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ON A CLASS OF REFLEXIVE TOEPLITZ OPERATORS

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Abstract. We will use a result of Farrell, Rubel and Shields to give sufficient conditions under which a Toeplitz operator with conjugate analytic symbol to be reflexive on Dirichlet-type spaces.

The space $\mathcal{H}_\alpha$ $(\alpha \geq 0)$ is a Hilbert space of analytic functions on the open unit disc with the inner product

$$< f, g >_\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + 1 + n)}{\Gamma(\alpha + 1)n!} \hat{f}(n)\overline{\hat{g}(n)}$$

where $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ and Γ denotes the standard Euler Γ- function. The space $\mathcal{H}_0$ is the Hardy space H^2 and the space $\mathcal{H}_1$ is the classical Dirichlet space D. Obviously, $\mathcal{H}_\alpha \subseteq H^2$ for each α. Also the space $\mathcal{H}_\alpha$ is norm equivalent to the Dirichlet-type space D_α corresponding to the weight sequences $(n+1)^\alpha$, see [9].

Recall that if $u \in L^\infty(\partial\mathbb{D})$, then the *Toeplitz operator* with symbol u is the operator $T_u : H^2 \to H^2$ defined by $T_u f = P(uf)$ where P denotes the orthogonal projection from L^2 onto H^2. It is known that if q is a polynomial and $u \in H^\infty$ then $q(T_g) = T_{q(g)}$ where $g(z) = u(\bar{z})$. Moreover, it is proved in [6, Proposition 1] that T_g maps $\mathcal{H}_\alpha$ into $\mathcal{H}_\alpha$ and $||T_g|| = ||g||_\infty$ where $||T_g||$ is the norm of T_g on the space $\mathcal{H}_\alpha$.

We also recall that for an operator T on a Hilbert space $\mathcal{H}$, $lat\,T$ is the lattice of all closed subspaces of $\mathcal{H}$ that are invariant for T. For any

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collection $\mathcal{L}$ of closed subspaces of $\mathcal{H}$, $Alg\,\mathcal{L}$ is the set of all bounded operators A on $\mathcal{H}$ such that $A\mathcal{M} \subseteq \mathcal{M}$ for every $\mathcal{M}$ in $\mathcal{L}$. It is not difficult to see that $Alg\,\mathcal{L}$ is a subalgebra of $\mathcal{B}(H)$, the set of all bounded operators on $\mathcal{H}$, that is closed in the weak operator topology (WOT). Observe that for any operator T and any polynomial $p, p(T) \in Alg\,lat\,T$. Thus, $W(T)$, the weak operator topology closed algebra generated by T and the identity, is contained in $Alg\,lat\,T$. An operator T on a Hilbert space $\mathcal{H}$ is *reflexive* if $Alg\,lat\,T = W(T)$. So reflexive operators have an enormous supply of invariant subspaces.

The first result about reflexive operators was appeared in a work of D. D. Sarason [11], where the reflexivity of normal operators and analytic Toeplitz operators was proved. J. A. Deddens [3] proved that every isometry is reflexive, also Deddens and P. A. Fillmore [4] treated that problem for operators acting on finite-dimensional spaces. Bercovici, Foias, Langsam, and Pearcy [1] showed that (BCP)-operators are reflexive. Olin and Thomson [10] proved the reflexivity of subnormal operators. It is shown in [12] that certain operators in Cowen-Douglas class are reflexive. Also, recently, in [7], we proved that if the multiplication operator by the independent variable z, acting on Banach spaces of formal Laurent series is invertible then it is reflexive. As a good source on reflexivity, see [8] and [2, Chapter 8]. In this paper we give a class of reflexive Toeplitz operators on Dirichlet-type spaces.

Now we include some preparatory material of general nature which will be needed in the main theorem. If G is a bounded domain in the plane let G^* be the complement of the closure of the unbounded component of the complement of the closure of G. We call G^* the Carathéodory hull of G and indeed can be described as the interior of the outer boundary of G, and in analytic terms it can be defined as the interior of the set of all points z_0 in the plane such that $|p(z_0)| \leq \sup_{z \in G} |p(z)|$ for all polynomials p. The components of G^* are simply connected; in fact, it is a simple matter to show that each of these components has a connected complement. We denote by G^1 the component of G^* that contains G. If G is bounded and $G = G^1$, then G is called a *Carathéodory region.* We can now state:

Theorem 1. (The Farrell-Rubel-Shields Theorem [5]). Let G be an open subset of the complex plane, and let f be a bounded analytic function on G. There is a uniformly bounded sequence of polynomials converging pointwise to f if and only if f has a bounded analytic extension to G^*.

