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FINITELY GENERATED PROJECTIVE MODULES OVER NOETHERIAN RINGS

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Abstract. It is well-known that every finitely generated torsion-free module over a principal ideal domain is free. This will be generalized. We deal with ideals of the finite, external direct product of certain rings. Finally, if M is a torsion-free, finitely generated module over a reduced, Noetherian ring A, then we prove that M_S is a projective module over A_S, where $S = A\backslash Z(A)$.

0. Introduction

Through out this paper, every *ring* is a commutative ring with an identity element.

Let A be a ring. We adopt the following notations.

(1) $Z(A)$= the set of all zero-divisors of A.
(2) Spec (A) = the set of all prime ideals of A.
(3) Min (A) = the set of all minimal prime ideals of A.
(4) Max (A)= the set of all maximal ideals of A.

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1. Finitely Generated Torsion-free Modules

Let $R = \mathbb{Z}_6$. Then $3^2 = 3$ in R, so 3 is an idempotent element of R. Let $M = R/3R$. Then M is a cyclic module over R, so it is a finitely generated multiplication module over R. Moreover, $\mathrm{Ann}_R(M) = 3R$. Then by [S88, Theorem 11], M is a projective R-module. Of course, R is a Noetherian ring.

It is well-known [M97, Theorem 2.4.1, p.19] that every finitely generated torsion-free module over a principal ideal domain is free. This will be generalized below.

Let $R_1, R_2, \cdots, R_n$ be rings and let $R = R_1 \times R_2 \times \cdots \times R_n$. For each $i \in \{1, 2, \cdots, n\}$, let

$$\overline{R_i} = \{0\} \times \cdots \times \{0\} \times R_i \times \{0\} \cdots \times \{0\}.$$

Let i be any element of $\{1, 2, \cdots, n\}$. Then $\overline{R}_i \subseteq R$. If $n > 1$, then the identities of $\overline{R}_i$ and R differ. Hence, if $n > 1$, then we cannot guarantee $\overline{R}_i M = M$ and hence we cannot say that M is an $\overline{R}_i$-module. However, $\overline{R}_i M$ is both an $\overline{R}_i$-module and an R-module.

$Z(\overline{R_1}) \subseteq Z(R)$. It is clear that $(1, 0, 0, \cdots, 0) \notin Z(\overline{R_1})$. However, noticing that

$$(1, 0, 0, \cdots, 0)(0, 1, 0, \cdots, 0) = (0, 0, 0, \cdots, 0),$$

we can see that $(1, 0, 0, \cdots, 0) \in Z(R)$. Hence, $Z(\overline{R_1}) \subsetneq Z(R)$.

THEOREM 1.1. *Let $R_1, R_2, \cdots, R_n$ be principal ideal domains and let $R = R_1 \times R_2 \times \cdots \times R_n$. For each $i \in \{1, 2, \cdots, n\}$, let*

$$\overline{R_i} = \{0\} \times \cdots \times \{0\} \times R_i \times \{0\} \cdots \times \{0\}.$$

Let M be finitely generated over R and let each $\overline{R}_i M$ be torsion-free over $\overline{R}_i$. Then the following statements are true.

(1) *There exists a positive integer k such that M is an R-submodule of $R_1^k \times R_2^k \times \cdots \times R_n^k$.*

(2) *M is projective over R.*

(3) *M is torsion-free over R.*

Proof. Assume that M is finitely generated over R and each $\overline{R}_iM$ is torsion-free over $\overline{R}_i$.

(1) There are elements $m_1, m_2, \cdots, m_r \in M$ such that

$$M = Rm_1 + Rm_2 + \cdots + Rm_r.$$

Let i be any element of $\{1, 2, \cdots, n\}$. Then

$$\begin{aligned}\overline{R}_iM &= \overline{R}_i(Rm_1 + Rm_2 + \cdots + Rm_r)\\ &= \overline{R}_iRm_1 + \overline{R}_iRm_2 + \cdots + \overline{R}_iRm_r\\ &= \overline{R}_im_1 + \overline{R}_im_2 + \cdots + \overline{R}_im_r.\end{aligned}$$

