (4.12), we find

$$\{2\beta + 6\alpha^2 + \frac{5}{4}c - \sigma\}\xi\alpha = 0.$$

From the last two equations, it is, using (4.13), verified that $\xi\alpha = 0$. From this and (4.4) it follows that $W\alpha = 0$. This completes the proof. ∎

The proof of Main Theorem.

According to Lemma 1, (4.8) and (4.9) are reduced respectively to

$$\frac{1}{2}\nabla\alpha^2 = (\alpha^2 + \frac{3}{4}c)U, \tag{4.18}$$

$$\alpha^2\nabla\beta - 2\alpha\beta\nabla\alpha + \frac{c}{2}(2\mu^2 + \sigma)U = 0. \tag{4.19}$$

Combining the last two equations, it follows that

$$\alpha\nabla\beta = \{2\beta\alpha + \frac{c}{2}(2\alpha + h)\}U. \tag{4.20}$$

Differentiating (4.18) covariantly and taking the skew-symmetric parts obtained, we find $(\alpha^2 + \frac{3}{4}c)du(X,Y) = 0$ for any vector fields X and Y. From this we verify that

$$du(X,Y) = 0. \tag{4.21}$$

In fact, if not, then we obtain $\alpha^2 + \frac{3}{4}c = 0$ on this subset. So we have

$$\nabla\alpha = 0, \quad 2\beta + \sigma + \frac{c}{2} = 0 \tag{4.22}$$

on the set by virtue of (4.5). Further, (4.19) is reformed as $3\nabla\beta = 2(2\mu^2 + \sigma)U$ on the set, which tells us that $2\mu^2 + \sigma = 0$. From this and the second equation of (4.22), it is seen that $\alpha^2 + \frac{c}{4} = 0$, a contradiction. Thus, (4.21) is accomplished everywhere on Ω.

From (4.21) we have $g(\nabla_\xi U, X) + g(\nabla_X \xi, U) = 0$, which together with (1.1), (1.8), (1.11), (1.12), (2.2) and (3.3) implies that $h = \alpha$. Accordingly (4.5) and (4.20) turn out respectivvely to

$$\alpha^2\mu^2 = \frac{c}{4}(\alpha^2 + \frac{c}{4}), \tag{4.23}$$

$$\frac{1}{2}\nabla\beta = (\beta + \frac{3}{4}c)U, \tag{4.24}$$

which together with (4.18) gives

$$\nabla\mu = \mu U. \tag{4.25}$$

Differentiating (4.23) along Ω, and using (4.18) and (4.25), we obtain $\alpha^2 + 3\mu^2 - \frac{c}{4} = 0$, which together with (4.23) gives $\alpha^4 + \frac{c}{2}\alpha^2 + \frac{3}{16}c^2 = 0$ and hence $\nabla\alpha = 0$. This together with (4.18) yields $\alpha^2 + \frac{3}{4}c = 0$, a contradiction. Therefore we conclude that $\Omega = \phi$. Accordingly we see that the subset Ω in M on which $A\xi - \eta(A\xi)\xi \neq 0$ is an empty set. Namely, in $M_n(c)$, $c \neq 0$, every real hypersurface satisfying $\nabla_\xi R_\xi = 0$ and $S\phi = \phi S$ is a Hopf hypersurface. Hence we have $U = 0$ and furthermore, the function α is constant on M ([7]).

Thus, (2.1) is led to $\alpha \nabla_\xi A = 0$, which together (1.4) and (1.9) implies that $\alpha(A\phi - \phi A) = 0$. Here, we note that the case $\alpha = 0$ corresponds to the case of tube of radius $\frac{\pi}{4}$ in $\mathbb{P}_n\mathbb{C}$ (See [2]).

But, in the case of $\mathbb{H}_n\mathbb{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf [1]). Owing to Okumura's work for $\mathbb{P}_n\mathbb{C}$ or Montiel and Romero's work for $\mathbb{H}_n\mathbb{C}$ mentioned in Introduction, we have completed the proof main theorem.

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