Theorem 2. If $u(z)$ is a univalent map on $\mathbb{D}$ such that $u(\mathbb{D})$ is a Carathéodory region then the operator $T_{u(\bar{z})}$ is reflexive on $\mathcal{H}_\alpha$.

Proof. For any $\lambda \in \mathbb{D}$, put $e_\lambda(z) = (1-\lambda z)^{-1}$. Each function e_λ is in $\mathcal{H}_\alpha$ and it is easy to see that $T_{p(\bar{z})}e_\lambda = p(\lambda)e_\lambda$ for every polynomial p. Also, if $\varphi(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is in H^∞ and $Q_N(z) = \sum\limits_{n=0}^{N} a_n z^n$, then $Q_N(\bar{z})$ converges to $\varphi(\bar{z})$ in $L^2(\partial\mathbb{D})$ as $N \to \infty$. Thus

$$T_{\varphi(\bar{z})}e_\lambda = P(\lim_N Q_N(\bar{z})e_\lambda) = \lim_N T_{Q_N(\bar{z})}e_\lambda = \lim_N Q_N(\lambda)e_\lambda = \varphi(\lambda)e_\lambda.$$

Moreover, if $f \in \mathcal{H}_\alpha$ and $< f, e_\lambda >= 0$ for every $\lambda \in \mathbb{D}$ then $f = 0$.

Let $X \in Alg\,lat\,T_{u(\bar{z})}$. Since the one dimensional span of e_λ is invariant under $T_{u(\bar{z})}$, it is invariant under X; consequently there is a function $\varphi(\lambda)$ such that

$$(Xe_\lambda)(z) = \varphi(\lambda)e_\lambda(z) \qquad z \in \mathbb{D}, \lambda \in \mathbb{D}.$$

But

$$(Xe_\lambda)(z) = (Xe_z)(\lambda) \qquad z \in \mathbb{D}, \lambda \in \mathbb{D}$$

and so

$$(Xe_z)(\lambda) = \varphi(\lambda)e_\lambda(z) \qquad z \in \mathbb{D}, \lambda \in \mathbb{D}.$$

Taking $z = 0$, we get $(X1)(\lambda) = \varphi(\lambda)$. This implies that φ is analytic on $\mathbb{D}$. Furthermore,

$$|\varphi(\lambda)|\ ||e_\lambda|| = ||Xe_\lambda|| \leq ||X||\ ||e_\lambda||$$

which implies that $\varphi \in H^\infty$. Since the linear span of e_λ is dense in $\mathcal{H}_\alpha$, $X = T_{\varphi(\bar{z})}$. If $G = u(\mathbb{D})$, then by the open mapping theorem the function $h = \varphi o u^{-1}$ is in $H^\infty(G)$ and the Farrel-Rubel-Shields Theorem guarantees the existence of a sequence of polynomials $\{q_n\}_n$ converging pointwise to h on G, such that $||q_n||_G \leq ||h||_G$ where $||f||_G := \sup\{|f(z)| : z \in G\}$. Hence

$$(q_n \, o \, u)(z) \longrightarrow (h \, o \, u)(z) = \varphi(z) \quad \forall z \in \mathbb{D}$$

and $||q_n \, o \, u||_\infty \leq ||q_n||_G \leq ||h||_G$ for all n.

Compactness of the unit ball of $\mathcal{B}(\mathcal{H}_\alpha)$ in the weak operator topology coupled with the fact that

$$||T_{q_n \, o \, u(\bar{z})}|| = ||q_n \, o \, u||_\infty \leq ||h||_G$$

imply the existence of a subsequence $\{n_i\}$ such that

$$T_{(q_{n_i} \, o \, u)(\bar{z})} = q_{n_i}(T_{u(\bar{z})}) \longrightarrow T \;\; (WOT)$$

for some operator T. Thus

$$T_{(q_{n_i} \, o \, u)(\bar{z})} e_\lambda \longrightarrow T e_\lambda$$

weakly for every $\lambda \in \mathbb{D}$, which in turn, implies that

$$(q_{n_i} \, o \, u)(\lambda) e_\lambda(z) = T_{q_{n_i} \, o \, u} e_\lambda(z) \longrightarrow (T e_\lambda)(z)$$

for every $z \in \mathbb{D}$. So $X e_\lambda = T_{\varphi(\bar{z})} e_\lambda = T e_\lambda$. Since the linear span of e_λ is dense in $\mathcal{H}_\alpha$, $X = T$. □

Example. If $|a| < 1$ and $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}, z \in \mathbb{D}$ is the Möbius transformation then $T_{\varphi_a(\bar{z})}$ is reflexive on $\mathcal{H}_\alpha$.

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