Time Discretization of Nonlinear Systems with Variable Time-Delayed Inputs using a Taylor Series Expansion

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This paper proposes a new method of discretization for nonlinear systems using a Taylor series expansion and the zero-order hold assumption. The method is applied to sampled-data representations of nonlinear systems with input time delays. The delayed input varies in time and its amplitude is bounded. The maximum time-delayed input is assumed to be two sampling periods. The mathematical expressions of the discretization method are presented and the ability of the algorithm is tested using several examples. A computer simulation is used to demonstrate that the proposed algorithm accurately discretizes nonlinear systems with variable time-delayed inputs.

Key Words : Time Discretization, Nonlinear System, Time Varying, Delayed Input

1. Introduction

Time delays occur during information processing and data transmission in many engineering systems. With recent network improvements, many systems have been developed that are controlled via networks ; these often transfer data from a remote site. The time delay that occurs while transmitting data through the network is the most important factor affecting the overall system performance (Diop et al., 2001).

Time-delay systems cannot be solved in continuous-time space due to the infinite dimensions created by the time delay. The same problem occurs in linear time-invariant systems, which introduce more complexities and difficulties. For this reason, many control methods developed for finite-dimension systems have experienced difficulties when applied to a time-delayed system directly. Therefore, new design methods to control time-delayed systems precisely must be developed.

Many studies have attempted to solve time delay problems. Luo and Chung (2002) proposed a delay-dependent criterion that guaranteed the asymptotic stability of a linear uncertain system with a time delay. Nihtila et al. (1997) proposed a design method that made use of a real-time delay estimator for the input delay found in a single-input single-output (SISO) system with finite dimensions by transferring the time delayed part of the system to the transport system, which consisted of a linear partial differential equation. Cho and Park (2004) proposed a new impedance controller for bilateral teleoperations subjected to time delays. In addition, Choi and Baek (2002) applied time delay control (TDC) to single-axis magnetic levitation systems.

Two methods are generally used for time-delay systems. In the first method, the controller acts in

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continuous-time space based on a continuous-time model. It is then transformed to a digital controller (Im et al., 2000; Chen, 1984; Carcia-Sanz et al., 2001; Hohmann et al., 2001). Although this method has been used in many existing studies, the digital controller has many limitations due to the infinite dimensions of a delayed item in the time-delayed system. In the second method, the digital controller design is based on a discrete-time space model after transforming the continuous-time model into discrete-time space. Here, the digital controller is less restricted because the problem of the infinite dimensions of the time-delayed item in discrete-time space is avoided. Therefore, the second method is well suited for discretizing time-delayed systems.

This paper proposes a discretization method for nonlinear systems with a time-delayed input using a well-known existing discretization algorithm (Franklin et al., 1988; Vaccaro, 1995). A Taylor series expansion is used for the variable time delay (Isidori, 1989; Vidyasagar, 1978). Although Park et al.(2004a, b) proposed a discretization method for systems with a constant time delay using methods such as the scaling and squaring technique, and this time discretization method was also applied to nonlinear control systems with delayed multi-inputs (Zhang and Chong, 2005), these studies did not consider variable time delays. Difficulties are encountered when applying the results of these studies to an actual system directly because the delayed input values tend to vary with time in real systems.

The remainder of this paper is divided into the following sections. Section 2 presents the existing discretization method for nonlinear systems using a Taylor series expansion. Section 3 derives a discretization algorithm for a nonlinear system with a variable time delay for cases where the delay times change within two sampling periods. Section 4 describes the scaling and squaring technique. Section 5 presents a computer simulation of the algorithm proposed here. Finally, Section 6 summarizes the conclusions of this study and the directions of future work.

2. Discretization of Nonlinear Systems Using a Taylor Series Expansion

2.1 Systems without a time-delay

Consider a typical nonlinear system expressed by the state-space equation

$$\frac{dx(t)}{dt}=f(x(t))+u(t)g(x(t)) \tag{1}$$

Assume that the solution to Eq. (1) is

$$\begin{aligned}x(t)=&A_0+A_1(t-t_k)\\&+A_2(t-t_k)^2+A_3(t-t_k)^3+\cdots\end{aligned} \tag{2}$$

Expanding Eq. (2) using a power series gives

$$\begin{aligned}&x'(t)=A_1+2A_2(t-t_k)+3A_3(t-t_k)^2+\cdots\cdots\\&x''(t)=2A_2+3\cdot 2A_3(t-t_k)+4\cdot 3A_4(t-t_k)^2+\cdots\cdots\\&x'''(t)=3\cdot 2A_3+4\cdot 3\cdot 2A_4(t-t_k)+\cdots\cdots\\&\cdots\cdots\end{aligned} \tag{3}$$

where the coefficients of Eq. (2) can be calculated as follows for $t=t_k$ in Eq. (3):

$$\begin{aligned}&A_0=x(t_k),\ A_1=x'(t_k)\\&A_2=\frac{x''(t_k)}{2!},\ A_3=\frac{x'''(t_k)}{3!},\ \cdots\end{aligned} \tag{4}$$

In the case $t-t_k=T$, the state value of $t=t_k$ can be obtained after substituting Eq. (4) into Eq. (2):

$$\begin{aligned}x(t)=&x(t_k)+x'(t_k)(t-t_k)\\&+\frac{x''(t_k)}{2!}(t-t_k)^2+\frac{x'''(t_k)}{3!}(t-t_k)^3+\cdots\cdots\\=&x(t_k)+\sum_{l=1}^{\infty}\frac{T^l}{l!}\frac{d^lx}{dt}\bigg|_{t_k}.\end{aligned} \tag{5}$$

