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FUZZY IDEALS OF K(G)-ALGEBRAS

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Abstract. Further properties on a fuzzy ideal of a right K(G)-algbera $\mathcal{G}$ are investigated. Using a family of ideals of a right K(G)-algebra $\mathcal{G}$ with additional conditions, a fuzzy ideal of $\mathcal{G}$ is established. Given a fuzzy set μ in $\mathcal{G}$, the least fuzzy ideal of $\mathcal{G}$ containing μ is described. Using a chain of ideals of $\mathcal{G}$, a fuzzy ideal of $\mathcal{G}$ is constructed, and their properties are investigated.

1. Introduction

The notion of K-algebras on a group $(G, \cdot)$ was introduced by K. H. Dar and M. Akram in [1, 2] as a non-associative and non-commutative algebraic system. A right K-algebra was built on the group G(briefly, right K(G)-algebra) by using the induced binary operation $\odot$ on $(G, \cdot)$ as $x \odot y = x \cdot y^{-1} = xy^{-1}$, for all $x, y \in G$. The notion of fuzzy set was introduced by Zadeh in his paper "L. A. Zadeh, *Fuzzy sets,* Inform. Control 8 (1965), 338–353". Since then there have been wide-ranging applications of the fuzzy set theory. Many research workers have fuzzified the various mathematical structures, such as topological spaces, functional analysis, loops, groups, rings, near rings, vector spaces, and automation. In [1], M. Akram et al. introduced the notions of fuzzy subalgebras and fuzzy (maximal) ideals of a right K(G)-algebra. In this paper, we investigate further properties on fuzzy ideals of right

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K(G)-algebras. Using a family of ideals of a right K(G)-algebra $\mathcal{G}$ with additional conditions, we establish a fuzzy ideal of $\mathcal{G}$. Given a fuzzy set μ in $\mathcal{G}$, we describe the least fuzzy ideal of $\mathcal{G}$ containing μ. Using a chain of ideals of $\mathcal{G}$, we make a fuzzy ideal of $\mathcal{G}$ and study their properties.

2. Preliminaries

Let $(G, \cdot, e)$ be a group with the identity e such that $x^2 \neq e$ for some $x(\neq e) \in G$. A *right algebra built on* G (briefly, *right K(G)-algebra*) is a structure $\mathcal{G} = (G, \cdot, \odot, e)$, where "$\odot$" is a binary operation on G which is derived from the operation "$\cdot$", that satisfies the following:

(k1) $(\forall a, x, y \in G)\ ((a \odot x) \odot (a \odot y) = (a \odot (y^{-1} \odot x^{-1})) \odot a)$,

(k2) $(\forall a, x \in G)\ (a \odot (a \odot x) = (a \odot x^{-1}) \odot a)$,

(k3) $(\forall a \in G)\ (a \odot a = e,\ a \odot e = a,\ e \odot a = a^{-1})$.

If G is abelian, then conditions (k1) and (k2) are written as follows:

(k1′) $(\forall a, x, y \in G)\ ((a \odot x) \odot (a \odot y) = y \odot x)$,

(k2′) $(\forall a, x \in G)\ (a \odot (a \odot x) = x)$,

respectively (see [2]).

3. Fuzzy ideals

Definition 3.1. [1] A nonempty subset A of a right K(G)-algebra $\mathcal{G}$ is called an *ideal* of $\mathcal{G}$ if it satisfies:

(i) $e \in A$,

(ii) $(\forall x, y \in G)\ (x \odot y \in A,\ y \odot (y \odot x) \in A \Rightarrow x \in A)$.

Definition 3.2. [1] A fuzzy set μ in a right K(G)-algebra $\mathcal{G}$ is called a *fuzzy ideal* of $\mathcal{G}$ if it satisfies:

(i) $(\forall x \in G)\ (\mu(e) \geq \mu(x))$,

(ii) $(\forall x, y \in G)\ (\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\})$.