Hence, $\overline{R}_iM$ is finitely generated over $\overline{R}_i$. Since $\overline{R}_iM$ is torsion-free over a principal ideal domain $\overline{R}_i$, it follows from [M97, Theorem 2.4.1, p.19] that $\overline{R}_iM$ is free over $\overline{R}_i$. Hence, there exists a positive integer k_i such that

$$\overline{R}_iM = \overline{R}_i^{k_i}.$$

Hence, taking $k = \max\{k_1, k_2, \cdots, k_n\}$, we can see that $\overline{R}_iM$ is R-isomorphic to $\overline{R_i^{k_i}}$, which is an R-submodule of R^k. Thus,

$$\begin{aligned}M &= RM\\ &= (\overline{R}_1 + \overline{R}_2 + \cdots + \overline{R}_n)M\\ &= \overline{R}_1M + \overline{R}_2M + \cdots + \overline{R}_nM\\ &= \overline{R_1^{k_1}} + \overline{R_2^{k_2}} + \cdots + \overline{R_n^{k_n}}\\ &\subseteq \overline{R_1^{k}} + \overline{R_2^{k}} + \cdots + \overline{R_n^{k}}\\ &= R_1^k \times R_2^k \times \cdots \times R_n^k.\end{aligned}$$

(2) By (1), we have

$$\begin{aligned} R^k &\cong R_1^k \times R_2^k \times \cdots \times R_n^k \\ &\cong (R_1^{k_1} \times R_2^{k_2} \times \cdots \times R_n^{k_n}) \oplus (R_1^{k-k_1} \times R_2^{k-k_2} \times \cdots \times R_n^{k-k_n}) \\ &\cong M \oplus (R_1^{k-k_1} \times R_2^{k-k_2} \times \cdots \times R_n^{k-k_n}). \end{aligned}$$

Hence, M is a projective R-module.

(3) By the proof of (2), there exists an R-monomorphism $\varphi : M \longrightarrow R^k$. Assume that $am = 0$, where $a \in R\backslash Z(R)$ and $m \in M$. Since $\varphi(m) \in R^k$, we can write $\varphi(m)$ as follows: $\varphi(m) = (a_1, a_2, \cdots, a_k)$, where each $a_i \in R$. Then

$$a(a_1, a_2, \cdots, a_k) = a\varphi(m) = \varphi(am) = \varphi(0) = 0.$$

This implies $aa_1 = 0, aa_2 = 0, \cdots, aa_k = 0$. Since $a \in R\backslash Z(R)$, we must have $a_1 = 0, a_2 = 0, \cdots, a_k = 0$. Hence, $\varphi(m) = 0$ and so $m = 0$. This shows that M is torsion-free over R. □

Corollary 1.2 [M97, Theorem 2.4.1, p.19]. *Every finitely generated torsion-free module over a PID is free.*

Proof. Let M be any finitely generated torsion-free module over a PID R. Then by Theorem 1.2(1), M is a submodule of a free module over the PID R. Hence, by [I81] or [P91, Corollary 6.4, p.58], M itself is free. □

2. Finite Direct Products of Principal Ideal Domains

Lemma 2.1 [M89, Exercise 1.2, p.6]. *Let $A_1, \cdots, A_n$ be rings and let $A = A_1 \times \cdots \times A_n$. Then*

$$SpecA = \bigcup_{i=1}^{n} \{A_1 \times \cdots \times A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n \mid P_i \text{ is a prime ideal of } A_i\}.$$

Proof. This will be proved by the Mathematical Induction on n.

Step I. Let $n = 1$. Then $\operatorname{Spec}(A_1) = \{P_1 \mid P_1 \text{ is a prime ideal of } A_1\}$.

Step II. We prove first that the result is true for $n = 2$. It is easy to prove that

$$\begin{aligned}\operatorname{Spec}(A) \supseteq &\{P_1 \times A_2 \mid P_1 \text{ is a prime ideal of } A_1\}\\ &\cup \{A_1 \times P_2 \mid P_2 \text{ is a prime ideal of } A_2\}.\end{aligned}$$

Conversely, let P be any member of $\operatorname{Spec}A$. Then $(1,0)(0,1) = (0,0) \in P$ implies $(1,0) \in P$ or $(0,1) \in P$. There are three cases to consider.

