J. Korean Math. Soc. **43** (2006), No. 3, pp. 677–690

A NOTE ON SOME CLOSURE TYPE PROPERTIES IN FUNCTION SPACES

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ABSTRACT. We show that a group of three closure properties including the selectively Pytkeev property coincide on $C_k(X)$ for a locally compact X. We also introduce a new game characterization of these properties for such spaces.

1. Introduction

In this paper we use the usual topological notation and terminology, as in [1]. All spaces are assumed to be Tychonoff (unless otherwise specified).

For a space X by $C(X)$ we denote the set of all continuous real-valued functions defined on X. $C_k(X)$ is the space $(C(X), \tau)$ where τ is the compact-open topology, a typical subbase of which consists of all the sets of the form $\{f \in C(X) : f[K] \subseteq U\}$ for a compact $K \subseteq X$ and an open subset U of the real line. We use the symbol o to denote the constantly zero function defined on X. One local neighborhood base of this topology at the point o consists of all the sets $O(K, \varepsilon) := \{f \in C(X) : f[K] \subseteq (-\varepsilon, \varepsilon)\}$, for a compact $K \subseteq X$ and ε a positive real number.

In 1996 Reznichenko (at a seminar at the Moscow State University) introduced a property now called the Reznichenko property (see [4]) or the weakly Fréchet-Urysohn property (see [6]):

A space X is said to have the Reznichenko property at the point $x \in X$ provided that $x \in \overline{A} \setminus A$ and $A \subseteq X$ imply the existence of a sequence $(A_n : n \in N)$ of pairwise disjoint finite subsets of A such that for every neighborhood V of x there is a $n_0 \in N$ with $\forall n \geq n_0$ $(V \cap A_n \neq \emptyset)$. If

Received August 12, 2005.
2000 Mathematics Subject Classification: 54C35, 91A44.
Key words and phrases: compact-open topology, selection principles, selectively Pytkeev, strictly Fréchet, selectively Reznichenko, k-covers, infinite games.

X has that property at all of its points then X is simply said to have the Reznichenko property.

The Reznichenko property in function spaces $C_k(X)$ was studied in [2], and in hyperspaces in [3].

We say that X has the Pytkeev property at the point $a \in X$ (see e.g. [6]) if $a \in \overline{A} \setminus A$ implies the existence of a π-network at a consisting of countably *infinite* subsets of A. If X has the Pytkeev property at each of its points, X is simply said to have the Pytkeev property.

It is often the case that for some topological properties one considers their selective versions (for example, see [7]). We are interested in the selective versions of the two closure properties mentioned above. We make this idea precise in what follows.

DEFINITION 1.1. A space X is said to have the selectively Reznichenko property at the point $x \in X$ if for each sequence $(A_n : n \in N)$ of subsets of X with $x \in \overline{A_n} \setminus A_n$ for all $n \in N$, there is a sequence $(B_n : n \in N)$ of pairwise disjoint sets such that each B_n is a finite subset of A_n and such that for each open neighborhood V of x there is an $n_0 \in N$ with $\forall n \geq n_0$ $(V \cap B_n \neq \emptyset)$. If X has that property at all of its points then X is simply said to have the selectively Reznichenko property.

In [5] the authors have actually considered the selective version of the Reznichenko property (see also [3]).

The paper [8] is entirely consecrated to the selectively Reznichenko property in the spaces $C_p(X)$ ($C(X)$ endowed with the topology of pointwise convergence), and there it was characterized both game-theoretically and dually via a suitable selection hypothesis involving certain classes of covers of the space X. Here we will make use of the class of open k-covers: $\mathcal{A}$ is a k-cover of a space X (or "$\mathcal{A}$ k-covers a space X") if for each compact $K \subseteq X$ there is a $A \in \mathcal{A}$ with $K \subseteq A$; it is an open k-cover if it consists of open sets. The class of all non trivial open k-covers of X (i.e. such that X does not belong to the cover) will be denoted by $\mathcal{K}(X)$ or just $\mathcal{K}$.

We give the selectively Reznichenko property a new frame to fit in, embodied in the following selection principle.

DEFINITION 1.2. For two sets $\mathcal{A}$ and $\mathcal{F}$ (we look at $\mathcal{F}$ as a list of properties) the formula $R(\mathcal{A}, \mathcal{F})$ abbreviates the statement: for each sequence $(A_n : n \in N)$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{A}_n : n \in N)$, where each $\mathcal{A}_n$ is a finite subset of A_n, such that the $\mathcal{A}_n$-s are pairwise disjoint and $\forall F \in \mathcal{F} \exists n \in N\ \forall m \geq n \exists A \in \mathcal{A}_m\ (A \in F)$.