Example 3.3. [1] Consider the right $K(S_3)$-algebra $\mathcal{G} = (S_3, \cdot, \odot, e)$ on the symmetric group $S_3 = \{e, a, b, x, y, z\}$ where $e = (1)$, $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, and $\odot$ is given by the following Cayley table:

$\odot$	e	x	y	z	a	b
e	e	x	y	z	a	b
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

Let μ be a fuzzy set in $\mathcal{G}$ defined by $\mu(e) = t_1$, $\mu(a) = \mu(b) = t_2$ and $\mu(x) = \mu(y) = \mu(z) = t_3$, where $t_1 \geq t_2 \geq t_3$ in $[0, 1]$. It is easy to check that μ is a fuzzy ideal of $\mathcal{G}$.

Lemma 3.4. [1] *Let μ be a fuzzy set in a right $K(G)$-algebra $\mathcal{G}$. Then μ is a fuzzy ideal of $\mathcal{G}$ if and only if the set $U(\mu; t) := \{x \in G \mid \mu(x) \geq t\}$, $t \in [0, 1]$, is an ideal of $\mathcal{G}$ when it is nonempty.*

We call $U(\mu; t)$ the *level ideal* of μ in $\mathcal{G}$.

Theorem 3.5. *Let A be an ideal of a right K(G)-algebra $\mathcal{G}$ and let $\mu : G \to [0, 1]$ be a fuzzy set defined by*

$$\mu(x) := \begin{cases} t_0 & \text{if } x \in A, \\ t_1 & \text{otherwise.} \end{cases}$$

for all $x \in G$ where $t_0, t_1 \in [0, 1]$ with $t_0 > t_1$. Then μ is a fuzzy ideal of $\mathcal{G}$ and $U(\mu; t_0) = A$.

Proof. Since $e \in A$, clearly $\mu(e) \geq \mu(x)$ for all $x \in G$. Let $x, y \in G$. If $x \odot y \notin A$, then $\mu(x \odot y) = t_1$ and so

$$\mu(x) \geq t_1 = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

Assume that $x \odot y \in A$. If $x \in A$ then $y \odot (y \odot x)$ may or may not belong to A. In any case,

$$\mu(x) = t_0 \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

If $x \notin A$ then $y \odot (y \odot x) \notin A$ because A is an ideal. Hence

$$\mu(x) = t_1 = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

Therefore μ is a fuzzy ideal of $\mathcal{G}$. It is clear that $U(\mu; t_0) = A$. □

Corollary 3.6. *Any ideal of a right $K(G)$-algebra $\mathcal{G}$ can be realized as a level ideal of some fuzzy ideal of $\mathcal{G}$.*

Proof. Straightforward. □

The following theorem shows that the concept of a fuzzy ideal in a right K(G)-algebra $\mathcal{G}$ is a generalization of an ideal. Using Theorem 3.5, the proof is straightforward.

Theorem 3.7. *Let A be a non-empty subset of a right $K(G)$-algebra $\mathcal{G}$ and let μ be a fuzzy set in $\mathcal{G}$ such that μ is into $\{0, 1\}$, so that μ is the characteristic function of A. Then μ is a fuzzy ideal of $\mathcal{G}$ if and only if A is an ideal of $\mathcal{G}$.*

The following lemma is obvious and we omit the proof.

Lemma 3.8. *Let Λ be a totally ordered set and let $\{A_t \mid t \in \Lambda\}$ be a family of ideals of a right $K(G)$-algebra $\mathcal{G}$ such that for all $s, t \in \Lambda$, $s > t$ if and only if $A_s \subset A_t$. Then $\bigcup_{t\in\Lambda} A_t$ and $\bigcap_{t\in\Lambda} A_t$ are ideals of $\mathcal{G}$.*

Let Λ be a non-empty subset of $[0, 1]$.