(1) Assume that $(1,0) \in P$ and $(0,1) \in P$. Then $(1,1) \in P$. Hence, $P = A$. This is a contradiction.

(2) Assume that $(1,0) \in P$ and $(0,1) \notin P$. Define a map $\lambda_1 : A_1 \longrightarrow A$ by $\lambda_1(a) = (a, 0)$, where $a \in A_1$. Then λ_1 is a monomorphism. Define a map $\lambda_2 : A_2 \longrightarrow A$ by $\lambda_2(b) = (0, b)$, where $b \in A_2$. Then λ_2 is a monomorphism.

Let (a_1, a_2) be any element of $\lambda_1^{-1}(P) \times \lambda_2^{-1}(P)$. Then $a_1 \in \lambda_1^{-1}(P)$ implies $(a_1, 0) = \lambda_1(a_1) \in P$. $a_2 \in \lambda_2^{-1}(P)$ implies $(0, a_2) = \lambda_2(a_2) \in P$. Hence, $(a_1, a_2) = (a_1, 0) + (0, a_2) \in P$. This shows that

$$\lambda_1^{-1}(P) \times \lambda_2^{-1}(P) \subseteq P.$$

Conversely, let (p_1, p_2) be any element of P. Then $(p_1, 0) = (p_1, 0)(1, 0) \in P$ and hence $(0, p_2) = (p_1, p_2) - (p_1, 0) \in P$. Hence, $(p_1, p_2) \in \lambda_1^{-1}(P) \times \lambda_2^{-1}(P)$. This shows that

$$P \subseteq \lambda_1^{-1}(P) \times \lambda_2^{-1}(P).$$

Thus,

$$P = \lambda_1^{-1}(P) \times \lambda_2^{-1}(P).$$

Since $(1,0) \in P$, we can see that $\lambda_1^{-1}(P) = A_1$. Since $(0,1) \notin P$, we can see that $\lambda_2^{-1}(P)$ is a prime ideal of A_2. Therefore,

$$P = A_1 \times \lambda_2^{-1}(P),$$

where $\lambda_2^{-1}(P)$ is a prime ideal of A_2.

(3) Assume that $(1,0) \notin P$ and $(0,1) \in P$. Then by a similar proof to (2), we can show that

$$P = \lambda_1^{-1}(P) \times A_2,$$

where $\lambda_1^{-1}(P)$ is a prime ideal of A_1.

This shows that

$$\begin{aligned}\mathrm{Spec}(A) \subseteq &\{P_1 \times A_2 \mid P_1 \text{ is a prime ideal of } A_1\}\\ &\cup \{A_1 \times P_2 \mid P_2 \text{ is a prime ideal of } A_2\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Spec}(A) = &\{P_1 \times A_2 \mid P_1 \text{ is a prime ideal of } A_1\}\\ &\cup \{A_1 \times P_2 \mid P_2 \text{ is a prime ideal of } A_2\}.\end{aligned}$$

Now, let $n > 1$. Assume that the result is true for $n-1$. Let

$$A' = A_2 \times \cdots \times A_n.$$

Then $A = A_1 \times A'$. By the previous argument, we have

$$\begin{aligned}\mathrm{Spec}(A) = &\{P_1 \times A' \mid P_1 \text{ is a prime ideal of } A_1\}\\ &\cup \{A_1 \times P' \mid P' \text{ is a prime ideal of } A'\}.\end{aligned}$$

Further, by the induction hypothesis, we have

$$\mathrm{Spec} A' =$$

$$\bigcup_{i=2}^{n} \{A_2 \times \cdots \times A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n \mid P_i \text{ is a prime ideal of } A_i\}.$$

Therefore,

$$\mathrm{Spec}A = \bigcup_{i=1}^{n} \{A_1 \times \cdots \times A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n \mid P_i \text{ is a prime ideal of } A_i\}.$$

□

LEMMA 2.2. *Let $A_1, \cdots, A_n$ be rings and let $A = A_1 \times \cdots \times A_n$. Then the set of all primary ideals of R is*

$$\bigcup_{i=1}^{n} \{A_1 \times \cdots \times A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n \mid P_i \text{ is a primary ideal of } A_i\}.$$