After applying a Taylor series to Eq. (5), the solution to Eq. (1) can be expressed using a uniformly convergent Taylor series (Park et al., 2004a, b), and each coefficient can be obtained using the continous partial differential equation for the right-hand side of Eq. (1),

$$\begin{aligned}x(k+1)=&x(k)+\sum_{l=1}^{\infty}\frac{T^l}{l!}\frac{d^lx}{dt^l}\bigg|_{t_k}\\&=x(k)=\sum_{l=1}^{\infty}A^{[l]}(x(k),\ u(k))\frac{T^l}{l!}\end{aligned} \tag{6}$$

where $x(k)$ gives the value of the state vector

x for $t=t_k=kT$, and $A^{[l]}(x, u)$ is a recursively determined function

$$A^{[1]}(x, u)=f(x)+ug(x)$$
$$A^{[l+1]}(x, u)=\frac{\partial A^{l}(x, u)}{\partial x}(f(x)+ug(x)) \quad (7)$$

with $l=1, 2, 3, \cdots$

The exact representation of Eq. (1) can be obtained from an expression of the state vector as an infinite series using the Taylor series expansion presented in Eq. (6), which also can be written as follows

$$\begin{aligned} x(k+1) &= \Phi_T(x(k), u(k)) \\ &= x(k)+\sum_{l=1}^{\infty}A^{[l]}(x(k), u(k))\frac{T^l}{l!} \end{aligned} \quad (8)$$

An approximated representation for Eq. (1) can also be obtained using an order N Taylor series of the state vector.

$$\begin{aligned} x(k+1) &= \Phi_T^N(x(k), u(k)) \\ &= x(k)=\sum_{l=1}^{N}A^{[l]}(x(k), u(k))\frac{T^l}{l!} \end{aligned} \quad (9)$$

where the subscript T of Φ_T^N is the sampling period of the sampled-data representation obtained from the discretization and the superscript N is the finite number of the series used in the approximated equation.

Remark 1. In general $A^{[l]}(x, u)$ is an l^{th} degree polynomial in the input variable u,

$$\begin{aligned} A^{[l]}(x, u) &= a_0^{[l]}(x)+a_1^{[l]}(x)u \\ &\quad +a_2^{[l]}(x)u^2+\cdots+a_l^{[l]}(x)u^l \end{aligned} \quad (10)$$

In view of representation (10), the series expansion (6) can be rewritten as

$$\begin{aligned} x(k+1) &= \Phi_T(x(k), u(k)) \\ &= x(k)+\sum_{l=1}^{\infty}\sum_{m=0}^{l}[u(k)]^m a_m^{[l]}(x(k))\frac{T^l}{l!} \end{aligned} \quad (11)$$

The series expansion (6) (or (8) and (9)) can also be expressed in operator form. Using the zero-order hold (ZOH) assumption, a new discretization approach can be naturally formulated within the context of a Lie series theory for nonlinear autonomous ordinary differential equations (ODEs). The following definition is deemed essential.

Definition 1. Given f, an analytic vector field on R^n and h, and an analytic scalar field on R^n, the Lie derivative of h with respect to f is defined in local coordinates as

$$L_f h(x)=\frac{\partial h}{\partial x_1}f_1+\cdots+\frac{\partial h}{\partial x_n}f_n$$

In light of Definition 1, the solution to the recursive relation (7) may be represented in terms of higher-order Lie derivatives as follows:

$$A_i^{[l]}(x, u)=(L_f+uL_g)^l x_i \quad (12)$$

where the subscript $i=1, \cdots, n$ denotes the i^{th} component and $L_f=\sum_{i=1}^{n}f_i(x)\frac{\partial}{\partial x_i}$ and $L_g=\sum_{i=1}^{n}g_i(x)\frac{\partial}{\partial x_i}$ are Lie derivative operators. This allows the series expansion (6) to be represented as a uniformly convergent Lie series for the Exact Sampled-Data Representation (ESDR),

$$\begin{aligned} x_i(k+1) &= \Phi_{i,T}(x(k), u(k)) \\ &= x_i(k)+\sum_{i=1}^{\infty}(L_f+uL_g)^l x_i|_{(x(k),u(k))}\frac{T^l}{l!} \end{aligned} \quad (13)$$

and similarly for the Approximate Sampled-Data Representation (ASDR)

$$\begin{aligned} x_i(k+1) &= \Phi_{i,T}^N(x(k), u(k)) \\ &= x_i(k)+\sum_{l=1}^{N}(L_f+uL_g)^l x_i|_{(x(k),u(k))}\frac{T^l}{l!} \end{aligned} \quad (14)$$

with $i=1, \cdots, n$.

2.2 Linear system with a time-delay

The previous section considered an algorithm for a system without a time delay using a Taylor series expansion. In this section, we discuss a discretization method using a Taylor series for a system with a time delay in the input. First, consider a constant time delay in a linear system:

$$\frac{dx(t)}{dt}=Ax(t)+bu(t-D) \quad (15)$$

where A and b are constant matrices of proper order. Assume that the time interval $(t_i, t_f)=[kT, (k+1)T)$ is the sampling interval and T is the sampling period. If the input has a constant value during the sampling period, i.e., the ZOH assumption holds, the input can be denoted as

$$\begin{aligned} u(t) &= u(kT)\equiv u(k) \\ &= \text{constant for } kT\le t<kT+T \end{aligned} \quad (16)$$