Theorem 3.9. *Let $\{A_t \mid t \in \Lambda\}$ be a family of ideals of a right $K(G)$-algebra $\mathcal{G}$ such that $G = \bigcup_{t\in\Lambda} A_t$ and for all $s, t \in \Lambda$, $s > t$ if and only if $A_s \subset A_t$. Define a fuzzy set μ in $\mathcal{G}$ by $\mu(x) = \bigvee\{t \in \Lambda \mid x \in A_t\}$ for all $x \in G$. Then μ is a fuzzy ideal of $\mathcal{G}$.*

Proof. Following Lemma 3.4, it is sufficient to show that $U(\mu; s)$ is an ideal of $\mathcal{G}$ for every $s \in [0,1]$. To do this, we divide into the following two cases:

$$\text{(i) } s = \bigvee\{t \in \Lambda \mid t < s\} \text{ and (ii) } s \neq \bigvee\{t \in \Lambda \mid t < s\}.$$

Case (i) implies that

$$x \in U(\mu; s) \Leftrightarrow x \in A_t \text{ for all } t < s \Leftrightarrow x \in \bigcap_{t<s} A_t,$$

so that $U(\mu; s) = \bigcap_{t<s} A_t$, which is an ideal of $\mathcal{G}$ by Lemma 3.8. For the case (ii), we claim that $U(\mu; s) = \bigcup_{t \geq s} A_t$. If $x \in \bigcup_{t \geq s} A_t$, then $x \in A_t$ for some $t \geq s$. It follows that $\mu(x) \geq t \geq s$ so that $x \in U(\mu; s)$. This proves that $\bigcup_{t \geq s} A_t \subset U(\mu; s)$. Now assume that $x \notin \bigcup_{t \geq s} A_t$. Then $x \notin A_t$ for all $t \geq s$. Since $s \neq \bigvee\{t \in \Lambda \mid t < s\}$, there exists $\varepsilon > 0$ such that $(s - \varepsilon, s) \cap \Lambda = \emptyset$. Hence $x \notin A_t$ for all $t > s - \varepsilon$, which means that if $x \in A_t$ then $t \leq s - \varepsilon$. Thus $\mu(x) \leq s - \varepsilon < s$, and so $x \notin U(\mu; s)$. Therefore $U(\mu; s) \subset \bigcup_{t \geq s} A_t$. Using Lemma 3.8, $U(\mu; s) = \bigcup_{t \geq s} A_t$ is an ideal of $\mathcal{G}$. Consequently, we conclude that μ is a fuzzy ideal of $\mathcal{G}$. □

Theorem 3.10. *Let μ be a fuzzy set in a right K(G)-algebra $\mathcal{G}$. Then a fuzzy set μ^* in $\mathcal{G}$ defined by*

$$\mu^*(x) = \bigvee\{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\}, \ \forall x \in G$$

is the least fuzzy ideal of $\mathcal{G}$ that contains μ, where $\langle U(\mu; t)\rangle$ means the least ideal of $\mathcal{G}$ containing $U(\mu; t)$.

Proof. For any $s \in \mathrm{Im}(\mu^*)$, let $s_n = s - \frac{1}{n}$ for any $n \in \mathbb{N}$. Let $x \in U(\mu^*; s)$. Then $\mu^*(x) \geq s$, which implies that

$$\bigvee\{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\} \geq s > s - \tfrac{1}{n} = s_n, \ \forall n \in \mathbb{N}.$$

Hence there exists $t^* \in \{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\}$ such that $t^* > s_n$. Thus $U(\mu; t^*) \subset U(\mu; s_n)$ and so $x \in \langle U(\mu; t^*)\rangle \subset \langle U(\mu; s_n)\rangle$ for all $n \in \mathbb{N}$. Consequently, $x \in \bigcap_{n \in \mathbb{N}} \langle U(\mu; s_n)\rangle$. On the other hand, if $x \in$

$\bigcap_{n\in\mathbb{N}} \langle U(\mu; s_n)\rangle$, then $s_n \in \{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\}$ for any $n \in \mathbb{N}$. Therefore

$$s - \tfrac{1}{n} = s_n \le \bigvee\{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\} = \mu^*(x),\ \forall n \in \mathbb{N}.$$