Proof. Adopt the proof of Lemma 2.1. □

THEOREM 2.3. *Let $A_1, \cdots, A_n$ be Noetherian rings and let $A = A_1 \times \cdots \times A_n$. Then every ideal of R is of the form*

$$\begin{aligned} A_1 \times \cdots \times A_{i_1-1} \times P_{i_1} \times A_{i_1+1} \times \cdots \times A_{i_2-1} \times P_{i_2} \times A_{i_2+1} \\ \times \cdots\cdots \\ \times A_{i_r-1} \times P_{i_r} \times A_{i_r+1} \times \cdots \times A_n, \end{aligned}$$

where $P_{i_1}, P_{i_2}, \cdots, P_{i_r}$ are primary ideals of $A_{i_1}, A_{i_2}, \cdots, A_{i_r}$, respectively.

Proof. Let $A_1, \cdots, A_n$ be Noetherian rings and let $A = A_1 \times \cdots \times A_n$. Then A is a Noetherian ring. Let I be any ideal of A. Then I has a primary decomposition. By Lemma 2.2, there exist primary ideals $P_{i_1}, P_{i_2}, \cdots, P_{i_r}$ of $A_{i_1}, A_{i_2}, \cdots, A_{i_r}$, respectively such that

$$\begin{aligned} I = (A_1 \times \cdots \times A_{i_1-1} \times P_{i_1} \times A_{i_1+1} \times \cdots \times A_n) \\ \cap (A_1 \times \cdots \times A_{i_2-1} \times P_{i_2} \times A_{i_2+1} \times \cdots \times A_n) \\ \cap \cdots\cdots \\ \cap (A_1 \times \cdots \times A_{i_r-1} \times P_{i_r} \times A_{i_r+1} \times \cdots \times A_n). \end{aligned}$$

The last expression is equal to

$$\begin{aligned} A_1 \times \cdots \times A_{i_1-1} \times P_{i_1} \times A_{i_1+1} \times \cdots \times A_{i_2-1} \times P_{i_2} \times A_{i_2+1} \\ \times \cdots\cdots \\ \times A_{i_r-1} \times P_{i_r} \times A_{i_r+1} \times \cdots \times A_n, \end{aligned}$$

as required. It is obvious that the converse holds. □

THEOREM 2.4. *Let $R_1, R_2, \cdots, R_n$ be principal ideal domains and let $R = R_1 \times R_2 \times \cdots \times R_n$. Then the following statements are true.*

(1) *R is a principal ideal ring and if $n > 1$ then R is not an integral domain.*
(2) *Every ideal of R is a direct summand of R.*
(3) *Every ideal of R is projective over R.*
(4) *R is hereditary.*

Proof. (1) Let I be any ideal of R. Then

$$I = IR = I\overline{R_1} + I\overline{R_2} + \cdots + I\overline{R_n}.$$

Each $I\overline{R_i}$ is an ideal of a PID $\overline{R_i}$, so there is an element $x_i \in R_i$ such that $I\overline{R_i} = \overline{R_i}\overline{x}_i$. Then

$$I = \overline{R}_1\overline{x}_1 + \overline{R}_2\overline{x}_2 + \cdots + \overline{R}_n\overline{x}_n = R(x_1, x_2, \cdots, x_n).$$

Hence, R is a principal ideal ring.

Assume that $n > 1$. Then

$$(1, 0, 0, \cdots, 0)(0, 1, 0, \cdots, 0) = (0, 0, 0, \cdots, 0).$$

Hence, R is not an integral domain.