In addition, the time delay is

$$D=qT+\gamma \tag{17}$$

where D is an integer of $q\in\{0, 1, 2, \cdots\}$ and a real number of $0<\gamma\leq T$. The delayed input assumed by the ZOH assumption and these expressions is determined by

$$u(t-D)=\begin{cases} u(kT-qT-T)\equiv u(k-q-1) & \text{if } kT\leq t<kT+\gamma \\ u(kT-qT)\equiv u(k-q) & \text{if } kT+\gamma\leq t<kT+T \end{cases} \tag{18}$$

The state values for the arbitrary time interval of $I=[t_i, t_f)$ and $u=u_c$=constant can be obtained at $t=t_f$ as follows:

$$x(t_f)=\exp(A(t_f-t_i))x(t_i)+u_c\int_{t_i}^{t_f}\exp(A(t_f-\tau))bd\tau \tag{19}$$

The input values with time delays in the sampling periods are determined according to the time interval; therefore, the state values for the time period can be obtained by applying these values,

$$x(kT+\gamma)=\exp(A\gamma)x(kT)+u(k-q-1)\int_{kT}^{kT+\gamma}\exp(A(kT+\gamma-\tau))bd\tau \tag{20}$$

$$x(kT+T)=\exp(A(T-\gamma))x(kT+\gamma)+u(k-q)\int_{kT+\gamma}^{kT+T}\exp(A(kT+T-\tau))bd\tau \tag{21}$$

By substituting Eq. (20) into Eq. (21),

$$\begin{aligned} x(kT+T)&=\exp(A(T-\gamma))x(kT+\gamma)\\ &+u(k-q)\int_{kT+\gamma}^{kT+T}\exp(A(kT+T-\tau))bd\tau\\ &=\exp(A(T-\gamma))\exp(A\gamma)x(kT)\\ &+u(k-q)\int_{kT+\gamma}^{kT+T}\exp(A(kT+T-\tau))bd\tau\\ &+\exp(A(T-\gamma))u(k-q-1)\int_{kT}^{kT+\gamma}\exp(A(KT+\gamma-\tau))bd\tau\\ &\Rightarrow x(kT+T)=\exp(AT)x(kT)+\Gamma_1u(k-q-1)+\Gamma_0u(k-q) \end{aligned} \tag{22}$$

where $\Gamma_1=\int_{T-\gamma}^{T}\exp(A\tau)\,bd\tau$ and $\Gamma_0=\int_0^{T-\gamma}\exp(A\tau)\,bd\tau$. Eq. (22) gives a sampled-data representation for a continuous-time space system with a time delay of D. By using Eq. (17), the values of the state vector at $(k+1)T$ can be calculated as a linear combination between the values of the state vector for kT and the past values of the input variable u for $(k-q-1)T$ and $(k-q)T$.

2.3 Nonlinear system with a time-delay

This section considers a discretization algorithm for a nonlinear system with a constant time-delay. Assume a nonlinear system described by

$$\frac{dx(t)}{dt}=f(x(t))+g(x(t))u(t-D) \tag{23}$$

If we apply the ZOH assumption to the above system, the delayed values of the input can be described over two different time intervals, as shown in Eq. (18). Therefore, the state values for $kT+\gamma$ can be obtained by applying the input values of the time interval $[kT, kT+\gamma)$,

$$x(kT+\gamma)=\Phi_\gamma(x(kT), u(k-q-1)) \tag{24}$$

where Φ_γ can be derived directly using Eq. (8).

In the same manner, the values of the state vector for $(k+1)T$ can be obtained by applying the input values of the time interval $[kT+\gamma, (k+1)T)$,

$$x(kT+T)=\Phi_{T-\gamma}(x(kT+\gamma), u(k-q)) \tag{25}$$

The sampled-data representation of the nonlinear system (23) can be obtained by using Eqs. (24) and (25) as follows:

$$\begin{aligned} x(k+1)&=\Phi_T^D(x(k), u(k-q-1), u(k-q))\\ &=\Phi_{T-\gamma}(\Phi_\gamma(x(k), u(k-q-1)), u(k-q)) \end{aligned} \tag{26}$$

If the finite series truncation order N is applied to Eq. (26), the approximated sampled-data representation is

$$x(k+1)=\Phi_T^{N,D}(x(k), u(k-q-1), u(k-q)) \tag{27}$$

Theorem 1. Let x^0 be an equilibrium point of the original nonlinear continuous-time system (1) that belongs to the continuous-time equilibrium manifold $E^C=\{x\in R^n \mid \exists u\in R : f(x)+g(x)u=0\}$ and $u=u^0$ be the corresponding equilibrium value of the input variable $f(x^0)+g(x^0)u^0=0$. Then, x^0 belongs to the discrete-time equilibrium manifold $E^d=\{x\in R^n \mid \exists u\in R : \Phi_T^D(x, u)=x\}$ of the ESDR (or ASDR) obtained under the proposed Taylor-Lie discretization method, with $u=u^0$ being the corresponding equilibrium value of the input variable $\Phi_T^D(x^0, u^0)=x^0$ (or $\Phi_T^{N,D}(x^0, u^0)=x^0$).

Proof. Note that the Taylor-Lie $A^{[l]}$ coefficient defined recursively by Eq. (7) vanishes at (x^0, u^0) since the latter belongs to the equilibrium manifold E^c, $A^{[l]}(x^0, u^0)=0$ for all $l\in\{1, 2, 3, \cdots\}$. It can be easily deduced that $\Phi_\gamma(x^0, u^0)=x^0$ and

$$\Phi_T^D(x^0, u^0)=\Phi_{T-\gamma}(\Phi_\gamma(x^0, u^0), u^0)=\Phi_\gamma(x^0, u^0)=x^0$$

Similar arguments apply to the $\Phi_T^{N,D}$ map of the ASDR. Therefore, x^0 belongs to the discrete-time equilibrium manifold E^d of the ESDR or ASDR for any finite truncation of order N.