If $a \in X$ denote by $\Omega_a = \Omega_a(X)$ the family of all $A \subseteq X$ such that $a \in \overline{A} \setminus A$, and fix a local base $\mathcal{B}_a$ at a. Then $R(\Omega_a, \mathcal{B}_a)$ is another way of saying that X has the selectively Reznichenko property at a.

The selective version of the Pytkeev property is defined in the following way. A space X has the selectively Pytkeev property at the point $a \in X$ if for each sequence $(A_n : n \in N)$ of elements of Ω_a, there is a sequence $(B_n : n \in N)$, where each B_n is an countably infinite subset of A_n, such that $\{B_n : n \in N\}$ is a π-network at a. If X has the selectively Pytkeev property at each of its points, X is simply said to have the selectively Pytkeev property.

As it is trivial to see, the selectively Pytkeev property at $a \in X$ can be given by the following equivalent formulation: for each sequence $(A_n : n \in N)$ of elements of Ω_a, there is a sequence $(B_n : n \in N)$, where each B_n is a countably infinite subset of A_n, such that for each $f \in \prod(B_n : n \in N)$ the set $\{f(n) : n \in N\}$ is an element of Ω_a. We now introduce a new selection hypothesis generalizing the selectively Pytkeev property.

DEFINITION 1.3. For two sets $\mathcal{A}$ and $\mathcal{B}$ the formula $P(\mathcal{A}, \mathcal{B})$ abbreviates the statement: for each sequence $(A_n : n \in N)$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{A}_n : n \in N)$, where each $\mathcal{A}_n$ is a countably infinite subset of A_n, such that for each $f \in \prod(\mathcal{A}_n : n \in N)$ the set $\{f(n) : n \in N\}$ is an element of $\mathcal{B}$.

It is clear that X has the selectively Pytkeev property at a point $a \in X$ if and only if $P(\Omega_a, \Omega_a)$ holds.

We will slightly modify the game corresponding in the usual way to the selection principle $P(\mathcal{A}, \mathcal{B})$. In the sequel $\underline{(x)}$ will stand for the finite sequence of length 1 $f : \{0\} \to \{x\}$, $f(0) = x$.

The infinitely long game $GameP(\mathcal{A}, \mathcal{B})$ is defined as follows: Two players, W(hite) and B(lack), play a round for each natural number n. In the first round player W plays $A_1 \in \mathcal{A}$ and B then responds by $a^1 = \underline{(a_1^1)}$, where $a_1^1 \in A_1$. In the n-th round W plays $A_n \in \mathcal{A}$ and B then responds by $a^n = (a_1^n, \ldots, a_n^n)$, where $a_i^n \in A_i$ for all $1 \leq i \leq n$. A play $(A_n, a^n : n \in N)$ is won by B if the sequence $(a_m^n : n \geq m)$ is injective for all $m \in N$, and if for each $f : N \to N$ with $f(n) \geq n$, the set $(a_n^{f(n)} : n \in N\}$ is an element of $\mathcal{B}$. Otherwise it is won by W. For a game G played between two players we use the symbol $^*G$ to abbreviate the statement: *the player that starts first does not have a winning strategy in this game.* Clearly, $^*GameP(\mathcal{A}, \mathcal{B})$ implies $P(\mathcal{A}, \mathcal{B})$. We shall write

$GamePyt(a)$ to shorten $GameP(\Omega_a, \Omega_a)$, or simply $GamePyt$ whenever the point a at which the game is played is understood.

The results of [8] could easily be modified to get corresponding results regarding $C_k(X)$ spaces, but that is not our intention here. However, we will need to borrow some ideas from it, and we shall start by the next definition and a few lemmas.

DEFINITION 1.4. A family $\mathcal{U}$ of subsets of a space X is said to be an "3-k-shrinkable cover" of X if there is a function g such that for each $U \in \mathcal{U}$ $g(U) = (V_U, Z_U)$, where $V_U \subseteq Z_U \subseteq U$, V_U is a cozero set, Z_U is a zero set and $\{V_U : U \in \mathcal{U}\}$ k- covers X; it is called an *open* "3-k-shrinkable cover" provided that its elements are open subsets of X. The collection of all non trivial open 3-k-shrinkable covers of X will be denoted by 3-$\mathcal{K}_{shr}(X)$ or just by 3-$\mathcal{K}_{shr}$ when it is clear to which X the notation refers.

In the sequel, "$f^{\leftarrow}A$" ("$f[A]$") will stand for the inverse image (image) of a set A under a function f.

