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MEAN CURVATURE OF NON-DEGENERATE SECOND FUNDAMENTAL FORM OF RULED SURFACES

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Abstract. In this paper, we classify non-developable ruled surfaces in a Euclidean 3-spaces satisfying some algebraic equations in terms of the second mean curvature, the mean curvature and the Gaussian curvature.

1. Introduction

The inner geometry of the second fundamental form has been a popular research topic for ages. It is readily seen that the second fundamental form of a surface is non-degenerate if and only if a surface is non-developable. On a non-developable surface M, we can regard the second fundamental form II of a surface M as a new Riemannian metric or pseudo-Riemannian metric on the Riemannian or pseudo-Riemannian manifold (M, II). In this case, we can define the Gaussian curvature and mean curvature of non-degenerate second fundamental form, denoted by K_{II} and H_{II} respectively, these are nothing but the Gaussian curvature and mean curvature of (M, II). By Briosch's formula in a Euclidean

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3-space $\mathbb{R}^3$ (cf.[13]) we are able to compute K_{II} of M by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form e, f, g, respectively (cf.[1],[2],[3],[5] etc). The curvature K_{II} is called the second Gaussian curvature.

On the other hand, the mean curvature H_{II} of non-degenerate second fundamental form is defined by ([4, pp.196-197])

$$H_{II} = H - \frac{1}{2}\Delta_{II}\sqrt{|K|}, \tag{1.1}$$

where K and H are the Gaussian and mean curvatures respectively, and Δ_{II} denotes the Laplacian operator of second fundamental form, that is,

$$\Delta_{II} = -\frac{1}{\sqrt{|h|}}\sum_{i,j}^{2}\frac{\partial}{\partial x^i}\left(\sqrt{|h|}h^{ij}\frac{\partial}{\partial x^j}\right), \tag{1.2}$$

where $e = h_{11}, f = h_{12}, g = h_{22}, h = \det(h_{ij}), (h^{ij}) = (h_{ij})^{-1}$ and $\{x_i\}$ is rectangular coordinate system in $\mathbb{R}^3$. The curvature H_{II} is said to be the second mean curvature.

For the study of the curvatures, D. Koutroufiotis([8]) has shown that a closed ovaloid is a sphere if $K_{II} = cK$ for some constant c or if $K_{II} = \sqrt{K}$, where K is the Gaussian curvature. Th. Koufogiorgos and T. Hasanis([7]) proved that the sphere is the only closed ovaloid satisfying $K_{II} = H$, where H is the mean curvature. Also, W. Kühnel([9]) studied surfaces of revolution satisfying $K_{II} = H$. One of the natural generalizations of surfaces of revolution is the helicoidal surfaces. In [1] C. Baikoussis and Th. Koufogiorgos proved that the helicoidal surfaces satisfying $K_{II} = H$ are locally characterized by constancy of the ratio of the principal curvatures. On the other hand, D. E. Blair and Th. Koufogiorgos ([2]) investigated a non-developable ruled surface in a Euclidean 3-space $\mathbb{R}^3$ satisfying the condition

$$aK_{II} + bH = \text{constant}, \quad 2a + b \neq 0, \tag{1.3}$$

along each ruling, and the second author ([14]) studied a non-developable ruled surface in a Euclidean 3-space $\mathbb{R}^3$ satisfying the conditions

$$aH + bK = \text{constant}, \quad a \neq 0, \tag{1.4}$$

$$aK_{II} + bK = \text{constant}, \quad a \neq 0, \tag{1.5}$$

along each ruling.