Theorem 1 essentially states that equilibrium properties are preserved under the proposed Taylor-Lie discretization method. This is a very important property and an advantageous feature of the proposed discretization method from a digital controller synthesis point of view.

3. Discretization of a Nonlinear System with a Variable Time-delayed Input

3.1 A time delay smaller than the sampling period

Consider the nonlinear system described by

$$\frac{dx(t)}{dt}=f(x(t))+g(x(t))u(t-D(t)) \quad (28)$$

Assume that Eq. (28) will be discretized to obtain a sampling period of $T=t_{k+1}-t_k>0$. The values of the time delay for the k^{th} sampling period can be expressed as

$$D_k=q_kT+\gamma_k \quad (29)$$

where $q_k=0$ and $0\leq\gamma_k<1$ is a real number. When the values of the delay are smaller than a single sampling period, the time interval for the k^{th} sampling period can be divided into two different sections, $[kT, kT+\gamma_k)$ and $[kT+\gamma_k, kT+T)$, based on the point in time that the delay occurs. This is because the maximum delayed input is located within a single sampling period. In this case, the values of γ_k become an important factor in the calculation of the state values because they can determine the delayed input values in the sampling periods. Therefore, when a time delay occurs in the k^{th} sampling period, the input values applied to the system can be expressed according to the interval

$$\begin{aligned} u(t-D_k)&=u(k-q_k-1),\ kT\leq t<(kT+\gamma_k) \\ &=u(k-q_k),\ (kT+\gamma_k)\leq t<(kT+T) \end{aligned} \quad (30)$$

The discretization of the nonlinear system given by Eq. (28) with an input represented by Eq. (30) using a Taylor series expansion is as follows:

$$\begin{aligned} x(kT+\gamma_k)&=x(kT) \\ &+\sum_{l=1}^{\infty}A^l(x(kT), u(k-q_k-1))\frac{\gamma_k^l}{l!}, \\ &kT\leq t(kT+\gamma_k) \end{aligned} \quad (31)$$

$$\begin{aligned} x(kT+T)&=x(kT+\gamma_k) \\ &+\sum_{l=1}^{\infty}A^l(x(kT+\gamma_k), u(k-q_k))\frac{(T-\gamma_k)^l}{l!} \\ &(kT+\gamma_k)\leq t<(kT+T) \end{aligned} \quad (32)$$

Substituting Eq. (31) into Eq. (32) gives

$$\begin{aligned} x(kT+T)&=x(kT+\gamma_k) \\ &+\sum_{l=1}^{\infty}A^l((x(kT)+\sum_{l=1}^{\infty}A^l(x(kT), \\ &u(k-q_k-1))\frac{\gamma_k^l}{l!}), u(k-q_k))\frac{(T-\gamma_k)^l}{l!} \end{aligned} \quad (33)$$

Approximating Eq. (33) to order N,

$$\begin{aligned} x(kT+T)&=x(kT+\gamma_k) \\ &+\sum_{l=1}^{N}A^l((x(kT)+\sum_{l=1}^{N}A^l(x(kT), \\ &u(k-q_k-1))\frac{\gamma_k^l}{l!}), u(k-q_k))\frac{(T-\gamma_k)^l}{l!} \end{aligned} \quad (34)$$

Therefore, a nonlinear system with a variable time-delayed input smaller than one sampling period can be discretized in discrete-time space to order N as shown in Eq. (34).

3.2 A time delay smaller than twice the sampling period

Reconsider the nonlinear system given by Eq. (28). In addition, assume that the time-delay for the k^{th} sampling period is $D_k=q_kT+\gamma_k$, where $q_k=0, 1, 2, \cdots$ is an integer and $0\leq\gamma_k<1$ is a real number. Moreover, assume that the delayed values in the present and previous sampling periods are already known. If the values of the variable time-delay are larger than the sampling scale, the input values will be beyond the sampling periods and affect the next sampling period.

As a result, a system that has a single input will be applied over two inputs according to the magnitude of the time delay. In this case, if the previous input before a given step is supplied after the input of that step, the input of the previous step will be neglected and only that of the present step will affect the system. Therefore, if the values of the variable time-delay are larger than one sampling period, a verification of which input affects the system for each sampling period is required.

In this study, the factors for each time delay are compared to solve this problem. The k^{th} time delay can be expressed as $D_k=q_kT+\gamma_k$, where q_k represents multiple sampling periods and γ_k identifies the location of the time delay in the sampling interval. It is possible to determine where the time-delayed inputs are applied to the system by checking the values of q_k. If two inputs exist in a single sampling interval, it is possible to verify which input values are applied to the system by comparing the values of γ_k. These processes are illustrated in Fig. 1. If $q_{k-1}=0$ and $q_k=0$, the time-delayed input for the k^{th} input is applied to the k^{th} period. Therefore, the input values in the k^{th} period can be determined from

$$\begin{aligned} u(t)&=u(k-q_k-1),\ kT\le t<kT+\gamma_k \\ u(t)&=u(k-q_k),\ kT+\gamma_k\le t<(k+1)T \end{aligned} \tag{35}$$