For a space X and a compact $K \subseteq X$ let $\mathcal{C}_K(X) := \{A \subseteq X : K \subseteq A\}$ and $\mathcal{C} \equiv \mathcal{C}(X) := \{\mathcal{C}_K : K \text{ is a compact subset of } X\}$. The following three lemmas are proved exactly as Lemma 1.2, Lemma 1.3 and Lemma 1.4, respectively, of [8], mostly by replacing the word *finite* by the word *compact*.

LEMMA 1.1. *Every open k-cover can be refined by an open 3-k-shrinkable cover.*

LEMMA 1.2. *If $X \in R(3 - \mathcal{K}_{shr}, \mathcal{C})$ then every open 3-k-shrinkable cover contains a countable 3-k-shrinkable subcover.*

LEMMA 1.3. *Let X be such a space that each open k-cover has a countable k-subcover. If $\varepsilon > 0$ and $o \in \overline{A} \subseteq C_k(X)$ then there is a $B \subseteq A$ and a function $s : B \to (0, \varepsilon)$ such that at least one of the families $\{|f|^{\leftarrow}[0, s(f)) : f \in B\}$ and $\{|f|^{\leftarrow}[0, s(f)] : f \in B\}$ is a 3-k-shrinkable open cover of X.*

We will also need the next selection principle.

DEFINITION 1.5. For two sets $\mathcal{A}$ and $\mathcal{F}$ (the latter should be regarded as a list of certain properties) the statement $C(\mathcal{A}, \mathcal{F})$ means that for each sequence $(\mathcal{U}_n : n \in N)$ of elements of $\mathcal{A}$ there is a sequence $(V_n : n \in N)$ such that each $V_n \in \mathcal{U}_n$ and $\forall F \in \mathcal{F}\, \exists n_0 \forall n \geq n_0\ (V_n \in F)$.

There is an infinitely long game $GameC(\mathcal{A}, \mathcal{F})$, played between two players W and B, corresponding to this selection principle in the usual way: in the n-th round W first picks $\mathcal{U}_n \in \mathcal{A}$ and then B responds by $V_n \in \mathcal{U}_n$; a play $(\mathcal{U}_n, V_n : n \in N)$ is won by B if $\forall F \in \mathcal{F}\, \exists n_0 \forall n \geq n_0\ (V_n \in F)$; else it is won by W.

A space X is said to be strictly Fréchet at the point $x \in X$ (see e.g. [9]) if for each sequence $(A_n : n \in N)$ of subsets of X with $x \in \overline{A_n} \setminus A_n$ for all $n \in N$, there is a sequence $(a_n : n \in N)$, with $a_n \in A_n$, converging to x. If X has that property at all of its points then X is simply said to be strictly Fréchet. Obviously X is strictly Fréchet at $a \in X$ if and only if $C(\Omega_a, \mathcal{B}_a)$ holds, where Ω_a and $\mathcal{B}_a$ are as previously mentioned.

The following two theorems can easily be shown using standard techniques as in [8]:

THEOREM 1.1. *$C_k(X)$ has the selectively Reznichenko property if and only if $R(3 - \mathcal{K}_{shr}, \mathcal{C})$ holds.*

THEOREM 1.2. *$C_k(X)$ is strictly Fréchet if and only if $C(3 - \mathcal{K}_{shr}, \mathcal{C})$ holds.*

The property of being strictly Fréchet implies the Pytkeev property and in [6] it was shown that the Pytkeev property implies the Reznichenko property. It is not clear if the same implications hold for their selective versions. However, the following can be proved.

PROPOSITION 1.1. *Let X be a T_1 space. If X is strictly Fréchet then it has the selectively Pytkeev property.*

Proof. Let $x \in X$, $A_n \subseteq X$, $n \in N$ and $x \in \overline{A_n} \setminus A_n$, for a strictly Fréchet space X.

Write $N := \cup\{Y_n : n \in N\}$ where $n \neq m \Rightarrow Y_n \cap Y_m = \emptyset$ and each Y_n is infinite. Define $f : N \to N$ by $f(n) = m \iff n \in Y_m$ and set $B_n := A_{f(n)}$. As X is strictly Fréchet, there is a sequence $(b_n : n \in N)$, with $b_n \in B_n$, converging to x. Let $C_n := (b_k : k \in Y_n)$. Then $C_n \subseteq A_n$: if $k \in Y_n$ then $n = f(k)$ so $b_k \in B_k = A_{f(k)} = A_n$. Also, each C_n is infinite because $(b_n : n \in N)$ converges to x and $x \neq b_n$ for all n. We claim that $\{C_n : n \in N\}$ is a π-network at x.