(2) Let I be any ideal of R. Then by Theorem 2.3, there exist primary ideals $P_{i_1}, P_{i_2}, \cdots, P_{i_r}$ of $R_{i_1}, R_{i_2}, \cdots, R_{i_r}$, respectively, such that

$$\begin{aligned} I = R_1 \times \cdots \times R_{i_1-1} \times P_{i_1} \times R_{i_1+1} \times \cdots \times R_{i_2-1} \times P_{i_2} \times R_{i_2+1} \\ \times \cdots\cdots \\ \times R_{i_r-1} \times P_{i_r} \times R_{i_r+1} \times \cdots \times R_n, \end{aligned}$$

Notice that if P is a primary ideal of a PID, then $P \subseteq \sqrt{P} = 0$; hence $P = 0$. Then

$$\begin{aligned} R_1 \times \cdots \times R_{i_1-1} \times 0 \times R_{i_1+1} \times \cdots \times R_{i_2-1} \times 0 \times R_{i_2+1} \\ \times \cdots\cdots \\ \times R_{i_r-1} \times 0 \times R_{i_r+1} \times \cdots \times R_n, \end{aligned}$$

Hence, $R = I \dotplus (\overline{R_{i_1}} \dotplus \overline{R_{i_2}} \dotplus \cdots \dotplus \overline{R_{i_r}})$. Therefore, every ideal of R is a direct summand of R.

(3) This follows from (2).

(4) This follows from (3). □

3. Finitely Generated Torsion-free Modules over Reduced Noetherian Rings

LEMMA 3.1. *Let R be a ring. If R is Noetherian, then $Min(R)$ is finite.*

Proof. Notice that

$$\sqrt{0_R} = \cap_{\mathfrak{p} \in \mathrm{Min}(R)} \mathfrak{p}.$$

If $\mathrm{Min}(R)$ is infinite, then the ideal $\sqrt{0_R}$ of R is an infinite intersection of primary ideals. This cannot happen in a Noetherian ring. □

It is well-known [H88, Cor. 2.4, p.3] that in a reduced ring R,

$$Z(R) = \cup_{\mathfrak{p}\in \mathrm{Min}(R)}\mathfrak{p}.$$

LEMMA 3.2. *Let R be a ring. If R is reduced and Noetherian, then every prime ideal $\mathfrak{p}$ of R with $\mathfrak{p} \subseteq Z(R)$ is a minimal prime ideal of R.*

Proof. Assume that R is reduced and Noetherian. Let $\mathfrak{p}$ be any prime ideal of R such that $\mathfrak{p} \subseteq Z(R)$. Since R is reduced, we can see that

$$\mathfrak{p} \subseteq Z(R) = \cup_{\mathfrak{q}\in \mathrm{Min}(R)}\mathfrak{q}.$$

Since R is Noetherian, it follows from Lemma 3.1 that $\mathrm{Min}(R)$ is finite. Hence, by the Prime Avoidance Theorem there exists $\mathfrak{q}$ in $\mathrm{Min}(R)$ such that $\mathfrak{p} \subseteq q$. By the minimality of $\mathfrak{q}$, we have $\mathfrak{q} = \mathfrak{p}$. Hence, $\mathfrak{p} \in \mathrm{Min}(R)$. □

LEMMA 3.3. *Let A be a reduced Noetherian ring. Let $S = A\backslash Z(A)$. Then there exist $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r \in$ Min (A) such that*

(1) $Z(A) = \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$,

(2) *Max* $(A_S) = \{\mathfrak{p}_1 A_S, \mathfrak{p}_2 A_S, \cdots, \mathfrak{p}_r A_S\}$, *(Hence, A_S is a semilocal ring with maximal ideals $\mathfrak{p}_1 A_S, \mathfrak{p}_2 A_S, \cdots, \mathfrak{p}_r A_S$.)*

(3) $A_S \cong A_S/\mathfrak{p}_1 A_S \oplus A_S/\mathfrak{p}_2 A_S \oplus \cdots \oplus A_S/\mathfrak{p}_r A_S$.

Proof. Since A is Noetherian and reduced, there exist distinct elements $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r \in \mathrm{Min}\ (A)$ such that $0 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$.

(1) If $r = 1$, then $\mathfrak{p}_1 = 0 \subseteq Z(A)$. Assume that $r > 1$. Then

$$0 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r \supseteq \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_r$$

implies $\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_r = 0$. $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r$ are all contained in $Z(A)$. Hence, $\mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r \subseteq Z(A)$.

Conversely, let $x \in Z(A)$. Then there exists a non-zero element $y \in A$ such that $xy = 0$. Suppose that $x \notin \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$. Then $x \notin \mathfrak{p}_1$. So, $xy = 0 \in \mathfrak{p}_1$ implies $y \in \mathfrak{p}_1$. By a similar proof, we can show that $y \in \mathfrak{p}_2, \cdots, y \in \mathfrak{p}_r$. Hence,

$$y \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r = 0.$$

This contradiction shows that $x \in \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$. Thus, $Z(A) \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$.