When $q_{k-1}=0$ and $q_k=1$, the $(k-1)^{th}$ input will affect the system because the k^{th} input cannot affect the k^{th} period. The input values for this period are shown in Fig. 2 and can be expressed as

$$u(t)=u(k-q_k),\ kT\le t<(k+1)T \tag{36}$$

When $q_{k-1}=1$ and $q_k=0$, there are two input values in a single sampling interval. Therefore, it is necessary to check which input affects the system first. For the condition $\gamma_{k-1}\ge\gamma_k$, the $(k-1)^{th}$ input will be neglected because the k^{th} input is applied to the system before the $(k-1)^{th}$ input arrives. Therefore, the input values are as shown in Fig. 3 and defined as

$$\begin{aligned} u(t)&=u(k-q_k-2),\ kT\le t<kT+\gamma_k \\ u(t)&=u(k-q_k),\ kT+\gamma_k\le t<(k+1)T \end{aligned} \tag{37}$$

Conversely, for the condition of $\gamma_{k-1}<\gamma_k$, both the $(k-1)^{th}$ and the k^{th} inputs affect the system. Hence, the input values are as shown in Fig. 4 and defined as

$$\begin{aligned} u(t)&=u(k-q_k-2),\ kT\le t<kT+\gamma_{k-1} \\ u(t)&=u(k-q_k-1),\ kT+\gamma_{k-1}\le t<kT+\gamma_k \\ u(t)&=u(k-q_k),\ kT+\gamma_k\le t<(k+1)T \end{aligned} \tag{38}$$

Finally, when $q_{k-1}=1$ and $q_k=1$, the $(k-1)^{th}$ input can only affect the k^{th} sampling period. Therefore, the input values are as shown in Fig. 5

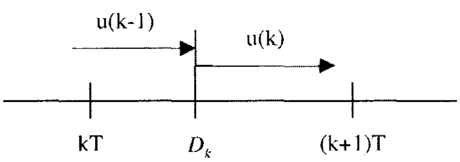

Fig. 1 Input values when $q(k-1)=0$ and $q(k)=0$

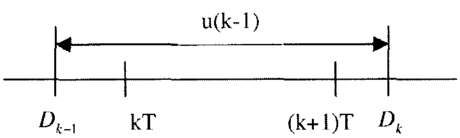

Fig. 2 Input values when $q(k-1)=0$ and $q(k)=1$

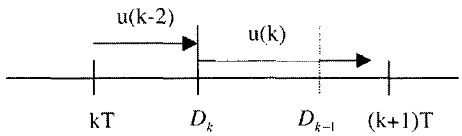

Fig. 3 Input values when $q(k-1)=1$ and $q(k)=0$, $r(k-1)\ge r(k)$

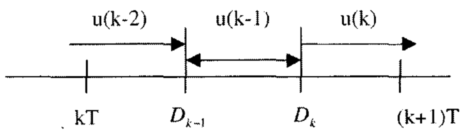

Fig. 4 Input values when $q(k-1)=1$ and $q(k)=0$, $r(k-1)<r(k)$

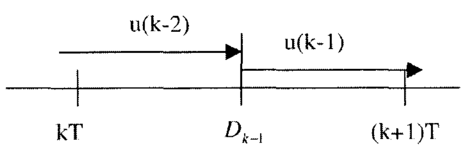

Fig. 5 Input values when $q(k-1)=1$ and $q(k)=1$

and defined as

$$u(t)=u(k-q_k-1),\ kT\le t<kT+\gamma_{k-1}$$
$$u(t)=u(k-q_k),\ kT+\gamma_{k-1}\le t<(k+1)T \qquad (39)$$

By substituting the results into the Taylor series expansion equation approximated to order N, the discretization equation for a nonlinear system with a variable time delay in which the values of the time delay are less than twice the sampling period can be obtained as follows. When $q_{k-1}=0$ and $q_k=0$, the values of the state vector can be expressed as

$$\begin{aligned}
&x(kT+\gamma_k)=x(kT)+\sum_{l=1}^{N}A^l(x(kT),\ u(k-q_k-1))\frac{\gamma_k^l}{l!},\\
&\qquad kT\le t<kT+\gamma_k\\
&x(kT+T)=x(kT+\gamma_k)\\
&\qquad +\sum_{l=1}^{N}A^l(x(kT+\gamma_k),\ u(k-q_k))\frac{(T-\gamma_k)^l}{l!},\\
&\qquad kT+\gamma_k\le t<(k+1)T
\end{aligned}\qquad (40)$$

When $q_{k-1}=0$ and $q_k=1$, the values of the state vector are

$$\begin{aligned}
&x(kT+T)=x(kT)\\
&\qquad +\sum_{l=1}^{N}A^l(x(kT),\ u(k-q_k))\frac{(T-\gamma_k)^l}{l!},\\
&\qquad kT\le t<(k+1)T
\end{aligned}\qquad (41)$$

When $q_{k-1}=1$ and $q_k=0$ for $\gamma_{k-1}\ge\gamma_k$,

$$\begin{aligned}
&x(kT+\gamma_k)=x(xT)+\sum_{l=1}^{N}A^l(x(kT),\ u(k-q_k-2))\frac{\gamma_k^l}{l!},\\
&\qquad kT\le t<kT+\gamma_k\\
&x(kT+T)=x(kT+\gamma_k)\\
&\qquad +\sum_{l=1}^{N}A^l(x(kT+\gamma_k),\ u(k-q_k))\frac{(T-\gamma_k)^l}{l!},\\
&\qquad kT+\gamma_k\le t<(k+1)T
\end{aligned}\qquad (42)$$