On the other hand, Y. H. Kim and the second author ([5]) extended ones to the Lorentz version of (1.3), (1.4) and (1.5). In [11] W. Sodsiri studied a non-developable ruled surface in $\mathbb{L}^3$ with non-null rulings such that the linear combination $aK_{II} + bH + cK$ is constant along ruling. G. Stamou([12]) classified non-developable ruled surface in a Euclidean 3-space on which the linear combination $aK_{II} + bH + cH_{II}$ is constant along each ruling, and F. Dillen and W. Sodsiri([3]) extended it to the Lorentz version. Recently, Y. H. Kim and the second author([6]) classified non-developable ruled surface in a Lorentz-Minkowski 3-space satisfying the equations

$$\begin{aligned} aH^2 + bHK_{II} + cK_{II}^2 &= d, \\ aK^2 + bKK_{II} + cK_{II}^2 &= d, \end{aligned} \tag{1.6}$$

where a, b, c, d are real numbers.

In this article, we investigate non-developable ruled surfaces in a Euclidean 3-space $\mathbb{R}^3$ satisfying the equations

$$aH^2 + bHH_{II} + cH_{II}^2 + dH + eH_{II} = k, \tag{1.7}$$

$$aK^2 + bKH_{II} + cH_{II}^2 + dK + eH_{II} = k, \tag{1.8}$$

along each ruling, where a, b, c, d, e, k are real numbers. If a surface satisfies the equations (1.7) and (1.8), then a surface is said to be HH_{II}-quadrics and KH_{II}-quadrics, respectively.

2. Main Results

In this section we study ruled HH_{II}-quadric surfaces and KH_{II}-quadric surfaces in a Euclidean 3-space $\mathbb{R}^3$.

Let M be a non-developable ruled surface in $\mathbb{R}^3$. Then the parametrization for M is given by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

where $\langle\beta,\beta\rangle = 1, \langle\beta',\beta'\rangle = 1$ and $\langle\alpha',\beta'\rangle = 0$. In this case α is the striction curve of x, and the parameter is the arc-length on the spherical curve β. And we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha' + t\beta'$ and $x_t = \beta$. Then, the components of the first fundamental form are given by

$$E = \langle\alpha',\alpha'\rangle + t^2, \ F = \langle\alpha',\beta\rangle, \ G = 1.$$

We put $D = \sqrt{EG - F^2}$. In terms of the orthonormal basis $\{\beta,\beta',\beta\times\beta'\}$ we obtain

$$\alpha' = F\beta + Q\beta\times\beta', \tag{2.1}$$

$$\beta'' = -\beta - J\beta\times\beta', \tag{2.2}$$

$$\alpha'\times\beta = Q\beta', \tag{2.3}$$

where $Q = \langle\alpha',\beta\times\beta'\rangle \neq 0, \quad J = \langle\beta'',\beta'\times\beta\rangle$. Thus, we get

$$D = \sqrt{Q^2 + t^2}, \tag{2.4}$$

from which the unit normal vector N is written as

$$N = \frac{1}{D}(\alpha'\times\beta + t\beta'\times\beta) = \frac{1}{D}(Q\beta' - t\beta\times\beta').$$

This leads to the components e, f and g of the second fundamental form

$$e = \frac{1}{D}(Q(F + QJ) - Q't + Jt^2), \ f = \frac{Q}{D} \neq 0, \ g = 0.$$

If we make use of (1.2) together with the functions D, Q and J, the Laplacian Δ_{II} of the second fundamental form II can be expressed as follows :

(2.5)

$$\Delta_{II} = -\frac{2D}{Q}\frac{\partial^2}{\partial s \partial t} + \frac{D}{Q^2}(2Jt - Q')\frac{\partial}{\partial t} + \frac{D}{Q^2}(Jt^2 - Q't + QF + Q^2J)\frac{\partial^2}{\partial t^2}.$$

Therefore, using the data described above, the mean curvature H, the Gaussian curvature K and the second mean curvature H_{II} are given respectively by

$$H = \frac{1}{2}\frac{Eg - 2Ff + Ge}{EG - F^2} = \frac{1}{2D^3}A, \tag{2.6}$$

$$K = \frac{eg - f^2}{EG - F^2} = -\frac{Q^2}{D^4} \tag{2.7}$$

and

$$H_{II} = \frac{1}{2Q^2D^3}B \tag{2.8}$$

where

$$\begin{aligned} A &= Jt^2 - Q't + Q(QJ - F), \\ B &= 2Jt^4 + (5Q^2J - 2QF)t^2 + 3Q^2Q't + Q^3F + 3Q^4J. \end{aligned} \tag{2.9}$$