Indeed, let $U \ni x$ be arbitrary open. There is a n_0 such that $\forall n \geq n_0\ (b_n \in U)$. Take a $m_0 \in N \setminus \{f(i) : i = \overline{1, n_0}\}$. Then $\{i : i = \overline{1, n_0}\} \subseteq \cup\{Y_{f(i)} : i = \overline{1, n_0}\}$ so $Y_{m_0} \subseteq N \setminus \{i : i = \overline{1, n_0}\}$. Hence, for each $n \in Y_{m_0}$ it must be that $n > n_0$ and thus $b_n \in U$. This means that $C_{m_0} \subseteq U$. $\square$

PROPOSITION 1.2. *If a T_1 space X is strictly Fréchet then it has the selectively Reznichenko property.*

Proof. Let $x \in X$ and $(A_n : n \in N)$ be a sequence of subsets of X with $x \in \overline{A_n} \setminus A_n$. It suffices to show that there is an injective sequence $(v_n : n \in N)$, $v_n \in A_n$, converging to x.

As before, write $N := \cup\{Y_n : n \in N\}$ where $n \neq m \Rightarrow Y_n \cap Y_m = \emptyset$ and each Y_n is infinite. Define $f : N \to N$ by $f(n) = m \iff n \in Y_m$ and set $B_n := A_{f(n)}$. X is strictly Fréchet so there is a sequence $(u_n : n \in N)$, $u_n \in B_n$, converging to x. Define recursively an injective sequence $(v_n : n \in N)$ in the following way:
take a $v_1 \in \{u_k : k \in Y_1\}$ arbitrarily. If $v_1, \dots, v_n$ have been defined choose a $v_{n+1} \in \{u_k : k \in Y_{n+1}\} \setminus \{v_1, \dots, v_n\}$.

For each $n \in N$, we have $v_n = u_k$ for a $k \in Y_n$, thus $n = f(k)$ and $u_k \in B_k = A_{f(k)} = A_n$. So $v_n \in A_n$ for all n.

Let $U \ni x$ be an open subset of X. Fix a $n_0 \in N$ with $\forall n \geq n_0 (u_n \in U)$. Put $m_0 = \max\{f(i) : i = \overline{1, n_0}\}$ and let $n > m_0$ be arbitrary. $v_n = u_{k_0}$ for a $k_0 \in Y_n$. $\{1, \dots, n_0\} \subseteq \cup\{Y_{f(j)} : j = \overline{1, n_0}\}$, $k_0 \in Y_n$ and $n > m_0$ imply $k_0 > n_0$, hence $u_{k_0} \in U$, i.e., $v_n \in U$. Therefore $(v_n : n \in N)$ is as required. □

To end this section we will show that in the realm of $C_p(X)$ and $C_k(X)$ spaces the selectively Pytkeev property is stronger that the selectively Reznichenko property. Again, a result of [8] comes in handy.

Let $wR(\mathcal{A}, \mathcal{F})$ denote the same statement as $R(\mathcal{A}, \mathcal{F})$ but without the condition that the $\mathcal{A}_n$-s of Definition 1.2 must be pairwise disjoint. Fix local bases $\mathcal{B}_o^p$ and $\mathcal{B}_o^k$ at the point o in spaces $C_p(X)$ and $C_k(X)$, respectively. In [8] it was shown that $C_p(X)$ has the selectively Reznichenko property if and only if $wR(\Omega_o(C_p(X)), \mathcal{B}_o^p)$ holds and practically the same line of reasoning shows that $C_k(X)$ has the selectively Reznichenko property if and only if $wR(\Omega_o(C_k(X)), \mathcal{B}_o^k)$ holds. Now we can prove the following proposition.

PROPOSITION 1.3. *If $C_k(X)$ ($C_p(X)$) has the selectively Pytkeev property then it has the selectively Reznichenko property.*

Proof. We prove the theorem for $C_k(X)$ spaces whereas practically the same proof works for $C_p(X)$ spaces.

Suppose that $C_k(X)$ has the selectively Pytkeev property and fix a local base $\mathcal{B}_o$ at the point o in the space $C_k(X)$. We prove that $wR(\Omega_o(C_k(X)), \mathcal{B}_o)$ holds.

Given a $(A_n : n \in N)$, $o \in \overline{A_n} \setminus A_n$, put $B_n := \{\max\{f_1, \dots, f_n\} : f_i \in A_i, \ i = \overline{1, n}\}$. We may suppose that the A_n-s consist of nonnegative functions. Then $o \in \overline{B_n} \setminus B_n$ for each n: let $K \subseteq X$ be compact and

$\varepsilon > 0$. Take a $f_i \in A_i \cap O(K,\varepsilon)$, $i = \overline{1,n}$. Then $\max\{f_1,\dots,f_n\} \in B_n \cap O(K,\varepsilon)$.