For $\gamma_{k-1}<\gamma_k$

$$\begin{aligned}
&x(kT+\gamma_{k-1})=x(kT)+\sum_{l=1}^{N}A^l(x(kT),\ u(k-q_k-2))\frac{\gamma_{k-1}^l}{l!},\\
&\qquad kT\le t<kT+\gamma_{k-1}\\
&x(kT+\gamma_k)=x(kT+\gamma_{k-1})\\
&\qquad +\sum_{l=1}^{N}A^l(x(kT+\gamma_{k-1}),\ u(k-q_k-1))\frac{(\gamma_k-\gamma_{k-1})^l}{l!},\\
&\qquad kT+\gamma_{k-1}\le t<kT+\gamma_k\\
&x(kT+T)=x(kT+\gamma_k)\\
&\qquad +\sum_{l=1}^{N}A^l(x(kT+\gamma_k),\ u(k-q_k))\frac{(T-\gamma_k)^l}{l!},\\
&\qquad kT+\gamma_k\le t<(k+1)T
\end{aligned}\qquad (43)$$

Finally, when $q_{k-1}=1$ and $q_k=1$,

$$\begin{aligned}
&x(kT+\gamma_{k-1})=x(kT)\\
&\qquad +\sum_{l=1}^{N}A^l(x(kT),\ u(k-q_k-1))\frac{\gamma_{k-1}^l}{l!}kT\le t<kT+\gamma_{k-1},\\
&\qquad kT+\gamma_{k-1}\le t<(k+1)T
\end{aligned}\qquad (44)$$

4. Scaling and Squaring Technique

The Taylor series method can provide accurate results. However, the order N must be very large in order to achieve the desired accuracy if the sampling interval T is also large. This is due to the probability that when T is very large, $A^{[l]}T^l/l!$ can become extremely large due to finite-precision arithmetic before it becomes small at higher powers when convergence takes over. For a linear system, this phenomenon occurs when calculating e^{AT} and $\int_0^T e^{AT}dt$, which causes a computer overflow error. A 'scaling and squaring' technique, which is also known as 'extrapolation to the limit' in the numerical analysis literature, can be applied to solve this type of problem. This technique is commonly used to calculate the exponential matrix $\exp(AT)$ for large sampling periods by subdividing the sampling interval T into two or more subintervals of equal length. An appropriate positive integer m is chosen such that $T/2^m$ is small enough to calculate the exponential matrix. In our case, the sampling period T is subdivided into 2^m equally spaced subintervals of length $T/2^m$ over which the exponential matrix is calculated. Squaring the matrix $\exp(AT/2^m)$ m times gives $\exp(AT)$:

$$\exp(AT)=\left(\left(\left(\exp\left(A\frac{T}{2^m}\right)\right)^2\right)^{\cdots}\right)^2 \qquad (45)$$

The scaling and squaring technique can be extended to nonlinear cases by applying the Taylor series method. When working on a particular analogue, one can use nonlinear operators and powers of operators as substitutes for matrices and matrix products. Subsequently, the key idea utilized in the nonlinear analogue of the scaling and squaring technique remains the same as presented for the linear case. When T is sufficiently large, one can divide the interval $[t_k,\ t_{k+1})$ into 2^m equally spaced subintervals and use a small

Taylor expansion of order N with a time step of $T/2^m$ for the 2^m intermediate subintervals as a substitution for the larger order N' used in the single-step Taylor method case. Assume that $\Omega(N', T): R^n \to R^n$ is the operator that corresponds to the Taylor expansion of order N' with a time step T, and when it acts on $x(kT)$, the outcome is

$$x(kT+T)=\Omega(\tilde{N}, T)x(kT) \qquad (46)$$

where $\Omega(\tilde{N}, T)(\bullet)=I+\sum_{l=1}^{\tilde{N}} A^{[l]}(x(k), u(k))\frac{T^l}{l!}$. Using operator notation, the resulting discrete-time system can be written as

$$x(kT+T)=\left[\Omega\left(N, \frac{T}{2^m}\right)\right]^{2m} x(kT) \qquad (47)$$

The above ASDR can be viewed as the direct result of combining Taylor's method and the scaling and squaring technique.

The choice of parameters N and m is important. Different values reflect different requirements of the discretization performance. The criterion for selecting an appropriate m involves comparing the magnitude of the sampling period T with the fastest time constant $1/\rho$ of the original continuous-time system. If T is small compared to $2/\rho$, we set $m=0$ and apply the single-step Taylor series method. Since T is small, a low-order N single-step Taylor discretization method is usually sufficient to meet the expected accuracy requirements. When T is larger than the fastest time constant $1/\rho$, we apply the scaling and squaring discretization technique. The sampling interval is then subdivided into 2^m subintervals and a low-order N single-step Taylor discretization method is applied to each subinterval. These subdivisions require that the following inequality hold:

$$\frac{T}{2^m}<\frac{2}{\rho} \qquad (48)$$

since the requirements for numerical convergence and stability must also be met. The positive integer m is assigned as

$$m=\max\left(\left[\log_2\left(\frac{T}{\theta}\right)\right]+1, 0\right) \qquad (49)$$

where $\theta<2/\rho$ is chosen arbitrarily and $[x]$ denotes the integer part of the number x. It is evident that smaller values of the arbitrarily selected number θ result in more stringent bounds on $T/2^m$.

5. Computer Simulations

5.1 Simple chemical processing system

A typical continuous stirred tank reactor (CSTR) system was simulated to verify the proposed algorithm. The system equation can be expressed as

$$\begin{aligned} x'(t)=&-x^2(t)-3x(t) \\ &+u(t-D(t))(1-x(t)) \end{aligned} \qquad (50)$$

The initial condition is $x(0)=0$, and the input and time-delay is applied using a sinusoidal wave. This simulation consists of six cases in which the period of the sinusoidal delay and the sampling period are changed. In addition, this study assumes that the results of the Taylor series method and the MATLAB ODE solver are the exact values of the system. It is possible to verify the validity of using the MATLAB ODE solver results as a reference from Park et al. (2004a, b).