Since n is arbitrary, it follows that $s \le \mu^*(x)$ so that $x \in U(\mu^*; s)$. Hence $U(\mu^*; s) = \bigcap_{n\in\mathbb{N}} \langle U(\mu; s_n)\rangle$, which is an ideal of $\mathcal{G}$. Using Lemma 3.4, we conclude that μ^* is a fuzzy ideal of $\mathcal{G}$. We now prove that μ^* contains μ. For any $x \in G$, let $s \in \{t \in [0,1] \mid x \in U(\mu; t)\}$. Then $x \in U(\mu; s)$ and so $x \in \langle U(\mu; s)\rangle$. Thus $s \in \{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\}$, which implies that

$$\{t \in [0,1] \mid x \in U(\mu; t)\} \subset \{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\}.$$

It follows that

$$\begin{aligned} \mu(x) &= \bigvee\{t \in [0,1] \mid x \in U(\mu; t)\} \\ &\le \bigvee\{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\} = \mu^*(x), \end{aligned}$$

which shows that μ^* contains μ. Finally let ν be a fuzzy ideal of $\mathcal{G}$ containing μ. Let $x \in G$. If $\mu^*(x) = 0$, then clearly $\mu^*(x) \le \nu(x)$. Assume that $\mu^*(x) = s \ne 0$. Then $x \in U(\mu^*; s) = \bigcap_{n\in\mathbb{N}} \langle U(\mu; s_n)\rangle$, and so $x \in \langle U(\mu; s_n)\rangle$ for all $n \in \mathbb{N}$. It follows that $\nu(x) \ge \mu(x) \ge s_n = s - \frac{1}{n}$ for all $n \in \mathbb{N}$ so that $\nu(x) \ge s = \mu^*(x)$ since n is arbitrary. This shows that $\mu^* \subset \nu$, and the proof is complete. □

Theorem 3.11. *Let μ be a fuzzy set in a right $K(G)$-algebra $\mathcal{G}$ and let $\operatorname{Im}(\mu) = \{t_k \mid k = 0, 1, 2, \cdots, n\}$, where $t_0 > t_1 > \cdots > t_n$. If $A_0 \subset A_1 \subset \cdots \subset A_n = G$ are ideals of $\mathcal{G}$ such that $\mu(A_k \setminus A_{k-1}) = t_k$ for $k = 0, 1, 2, \cdots, n$, where $A_{-1} = \emptyset$, then μ is a fuzzy ideal of $\mathcal{G}$.*

Proof. Since A_0 is an ideal of $\mathcal{G}$, $e \in A_0$ and $\mu(e) = \mu(A_0 \setminus A_{-1}) = t_0$, which gives $\mu(e) \ge \mu(x)$ for all $x \in G$. To prove that μ satisfies the condition (ii) of Definition 3.2, we consider the following four cases:

1. $x \odot y \in A_k \setminus A_{k-1},\quad y \odot (y \odot x) \in A_k \setminus A_{k-1}$,
2. $x \odot y \in A_k \setminus A_{k-1},\quad y \odot (y \odot x) \notin A_k \setminus A_{k-1}$,

3. $x \odot y \notin A_k \setminus A_{k-1}, \quad y \odot (y \odot x) \in A_k \setminus A_{k-1}$,
4. $x \odot y \notin A_k \setminus A_{k-1}, \quad y \odot (y \odot x) \notin A_k \setminus A_{k-1}$.

In the first case, we have $x \in A_k$, because A_k is an ideal. Thus

$$\mu(x) \geq t_k = \mu(x \odot y) = \mu(y \odot (y \odot x)) = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

In the second case, $y \odot (y \odot x) \in A_{k-1} \subset A_k$ or $y \odot (y \odot x) \in A_m \setminus A_{m-1} \subset A_m \setminus A_k$ for some $m > k$. This together with $x \odot y \in A_k \setminus A_{k-1}$ implies $x \in A_k$ or $x \in A_m \setminus A_k$. Hence

$$\mu(x) \geq t_k = \mu(x \odot y) = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$$

for $x \in A_k$ and $y \odot (y \odot x) \in A_{k-1}$. Similarly,

$$\mu(x) \geq t_m = \mu(y \odot (y \odot x)) = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$$

for $x \in A_m \setminus A_k$ and $y \odot (y \odot x) \in A_m \setminus A_{m-1}$. In the last two cases the process of verification is analogous. $\square$