We shall write $B \leq^* A$, for subsets A, B of $C(X)$, if $\forall f \in B\, \exists l \in A \forall x \in X(l(x) \leq f(x))$. Whenever $\{f\} \leq^* A$ choose a $g(A,f) \in A$ with $\forall x \in X(g(A,f)(x) \leq f(x))$. Obviously $B_m \leq^* B_n \leq^* A_n$ whenever $m \geq n$.

Note that for two nonnegative functions $f, g \in C(X)$ with $\{f\} \leq^* \{g\}$, the condition $f \in O(K,\varepsilon)$ implies $g \in O(K,\varepsilon)$, for each $K \subseteq X$ and $\varepsilon > 0$.

As $C_k(X)$ has the selectively Pytkeev property there is an infinite "matrix" $(u_n^m : n, m \in N)$, with $u_n^m \in B_n$, such that $\{\{u_n^m : m \in N\} : n \in N\}$ is a π-network at o.

For each m let $V_m := \{u_n^m : n \in N\}$. Then $o \in \overline{V_m} \setminus V_m$. Again, by the selectively Pytkeev property there is a sequence $(k_m : m \in N)$ of strictly increasing functions $k_m : N \to N$ such that $\{\{u_{k_m(n)}^m : n \in N\} : m \in N\}$ is a π-network at o. Define v_n^m, $n, m \in N$ as follows:

$v_n^m = g(B_n, u_{k_m(l)}^m)$ if $k_m(l-1) < n \leq k_m(l)$ for a $l > 1$;

$v_n^m = g(B_n, u_{k_m(1)}^m)$ if $n \leq k_m(1)$.

Then $\{\{v_n^m : n \in N\} : m \in N\}$ is a π-network at o. Indeed, given a compact $K \subseteq X$ and $\varepsilon > 0$, there is a $m_0 \in N$ such that $\forall n\ (u_{k_{m_0}(n)}^{m_0} \in O(K,\varepsilon)$. We show $\forall n\ (v_n^{m_0} \in O(K,\varepsilon))$. If $k_m(l-1) < n \leq k_m(l)$ for a $l > 1$, then $v_n^{m_0} = g(B_n, u_{k_{m_0}(l)}^{m_0})$, thus $\{u_{k_{m_0}(l)}^{m_0}\} \leq^* \{v_n^{m_0}\}$. But then $u_{k_{m_0}(l)}^{m_0} \in O(K,\varepsilon)$ implies $v_n^{m_0} \in O(K,\varepsilon)$.

Now, put $\mathcal{B}_n := \{v_n^i : i = \overline{1,n}\}$. Obviously $\mathcal{B}_n$ is a finite subset of B_n. Then given a compact $K \subseteq X$ and $\varepsilon > 0$ we have that $\mathcal{B}_n \cap O(K,\varepsilon) \neq 0$ for all but finitely many n: take a $m_0 \in N$ such that $\forall n\ (v_n^{m_0} \in O(K,\varepsilon))$; if $n \geq m_0$ then $v_n^{m_0} \in O(K,\varepsilon) \cap \mathcal{B}_n$.

Each $\mathcal{A}_n := \{g(A_n, v_n^i) : i = \overline{1,n}\}$ is a finite subset of A_n. It is easy to see that the sequence $(\mathcal{A}_n : n \in N)$ is as required. □

2. Results

We trace the proof of Proposition 2.3 of [8] to obtain the next easy fact:

Proposition 2.1. *If $\forall x \in \mathcal{A}\ \exists y \subseteq x\ (y \in \mathcal{A} \ \wedge\ card(y) = \omega)$ and if $C(\mathcal{A},\mathcal{F})$ holds then W has no winning strategy in the game $GameC(\mathcal{A},\mathcal{F})$.*

Proof. For a proof by contradiction let o be a winning strategy for W in Game$C(\mathcal{A},\mathcal{F})$. We may suppose that it directs the player W to play countable sets.