Figure 6 shows the state values and relative errors for the system in which the input is given by $u(t-D(t))=0.9\sin(((t-D(t))/4)$ and the time delay is $D(t)=0.04\ \sin(t/4)+0.05$ for a sampling period of $T=0.05$s. As shown in the figure, the maximum error of the state values did not exceeded 1%. Figures 7 to 9 present the results for the same input applied to systems where the

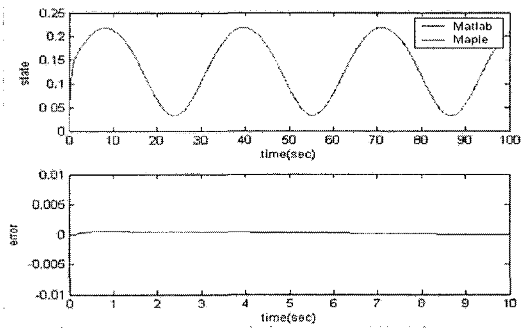

Fig. 6 State errors and values of the CSTR; $D(t)=0.04\ \sin(t/4)+0.05$, $T=0.05$s

time delay is given by $D(t)=0.04\ \sin(t/n)+0.05$, where $n=8$, 12, and 16, respectively. Figures 10 and 11 give the state values and relative errors for systems with the same input and time delay used in Figure 6, but with sampling periods of $T=0.01$ and 0.005s, respectively. The RMS values

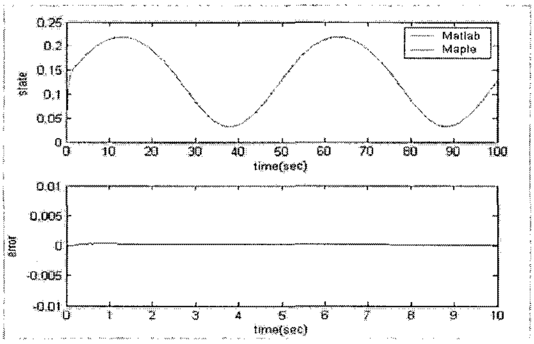

Fig. 7 State errors and values of the CSTR ; $D(t)=0.04\ \sin(t/4)+0.05$, $T=0.05$s

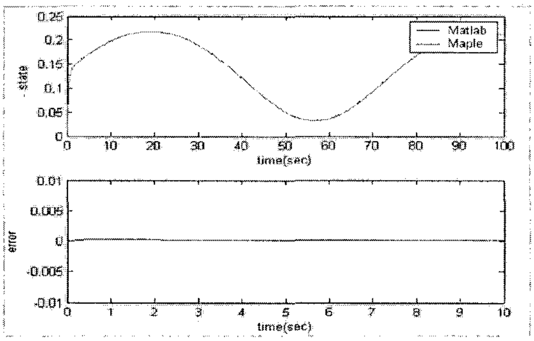

Fig. 8 State errors and values of the CSTR ; $D(t)=0.04\ \sin(t/12)+0.05$, $T=0.05$s

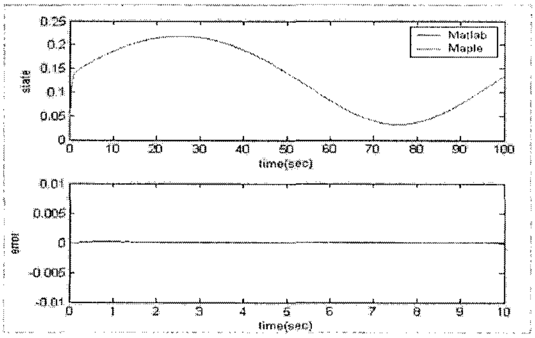

Fig. 9 State errors and values of the CSTR ; $D(t)=0.04\ \sin(t/16)+0.05$, $T=0.05$s

were 0.0018, 0.0014, and 0.0009 for sampling periods $T=0.05$, 0.01, and 0.005s, respectively; therefore, the RMS value was decreased by the shorter sampling periods.

5.2 Second-order nonlinear system

This section presents a slightly more complex second-order nonlinear system with a variable time-delay. The system equation is

$$x''(t)=x'(t)(1-x^2(t))-x(t)+u(t-D(t)) \quad (51)$$

The initial conditions are $x(0)=0.1$ and $x'(0)=0$, and the input and time-delay are applied using a sinusoidal wave. In order to use the discretization Taylor series algorithm, the system is changed to a state-space equation. If we assume the state of the system is

$$X_1=x,\ X_2=x' \quad (52)$$

the state variables can be expressed as

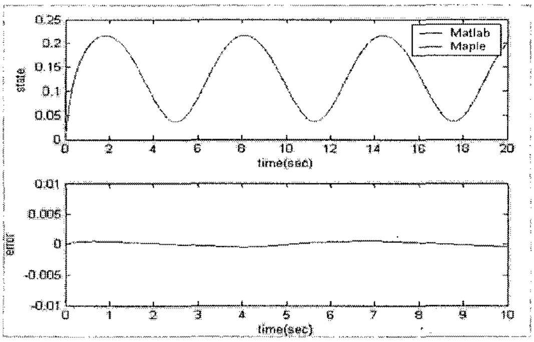

Fig. 10 State errors and values of the CSTR ; $D(t)=0.04\ \sin(t/4)+0.05$, $T=0.01$s