Suppose that a non-developable ruled surface is HH_{II}-quadric. Then by (1.7), (2.6) and (2.7) we have

$$(aQ^4A^2 + bQ^2AB + cB^2 - 4kQ^4D^6)^2 = (-2dQ^4A - 2eQ^2B)^2D^6. \tag{2.10}$$

From (2.4) and (2.9) the equation (2.10) becomes the polynomial with the variable t whose coefficients are functions of variable s. Then, by the coefficient of the highest order t^{16}, we have

$$16c^2J^4 = 0,$$

from which $J = 0$ because of $c \neq 0$. Therefore, we can rewrite (2.9) in the form

$$
\begin{aligned}
A &= -Q't - QF, \\
B &= -2QFt^2 + 3Q^2Q't + Q^3F.
\end{aligned} \tag{2.11}
$$

By (2.11) and the coefficient of t^{12} of (2.10), we have

$$16k^2Q^8 = 0,$$

from which $k = 0$. From (2.11) and the coefficient of t^{10} of (2.10) we have

$$16e^2Q^6F^2 = 0,$$

which implies $F = 0$ because $e \neq 0$. In this case, we can also obtain the following:

$$4(d-3e)^2Q^8Q'^2 = 0.$$

If $d - 3e \neq 0$, $Q' = 0$. Thus, from (2.6) a surface M is minimal, that is, a helicoid.

If $d - 3e = 0$, Q' is arbitrary function and from (2.6) and (2.8) $H_{II} = -3H$. In this case, without loss of generality, we may assume $\beta(0) = (1, 0, 0)$. Then, by (2.2) $\beta'' = -\beta$ implies

$$\beta(s) = (d_1 \sin s, d_2 \sin s, \cos s + d_3 \sin s)$$

for some constants d_1, d_2, d_3 satisfying $d_1^2 + d_2^2 + d_3^2 = 1$. Since $\langle \beta, \beta \rangle = 1$, we have $d_1^2 + d_2^2 = 1$ and $d_3 = 0$. From this we can obtain

$$\beta(s) = (d_1 \sin s, \pm\sqrt{1-d_1^2} \sin s, \cos s),$$

where $-1 \leq d_1 \leq 1$. On the other hand, by (2.1) we have

$$\alpha(s) = \left(\mp\sqrt{1-d_1^2}, d_1, 0\right) f(s) + \mathbb{C},$$

where $f(s) = \int^s Q(u)du$ and $\mathbb{C} = (c_1, c_2, c_3)$ is a constant vector. Thus, the surface M has the parametrization of the form

(2.12)

$$x(s,t) = \left(\mp\sqrt{1-d_1^2}f(s) + td_1 \sin s + c_1,\right.$$
$$\left. d_1 f(s) \mp t\sqrt{1-d_1^2}\sin s + c_2, t\cos s + c_3\right),$$

where $-1 \leq d_1 \leq 1$, $f(s) = \int^s Q(u)du$ and $\mathbb{C} = (c_1, c_2, c_3)$ is a constant vector in $\mathbb{R}^3$.

Thus, we have

Theorem 2.1. *Let M be a non-developable ruled surface in a Euclidean 3-space, and a, b, c, d, e, k be constants such that $c^2 + e^2 \neq 0$. Suppose that M satisfies $aH^2 + bHH_{II} + cH_{II}^2 + dH + eH_{II} = k$ along each ruling of M. Then M satisfies the following properties:*

1. *If $d - 3e \neq 0$, then M is a helicoid.*
2. *If $d - 3e = 0$, then M is given by (2.12).*

Furthermore, the surface M satisfies the equation $H_{II} = -3H$.