For each $n \in N$ let $S_n := \{\mathcal{U} \in \mathcal{A} :$ there is a finite sequence s of length $2n-2$ such that $s^\frown\underline{(\mathcal{U})} \in o\}$. As W plays countable moves and B responds to him by elements of W-s moves, it is obvious that each S_n is countable. Write $N = \bigcup\{Y_n : n \in N\}$ where $\forall n\ (\mathrm{card}(Y_n) = \omega \wedge (n \neq m \Rightarrow Y_n \cap Y_m = \emptyset))$. We may also write $S_n = \{\mathcal{V}_i : i \in Y_n\}$. If $i \in N$ and $\mathcal{U} \in \mathcal{A}$ are such that $\mathcal{U} \in S_i$ choose a $t(\mathcal{U}, i) \in Y_i$ with $\mathcal{U} = \mathcal{V}_{t(\mathcal{U},i)}$. Apply the fact that $C(\mathcal{A},\mathcal{F})$ holds to the sequence $(\mathcal{V}_n : n \in N)$ of elements of $\mathcal{A}$ to obtain a sequence $(U_n : n \in N)$ such that: each $U_n \in \mathcal{V}_n$ and for each $F \in \mathcal{F}$ there is a n_0 with $\forall n \geq n_0\ (U_n \in F)$.

Now we define a play s played according to o and lost by W.

If $\mathcal{U}_1$ is W-s first move according to o then let $s_1 := \underline{(\mathcal{U}_1)} \in o$. If $s_i \in o$ have been defined for $1 \leq i < n$ so that $1 \leq i \leq j < n \Rightarrow s_i \subseteq s_j$ and that length$(s_i) = 2i-1$ and if $s_{n-1} = (\mathcal{U}_1, A_1; \ldots; \mathcal{U}_{n-1}) \in o$ then $U_{t(\mathcal{U}_{n-1},n-1)} \in \mathcal{V}_{t(\mathcal{U}_{n-1},n-1)} = \mathcal{U}_{n-1}$ (length$(s_{n-1}) = 2n-3$ and $s_{n-1} \in o$ imply $\mathcal{U}_{n-1} \in S_{n-1}$ so the number $t(\mathcal{U}_{n-1}, n-1)$ is defined). Therefore $(\mathcal{U}_1, A_1; \ldots; \mathcal{U}_{n-1}, U_{t(\mathcal{U}_{n-1},n-1)}) \in o$, hence there is a unique $\mathcal{U}_n$ with $(\mathcal{U}_1, A_1; \ldots; \mathcal{U}_{n-1}, U_{t(\mathcal{U}_{n-1},n-1)}, \mathcal{U}_n) \in o$. Let $s_n := (\mathcal{U}_1, A_1; \ldots; \mathcal{U}_{n-1}, U_{t(\mathcal{U}_{n-1},n-1)}; \mathcal{U}_n)$.

As $(s_n : n \in N)$ is an increasing sequence of legal positions of the game Game$C(\mathcal{A},\mathcal{F})$ played according to o with lengths going to infinity, $s := \bigcup\{s_n : n \in N\}$ is a play of this game played according to o. Also, by the construction of the s_n-s, $s = (\mathcal{U}_1, U_{t(\mathcal{U}_1,1)}, \ldots; \mathcal{U}_k, U_{t(\mathcal{U}_k,k)}; \ldots)$. If $F \in \mathcal{F}$ is arbitrary then there is a $k_0 \in N$ such that $\forall k \geq k_0\ (U_k \in F)$. Choose a $k_1 \in N$ such that $\{0,1,\ldots,k_0\} \subseteq \bigcup_{i=0}^{k_1} Y_i$. Now, if $k > k_1$ then $t(\mathcal{U}_k, k) > k_0$ (because $t(\mathcal{U}_k, k) \in Y_k$). Thus $U_{t(\mathcal{U}_k,k)} \in F$. This just means that s is lost by W.

In this way, o cannot be winning for W which contradicts the assumption we have made at the beginning of the proof. □

LEMMA 2.1. *For a locally compact space X the following implication holds: $R(3-\mathcal{K}_{shr},\mathcal{C}) \Rightarrow C(\mathcal{K},\mathcal{C})$.*

Proof. By virtue of Lemma 1.1 and Lemma 1.2 every open k-cover must have a countable k-subcover so we are free to use Lemma 1.3. If X is compact then $\mathcal{K} = \emptyset$, so we may suppose that X is not compact.

Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of elements of $\mathcal{K}$. Set $A := \{f \in C(X) : |f|^{\leftarrow}[0,1)$ has compact closure in X$\}$.

Then $o \in \overline{A} \subseteq C_k(X)$. Indeed, if $K \subseteq X$ is compact take an open $U \supseteq K$ with compact closure (X is locally compact so there is one) and choose a continuous $f : X \to [0,1]$ with $f[K] \subseteq \{0\}$, $f[X \setminus U] \subseteq \{1\}$. Then $|f|^{\leftarrow}[0,1) \subseteq U$ so the closure of $|f|^{\leftarrow}[0,1)$ is a closed subset of the compact set $\overline{U}$ thus itself compact. This proves $f \in A$. Also, $f \in O(K,\varepsilon)$ for each $\varepsilon > 0$.