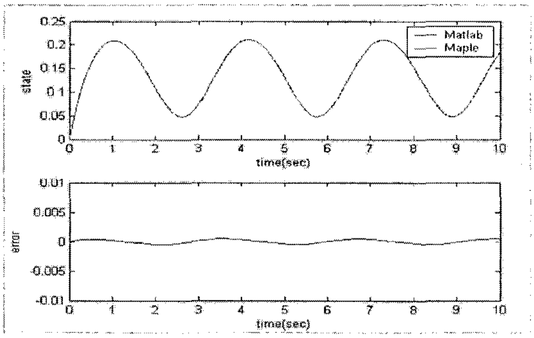

Fig. 11 State errors and values of the CSTR ; $D(t)=0.04\ \sin(t/4)+0.05$, $T=0.005$s

$$X_1' = f_1(X) + g_1(X)u = X_2$$
$$X_2' = f_2(X) + g_2(X)u = X_2(1 - X_1^2) - X_1 + u \quad (53)$$

The same simulation used for the CSTR was applied to the second-order nonlinear system. Figures 12 and 13 show the results of the computer simulation for a system in which the input is given by $u(t-D(t)) = \sin(10(t-D(t))$ and the time delay is $D(t) = 0.0009\sin(t) + 0.001$ for a sampling period of $T=0.001$s. The figures show that the state values in continuous-time space are close to those in discrete-time space. In addition, the errors of the state values between the two different time domains are quite small. The RMS value is 9.2093×10^{-5} for state X_1 and 4.0962×10^{-4} for state X_2.

The simulation results demonstrate that the values of the state error decrease with the sampling period of the system. If the sampling period remains constant, the state errors decrease with smaller values of the time delay. This is because the ZOH assumption was applied to the time-delayed values in the simulation, i.e., some errors arose when the values of the time delay did not follow the sampling period in continuous-time space exactly because the values of the variable time-delay in discrete-time space were constant over the sampling period and the values were maintained at the point of time of the sampling. In addition, the errors of the delayed values affected the calculation of the state values, creating state errors. Therefore, the performance of the discretization algorithms for variable time-delay values in a nonlinear system is very important.

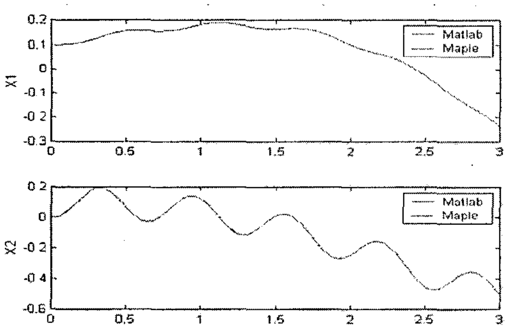

Fig. 12 State values of the second-order nonlinear system ($T=0.001$s)

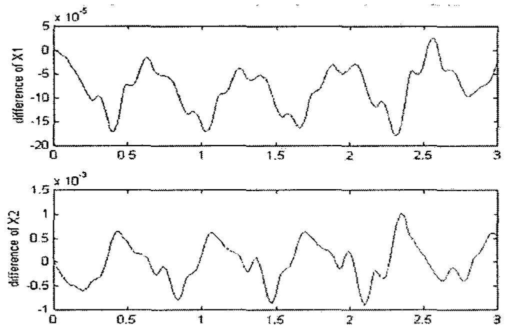

Fig. 13 State differences of the second-order nonlinear system ($T=0.001$s)

5.3 Scaling and squaring technique

In this section, we show that the scaling and squaring technique gives more precise discretization results when the sampling period of the system is large. Consider a nonlinear CSTR system as follows:

$$x'(t) = -x^2(t) - 3x(t) + u(t-D(t))(1-x(t)) \quad (54)$$

The sampling period is $T=2.0$s. According to Park et al. (2004a), an ideal sampling period for a CSTR is less than 0.6s, so this presents a suitable problem to verify the performance of the scaling and squaring technique. Figure 14 shows the results of the discretization using the scaling and squaring technique. The maximum error is less than 10^{-4}, indicating the usefulness of the technique when a system has a large sampling period.

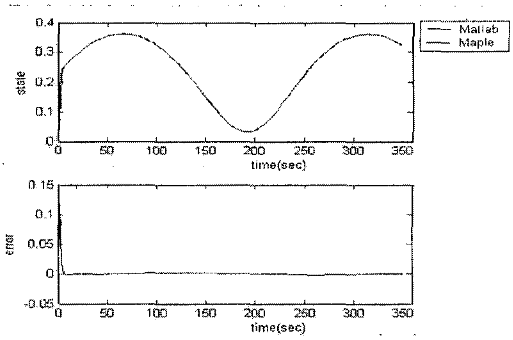

Fig. 14 Results obtained using the scaling and squaring technique

6. Conclusions

This paper proposed a discretization method that used a Taylor series expansion for nonlinear systems with variable time-delay inputs. Computer simulations with various examples were used to verify the proposed algorithm. The time-delay was typically restricted to less than twice the sampling period for each simulation, and the zero-order hold assumption was applied in the discretization of the variable time-delay for each sampling period. The results showed that the values of the state error decreased with smaller system sampling periods. When the sampling period was held constant, the state errors decreased with smaller time delay values. The maximum state errors did not exceed 1% for state values in continuous-time space, and the errors were approximately 0.01% of those obtained for a system with more appropriate sampling period and time delay values. The results demonstrated that the discretization algorithms proposed here produced satisfactory results, as verified by the RMS values of the state error for different cases. In the future, we plan to apply the first-order hold assumption to variable time delay values to reduce the maximum state error.

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