Corollary 3.12. *Let μ be a fuzzy set in a right K(G)-algebra $\mathcal{G}$ and let $\operatorname{Im}(\mu) = \{t_k \mid k = 0, 1, 2, \cdots, n\}$, where $t_0 > t_1 > \cdots > t_n$. If $A_0 \subset A_1 \subset \cdots \subset A_n = G$ are ideals of $\mathcal{G}$ such that $\mu(A_k) \geq t_k$ for $k = 0, 1, 2, \cdots, n$, then μ is a fuzzy ideal of $\mathcal{G}$.*

Corollary 3.13. *If $\operatorname{Im}(\mu) = \{t_k \mid k = 0, 1, 2, \cdots, n\}$, where $t_0 > t_1 > \cdots > t_n$, is the image of a fuzzy ideal μ of a right K(G)-algebra $\mathcal{G}$, then $U(\mu; t_k)$ are ideals of $\mathcal{G}$ such that $\mu(U(\mu; t_0)) = t_0$ and $\mu(U(\mu; t_k) \setminus U(\mu; t_{k-1})) = t_k$ for $k = 1.2. \cdots, n$.*

Proof. Note that $U(\mu; t_k)$ are ideals of $\mathcal{G}$ by Lemma 3.4, and obviously $\mu(U(\mu; t_0)) = t_0$. Since $\mu(U(\mu; t_1)) \geq t_1$, we have $\mu(x) = t_0$ for $x \in U(\mu; t_0)$ and $\mu(x) = t_1$ for $x \in U(\mu; t_1) \setminus U(\mu; t_0)$. Repeating this procedure, we conclude that $\mu(U(\mu; t_k) \setminus U(\mu; t_{k-1})) = t_k$ for $k = 1, 2, \cdots, n$. $\square$

Proposition 3.14. *Let μ be a fuzzy ideal of a right K(G)-algebra $\mathcal{G}$ with the image $\operatorname{Im}(\mu) = \{t_i \mid i \in \Lambda\}$ where Λ is any index set. If $\Omega := \{U(\mu; t) \mid t \in \operatorname{Im}(\mu)\}$, then*

(i) *there exists a unique* $t_0 \in \mathrm{Im}(\mu)$ *such that* $t_0 \geq t$ *for all* $t \in \mathrm{Im}(\mu)$,

(ii) G *is the set-theoretic union of all* $U(\mu; t) \in \Omega$,

(iii) *the members of* Ω *form a chain,*

(iv) Ω *contains all level ideals of* μ *if and only if* μ *attains its infimum on all ideals of* $\mathcal{G}$.

Proof. (i) Follows from the fact that $t_0 = \mu(e) \geq \mu(x)$ for all $x \in G$.

(ii) If $x \in G$, then $\mu(x) := t(x) \in \mathrm{Im}(\mu)$. This implies $x \in U(\mu; t(x)) \subset \bigcup U(\mu; t) \subset G$ where $t \in \mathrm{Im}(\mu)$, which proves (ii).

(iii) Since $U(\mu; t_i) \subset U(\mu; t_j) \Leftrightarrow t_i \geq t_j$ for $i, j \in \Lambda$. Thus the set Ω is totally ordered by inclusion.

(iv) Suppose that Ω contains all level ideals of μ. Let A be an ideal of $\mathcal{G}$. If μ is constant on A, then we are done. Assume that μ is not constant on A. We consider two cases: $A = G$ and $A \subsetneq G$. For $A = G$ let $s = \vee \mathrm{Im}(\mu)$. Then $s \leq t \in \mathrm{Im}(\mu)$, i.e., $U(\mu; t) \subset U(\mu; s)$ for all $t \in \mathrm{Im}(\mu)$. But $U(\mu; 0) = G \in \Omega$ because Ω contains all level ideals of μ. Hence there exists $r \in \mathrm{Im}(\mu)$ such that $U(\mu; r) = G$. It follows that $G = U(\mu; r) \subset U(\mu; s)$ so that $U(\mu; s) = U(\mu; r) = G$ because every level ideal of μ is an ideal of $\mathcal{G}$. Now it is sufficient to show that $s = r$. If $s < r$ then there exists $l \in \mathrm{Im}(\mu)$ such that $s \leq l < r$. This implies $G = U(\mu; r) \subset U(\mu; l)$ which is a contradiction. Therefore $s = r \in \mathrm{Im}(\mu)$. In the case $A \subsetneq G$ we consider the fuzzy set μ_A defined by