Remark 1. 1. For specific function $f(s)$ and appropriate intervals of s and t in (2.12), we have the graph shown in Figure 1.

2. If $d_1 = 0$ or $d_1 = \pm 1$, then the surface M is a right conoid. Therefore, a right conoid satisfies the equation $H_{II} = -3H$.

Remark 2. On a non-developable ruled surface in a Euclidean 3-space,

1. $H_{II} = 0$ if and only if $H = 0$.
2. $H_{II} = H$ if and only if $H_{II} = H = 0$.

Theorem 2.2. *Let M be a non-developable ruled surface in a Euclidean 3-space, and a, b, c, d, e, k be constants such that $c^2 + e^2 \neq 0$. Suppose that M satisfies $aK^2 + bKH_{II} + cH_{II}^2 + dK + eH_{II} = k$ along each ruling of M. Then M is a helicoid.*

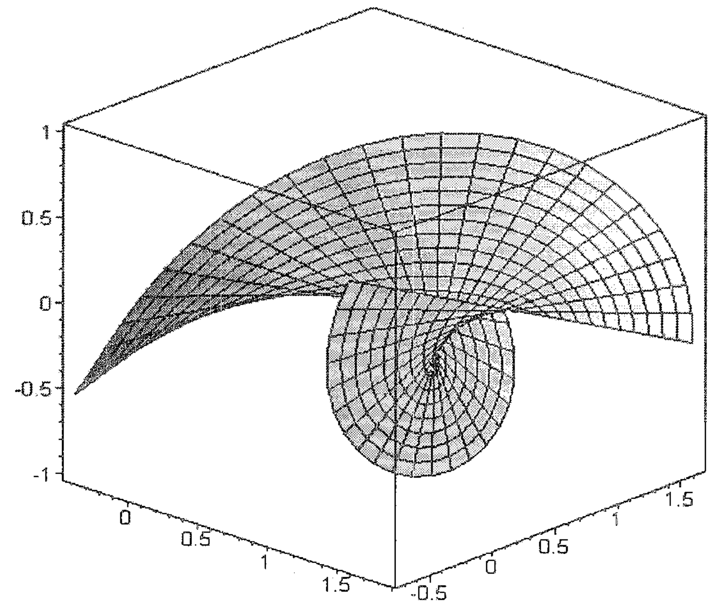

FIGURE 1. $Q(s) = \frac{1}{s}$, $d_1 = \frac{1}{2}$, $-1 \le t \le 1$, $1 \le s \le 5$

Proof. Let M be a non-developable ruled surface in $\mathbb{R}^3$. Then the parametrization for M is given by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

where $\langle \beta, \beta \rangle = 1$, $\langle \beta', \beta' \rangle = 1$ and $\langle \alpha', \beta' \rangle = 0$.

Suppose that a non-developable ruled surface is KH_{II}-quadric. Then, by using (2.7) and (2.8) the equation (1.8) implies

$$(2.13)\quad (4aQ^8 + cB^2D^2 - 4dQ^6D^4 - 4kQ^4D^8)^2 = 4Q^4B^2(bQ^2 - eD^4)^2D^2.$$

From (2.4) and the second equation of (2.9) the equation (2.13) becomes the polynomial with the variable t whose coefficients are functions of variable s. Then, by the coefficient of the highest order t^{20}, we have

$$16c^2J^4 = 0,$$

from which $J = 0$ because $c \neq 0$. In this case we can also obtain $k = 0$. Furthermore, by the coefficient of t^{14} of the equation (2.13), we have

$$16e^2Q^6F^2 = 0,$$

from which $F = 0$ because of $e \neq 0$. From $J = F = 0$, we have $B = 3Q^2Q't$. By the coefficient of t^{12} of the equation (2.13) we have $36e^2Q^8Q'^2 = 0$. Thus the function $Q' = 0$, which implies $B = 0$. Thus, by (2.13) $d = 0$, $a = 0$. Thus, from (2.6) M is minimal, that is, a helicoid. □

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