By Lemma 1.3 there is a $B \subseteq A$ and a $s : B \to [0, \frac{1}{2})$ such that either $\{|f|^{\leftarrow}[0, s(f)) : f \in B\}$ or $\{|f|^{\leftarrow}[0, s(f)] : f \in B\}$ is a 3-k-shrinkable open cover of X. Denote the one that is by $\mathcal{V}$. Clearly, the elements of $\mathcal{V}$ have compact closures. As X is not compact, $\mathcal{V}$ is not trivial.

Apply $R(3 - \mathcal{K}_{shr}, \mathcal{C})$ to the sequence $(\mathcal{V}_n : n \in \mathbf{N})$ where $\mathcal{V}_n = \mathcal{V}$ for all n to obtain a sequence $(\mathcal{G}_n : n \in N)$, where each $\mathcal{G}_n$ is a finite subset of $\mathcal{V}$, such that for each compact $K \subseteq X$ there is an $n \in N$ with $\forall k > n \, \exists G \in \mathcal{G}_k \; (K \subseteq G)$.

For each n $\mathcal{G}_n$ is finite and its elements have compact closures so $\overline{\cup \mathcal{G}_n} = \cup\{\overline{G} : G \in \mathcal{G}_n\}$ is compact. Therefore there is a $U_n \in \mathcal{U}_n$ such that $\overline{\cup \mathcal{G}_n} \subseteq U_n$. It is trivial to check that the sequence $(U_n : n \in N)$ is the required one. □

In the sequel $x \preceq y$ will mean $\forall X \in x \, \exists Y \in y \; (X \subseteq Y)$.

PROPOSITION 2.2. *${}^*GameC(3 - \mathcal{K}_{shr}, \mathcal{C}) \Rightarrow {}^*GameP(\mathcal{K}, \mathcal{K})$.*

Proof. Let σ be a strategy for W in the game Game$P(\mathcal{K}, \mathcal{K})$. We define a strategy δ for W in the game Game$C(3 - \mathcal{K}_{shr}, \mathcal{C})$. Let by Lemma 1.1 f be a function assigning to each k-cover x a 3-k-shrinkable cover $f(x)$ refining x. For sets $x_1, \ldots, x_n$ we write $\wedge(x_1, \ldots, x_n)$ for $\{\cap_{i=1}^n y_i : \; y_i \in x_i, \; i = \overline{1,n}\}$. Let g be a function assigning to each pair $(\mathcal{U}, A)$, where $\mathcal{U}$ is a family of subsets of X and $A \subseteq X$ such that $\{A\} \preceq \mathcal{U}$, a $g(\mathcal{U}, A) \in \mathcal{U}$ such that $A \subseteq g(\mathcal{U}, A)$. Put $\sigma_n := \{s \in \sigma : \text{length}(s) = n\}$.

Now we construct inductively on $n = 1, 2, \ldots$ pairs (δ_n, l_n) such that:

(i) δ_n is a set of legal positions of length n of the game Game$C(3 - \mathcal{K}_{shr}, \mathcal{C})$ and δ_1 is a singleton,

(ii) For each $s \in \delta_{2n-1}$ and each legal position $s_1 \supseteq s$ of the game Game$C(3 - \mathcal{K}_{shr}, \mathcal{C})$ with length$(s_1) = 2n$ there is a unique a such that $s_1 {}^\frown (a) \in \delta_{2n+1}$.

(the conditions (i) and (ii) guarantee that $\delta := \bigcup_{n \in N} \delta_n$ is a strategy for W in the game Game$C(3 - \mathcal{K}_{shr}, \mathcal{C})$)

(iii) $l_n : \delta_n \to \sigma_n$ and; if $s \in \delta_{2n+1}$, $s = (B_1, b_1; \ldots; B_n, b_n; B_{n+1})$, $l_{2n}(s) = (A_1, a^1; \ldots; A_n, a^n; A_{n+1})$, where $a^i = (a^i_1, \ldots, a^i_i)$, then

(1.s.$2n+1$)... $\forall i \in \{2, \dots, n+1\}\ \forall j \in \{1, \dots, i\}\ (B_i \preceq (A_j \setminus \{a_j^k : 1 \le j \le k \le i-1\}))$ and $B_1 \preceq A_1$;