$$\mu_A(x) := \begin{cases} m & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then μ_A is a fuzzy ideal of $\mathcal{G}$ (see Theorem 3.5). Let $\Omega_A = \{U(\mu_A; t_i) \mid i \in J\}$, where $J := \{i \in \Lambda \mid \mu(y) = t_i \text{ for some } y \in A\}$. Noticing that Ω_A contains all level ideals of μ_A, then there exists $z \in A$ such that $\mu_A(z) = \bigvee\{\mu_A(x) \mid x \in A\}$, which implies that $\mu(z) = \bigvee\{\mu(x) \mid x \in A\}$. This proves that μ attains its infimum on all ideals of $\mathcal{G}$. Conversely, assume that μ attains its infimum on all ideals of $\mathcal{G}$. Let $U(\mu; t)$ be a level ideal of μ. If $t = t_i$ for some $i \in \Lambda$, then clearly $U(\mu; t) \in \Omega$.

Assume that $t \neq t_i$ for all $i \in \Lambda$. Then there does not exist $x \in G$ such that $\mu(x) = t$. Let $A = \{x \in G \mid \mu(x) > t\}$. Obviously, $e \in A$. Let $x, y \in G$ satisfy $x \odot y \in A$ and $y \odot (y \odot x) \in A$. Then $\mu(x \odot y) > t$ and $\mu(y \odot (y \odot x)) > t$. It follows from Definition 3.2(ii) that

$$\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\} > t$$

so that $x \in A$. This shows that A is an ideal of $\mathcal{G}$. By hypothesis, there exists $y \in A$ such that $\mu(y) = \bigvee\{\mu(x) \mid x \in A\}$. But $\mu(y) \in \mathrm{Im}(\mu)$ implies $\mu(y) = t_i$ for some $i \in \Lambda$. Hence $\bigvee\{\mu(x) \mid x \in A\} = t_i > t$. Note that there does not exist $z \in G$ such that $t \leq \mu(z) < t_i$. It follows that $U(\mu; t) = U(\mu; t_i) \in \Omega$. Therefore Ω contains all level ideals of μ. □

Since $\mathrm{Im}(\mu)$ is a bounded subset of $[0, 1]$, we can consider $\mathrm{Im}(\mu)$ as a sequence which is either increasing or decreasing.

Theorem 3.15. *Let $\mathcal{G}$ be a right $K(G)$-algebra in which every descending chain $A_1 \supset A_2 \supset \cdots$ of ideals of $\mathcal{G}$ terminates at finite step. If μ is a fuzzy ideal of $\mathcal{G}$ such that a sequence of elements of* $\mathrm{Im}(\mu)$ *is strictly increasing, then μ has finite number of values.*

Proof. Assume that $\mathrm{Im}(\mu)$ is not finite. Let $0 \leq t_1 < t_2 < \cdots \leq 1$ be a strictly increasing sequence of elements of $\mathrm{Im}(\mu)$. Then every $U(\mu; t_i)$ is an ideal of $\mathcal{G}$. For $x \in U(\mu; t_i)$ we have $\mu(x) \geq t_i > t_{i-1}$, which implies that $x \in U(\mu; t_{i-1})$. Thus $U(\mu; t_i) \subset U(\mu; t_{i-1})$. But for $t_{i-1} \in \mathrm{Im}(\mu)$ there exists $x_{i-1} \in G$ such that $\mu(x_{i-1}) = t_{i-1}$. This gives $x_{i-1} \in U(\mu; t_{i-1})$ and $x_{i-1} \notin U(\mu; t_i)$. Hence $U(\mu; t_i) \subsetneq U(\mu; t_{i-1})$, and so we obtain a strictly descending chain $U(\mu; t_1) \supsetneq U(\mu; t_2) \supsetneq U(\mu; t_3) \supsetneq \cdots$ of ideals of $\mathcal{G}$ which is not terminating. This contradiction completes the proof. □

Now we consider the converse of Theorem 3.15.