Therefore, $Z(A) = \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$.

(2) By (1),

$$S = A \backslash Z(A) = (A \backslash \mathfrak{p}_1) \cap (A \backslash p_2) \cap \cdots \cap (A \backslash \mathfrak{p}_r).$$

Since $\mathfrak{p}_1 \cap S = \emptyset$, $\mathfrak{p}_1 A_S$ is a minimal prime ideal of A_S. Similarly, $\mathfrak{p}_2 A_S, \cdots, \mathfrak{p}_r A_S$ are minimal prime ideals of A_S.

Now, let M be any maximal ideal of A_S. Then there exists a prime ideal $\mathfrak{p}$ of A with $\mathfrak{p} \cap S = \emptyset$ such that $\mathfrak{p} A_S = M$. Since $\mathfrak{p} \cap S = \emptyset$, $\mathfrak{p} \subseteq Z(A) = \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$. By the Prime Avoidance Theorem [S90, Theorem 3.61, p.56], there is an element $i \in \{1, 2, \cdots, r\}$ such that $\mathfrak{p} \subseteq \mathfrak{p}_i$. By the minimality of $\mathfrak{p}_i$, we must have $\mathfrak{p}_i = \mathfrak{p}$. So, $M = \mathfrak{p} A_S = \mathfrak{p}_i A_S$. This shows that

$$\text{Max } (A_S) \subseteq \{\mathfrak{p}_1 A_S, \mathfrak{p}_2 A_S, \cdots, \mathfrak{p}_r A_S\}.$$

Conversely, let I be an ideal of A_S such that $\mathfrak{p}_1 A_S \subseteq I \subseteq A_S$. Assume that $I \neq A_S$. Then there is a maximal ideal N of A_S such that $I \subseteq N$. By the previous argument, there is an element $i \in \{1, 2, \cdots, r\}$ such that $N = \mathfrak{p}_i A_S$. So,

$$\mathfrak{p}_1 A_S \subseteq I \subseteq N = \mathfrak{p}_i A_S.$$

By the minimality of $\mathfrak{p}_i A_S$, we must have

$$\mathfrak{p}_i A_S = \mathfrak{p}_1 A_S.$$

Hence, $I = \mathfrak{p}_1 A_S$. This shows that $\mathfrak{p}_1 A_S$ is a maximal ideal of A_S. Similarly, we can show that $\mathfrak{p}_2 A_S, \cdots, \mathfrak{p}_2 A_S$ are maximal ideals of A_S. Therefore,

$$\{\mathfrak{p}_1 A_S, \mathfrak{p}_2 A_S, \cdots, \mathfrak{p}_r A_S\} \subseteq \text{Max}\ (A_S).$$

Consequently, Max $(A_S) = \{\mathfrak{p}_1 A_S, \mathfrak{p}_2 A_S, \cdots, \mathfrak{p}_r A_S\}$.

(3) Notice that

$$\mathfrak{p}_1 A_S \cap \mathfrak{p}_2 A_S \cap \cdots \cap \mathfrak{p}_r A_S = (\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r) A_S = 0 A_S = 0.$$

Further, by Zorn's lemma and by (2), we see that

$$\mathfrak{p}_i A_S + \mathfrak{p}_j A_S = A_S$$

for every i, j with $i \neq j$. Then by the Chinese Remainder Theorem, we have

$$A_S \cong A_S / \mathfrak{p}_1 A_S \oplus A_S / \mathfrak{p}_2 A_S \oplus \cdots \oplus A_S / \mathfrak{p}_r A_S .$$

□

THEOREM 3.4. *Let A be a reduced Noetherian ring. Let $S = A \backslash Z(A)$. If M is finitely generated torsion-free over A, then M_S is projective over A_S.*

Proof. Let M be finitely generated torsion-free over A. Then M_S is finitely generated torsion-free over A_S. By Lemma 3.3 and Theorem 1.1, M_S is projective over A_S. □

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