Theorem 3.16. *Let μ be a fuzzy ideal of a right $K(G)$-algebra $\mathcal{G}$ that has the finite image. Then every descending chain of ideals of $\mathcal{G}$ terminates at finite step.*

Proof. Assume that a strictly descending chain $A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \cdots$ of ideals of $\mathcal{G}$ which does not terminate at finite step. Let μ be a fuzzy set in $\mathcal{G}$ defined by

$$\mu(x) := \begin{cases} \frac{n}{n+1} & \text{for } x \in A_n \setminus A_{n+1};\ n = 0, 1, 2, \cdots, \\ 1 & \text{for } x \in \bigcap A_n;\ n = 0, 1, 2, \cdots, \end{cases}$$

where $A_0 = G$. Then μ has an infinite number of different values. Obviously, $\mu(e) \geq \mu(x)$ for all $x \in G$. Let $x, y \in G$. Assume that $x \odot y \in A_n \setminus A_{n+1}$ and $y \odot (y \odot x) \in A_k \setminus A_{k+1}$ for some k and n. Without loss of generality, we may assume that $n \leq k$. Then $y \odot (y \odot x) \in A_n$, and in the consequence, $x \in A_n$ because A_n is an ideal. Hence $\mu(x) \geq \frac{n}{n+1} = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$. If $x \odot y, y \odot (y \odot x) \in \bigcap A_n$, then $x \in \bigcap A_n$. Thus

$$\mu(x) = 1 = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

If $x \odot y \notin \bigcap A_n$ and $y \odot (y \odot x) \in \bigcap A_n$, then $x \odot y \in A_k \setminus A_{k+1}$ for some k. Hence $x \in A_k$, and so $\mu(x) \geq \frac{k}{k+1} = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$. If $x \odot y \in \bigcap A_n$ and $y \odot (y \odot x) \notin \bigcap A_n$, then $y \odot (y \odot x) \in A_t \setminus A_{t+1}$ for some t, which implies $x \in A_t$. Therefore $\mu(x) \geq \frac{t}{t+1} = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$. This proves that μ is a fuzzy ideal of $\mathcal{G}$. The obtained contradiction completes our proof. □

Theorem 3.17. *Every ascending chain of ideals of a right $K(G)$-algebra $\mathcal{G}$ terminates at finite step if and only if for every fuzzy ideal μ of $\mathcal{G}$, the image* $\mathrm{Im}(\mu)$ *of μ is a well ordered subset of* $[0, 1]$.

Proof. Assume that $\mathrm{Im}(\mu)$ is not well ordered. Then there exists a strictly decreasing sequence $\{t_n\}$ such that $t_n = \mu(x_n)$ for some $x_n \in G$. But in this case ideals $B_n := \{x \in G \mid \mu(x) \geq t_n\}$ form a strictly ascending chain, which is a contradiction. Conversely, suppose that there exists a strictly ascending chain $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots$ of ideals of $\mathcal{G}$. Then $A = \bigcup_{n \in \mathbb{N}} A_n$ is an ideal of $\mathcal{G}$. Let μ be a fuzzy set in $\mathcal{G}$ defined by

$$\mu(x) := \begin{cases} 0 & \text{for } x \notin A, \\ \frac{1}{k} & \text{where } k = \min\{n \in \mathbb{N} \mid x \in A_n\}. \end{cases}$$

Since $e \in A_n$ for all $n \in \mathbb{N}$, it follows that $\mu(e) = 1 \geq \mu(x)$ for all $x \in G$. Let $x, y \in G$. If $x \odot y$ and $y \odot (y \odot x)$ are not in A, then obviously $\mu(x) \geq 0 = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$. Now let $x \odot y$, $y \odot (y \odot x) \in A$. If $x \odot y$, $y \odot (y \odot x) \in A_n \setminus A_{n-1}$ for some $n \in \mathbb{N}$, then $x \in A_n$. Hence $\mu(x) \geq \frac{1}{n} = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$. If $x \odot y \in A_n \setminus A_{n-1}$ and $y \odot (y \odot x) \notin A_n \setminus A_{n-1}$, then $y \odot (y \odot x) \in A_{n-1} \subsetneq A_n$ or $y \odot (y \odot x) \in A_m \setminus A_{m-1}$ for some $m > n$. In the first case, $x \in A_n$ and $\mu(x) \geq \frac{1}{n} = \mu(x \odot y) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$. The second case implies $x \in A_m$ and $\mu(x \odot y) = \frac{1}{n} > \frac{1}{m} = \mu(y \odot (y \odot x))$, which gives $\mu(x) \geq \frac{1}{m} = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$. In a similar way we obtain $\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}$ for $x \odot y \notin A_n \setminus A_{n-1}$ and $y \odot (y \odot x) \in A_n \setminus A_{n-1}$. This proves that μ is a fuzzy ideal of $\mathcal{G}$. Since the chain of ideals $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots$ is not terminating, μ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy ideal is well ordered. The proof is complete. □

We note that a set is well ordered if and only if it does not contain any infinite descending sequence.

Theorem 3.18. *Let $\mathcal{G}$ be a right K(G)-algebra and let $S = \{t_i \mid i = 1, 2, 3, \cdots\} \cup \{0\}$ where $\{t_n\}$ is a strictly decreasing sequence in $(0, 1)$. Then the following assertions are equivalent.*

(i) *For every ascending sequence $A_1 \subset A_2 \subset \cdots$ of ideals of $\mathcal{G}$ there exists a natural number n such that $A_i = A_n$ for all $i \geq n$.*

(ii) *For each fuzzy ideal μ of $\mathcal{G}$, $\mathrm{Im}(\mu) \subset S$ implies that there exists a natural number n_0 such that $\mathrm{Im}(\mu) \subset \{t_i \mid i = 1, 2, \cdots, n_0\} \cup \{0\}$.*

Proof. If (i) is valid, then we know from Theorem 3.17 that $\mathrm{Im}(\mu)$ is a well ordered subset of $[0, 1]$ and so (ii) is valid. Assume that (ii) is true.

If the condition (i) is not valid, then there exists a strictly ascending chain of ideals $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots$. Define a fuzzy set μ in $\mathcal{G}$ by

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in A_1, \\ t_n & \text{if } x \in A_n \setminus A_{n-1},\ n = 2, 3, 4, \cdots, \\ 0 & \text{if } x \in G \setminus \bigcup_{n=1}^{\infty} A_n. \end{cases}$$

Since $e \in A_1$, we have $\mu(e) = t_1 \geq \mu(x)$ for all $x \in G$. Let $x, y \in G$. If either $x \odot y$ or $y \odot (y \odot x)$ belongs to $G \setminus \bigcup_{n=1}^{\infty} A_n$, then either $\mu(x \odot y)$ or $\mu(y \odot (y \odot x))$ is equal to 0. Hence

$$\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

If $x \odot y$, $y \odot (y \odot x) \in A_1$, then $x \in A_1$ and so

$$\mu(x) = t_1 = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

If $x \odot y$, $y \odot (y \odot x) \in A_n \setminus A_{n-1}$, then $x \in A_n$. Thus

$$\mu(x) \geq t_n = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

Assume that $x \odot y \in A_1$ and $y \odot (y \odot x) \in A_n \setminus A_{n-1}$ for $n = 2, 3, 4, \cdots$. Then $x \in A_n$ and hence

$$\mu(x) \geq t_n = \min\{t_1, t_n\} = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

Similarly for $x \odot y \in A_n \setminus A_{n-1}$ and $y \odot (y \odot x) \in A_1$ for $n = 2, 3, 4, \cdots$, we obtain

$$\mu(x) \geq t_n = \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}.$$

Hence μ is a fuzzy ideal of $\mathcal{G}$. This contradicts our assumption. □

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