(2.s.$2n+1$)... $\forall i \in \{1, \dots, n\}\ (b_i \subseteq \cap_{j=1}^i a_j^i)$ and $\forall i \in \{1, \dots, n\}$ $(\mathrm{card}(\{a_i^i, a_i^{i+1}, \dots, a_i^n\}) = n - i + 1)$.

and if $s \in \delta_{2n}$, $s = (B_1, b_1; \dots; B_n, b_n)$, $l_{2n}(s) = (A_1, a^1; \dots; A_n, a^n)$, where $a^i = (a_1^i, \dots, a_i^i)$, then

(1.s.$2n$)... $\forall i \in \{2, \dots, n\}\ \forall j \in \{1, \dots, i\}\ (B_i \preceq (A_j \setminus \{a_j^k : 1 \le j \le k \le i-1\}))$ and $B_1 \preceq A_1$;

(2.s.$2n$)... $\forall i \in \{1, \dots, n\}\ (b_i \subseteq \cap_{j=1}^i a_j^i)$ and $\forall i \in \{1, \dots, n\}$ $(\mathrm{card}(\{a_i^i, a_i^{i+1}, \dots, a_i^n\}) = n - i + 1)$.

(iv) The following extension condition holds:

$\forall s \in \delta_n\ \forall t \in \delta_{n+1}\ (s \subseteq t \Rightarrow l_n(s) \subseteq l_{n+1}(t))$.

We define (δ_1, l_1): if A is such that $\sigma_1 = \{\underline{(A)}\}$ then we put $\delta_1 = \{\underline{(f(A))}\}$ as well as $l_1(\underline{(f(A))}) := \underline{(A)}$.

We define (δ_2, l_2): δ_2 consists of pairs $(f(A), b)$, where $b \in f(A)$, and for each such pair we put $l_2((f(A), b) := (A, \underline{(g(A, b))})$. If all pairs (δ_i, l_i) for $i = \overline{1, 2n}$ have been defined subject to the conditions (i)-(iv), then we define $(\delta_{2n+1}, l_{2n+1})$:

Let $s = (B_1, b_1; \dots; B_n, b_n) \in \delta_{2n}$ and $l_n(s) = (A_1, a^1; \dots; A_n, a^n)$. There is a unique A_{n+1} such that $(A_1, a^1; \dots; A_n, a^n; A_{n+1}) \in \sigma_{2n+1}$. We set $s_1 = s \frown \underline{(B)} \in \delta_{2n+1}$ and $l_{2n+1}(s_1) := (A_1, a^1; \dots; A_n, a^n; A_{n+1}) \in \sigma_{2n+1}$, where $B := f\left(\wedge(A_i \setminus \{a_i^j : 1 \le i \le j \le n\} : i = \overline{1, n+1})\right)$. It is easy to see that the conditions (i)-(iv) still hold.

We also define $(\delta_{2n+2}, l_{2n+2})$:

Let $s = (B_1, b_1; \dots; B_n, b_n; B_{n+1}) \in \delta_{2n+1}$, $l_{2n+1}(s) = (A_1, a^1; \dots; A_n, a^n; A_{n+1})$ and $b \in B_{n+1}$. We set $s_1 = s \frown \underline{(b)} \in \delta_{2n+2}$, $a^{n+1} := (g(A_i \setminus \{a_i^j : 1 \le i \le j \le n\}, b) : i = \overline{1, n+1})$ and $l_{2n+2}(s_1) := l_{2n+1}(s) \frown \underline{(a^{n+1})} \in \sigma_{2n+2}$. (i)-(iv) are again preserved.

Now, $\delta := \cup\{\delta_n : n \in N\}$ is a strategy for W in the game Game$C(3 - \mathcal{K}_{shr}, \mathcal{C})$ so, as it cannot be winning , there is a play $l = (B_1, b_1; \dots; B_n, b_n; \dots)$ of this game won by B. By the extension condition there is a play $p = (A_1, a^1; \dots)$ of the game Game$P(\mathcal{K}, \mathcal{K})$ such that $l_{2n}((B_1, b_1; \dots; B_n, b_n)) = (A_1, a^1; \dots; A_n, a^n)$ for all $n \in N$. Then p must be won by B:

Suppose this is not true. Then, if we write $a^n = (a_1^n, \dots, a_n^n)$, by the construction of p the sequence $(a_m^n : n \ge m)$ is injective for each $m \in N$. Therefore, as p is not won by B, there must be a $f : N \to N$ with $f(n) \ge n$ for all n, such that $\{a_n^{f(n)} : n \in N\} \notin \mathcal{K}$. Thus there is a compact

$K \subseteq X$ with $K \subseteq a_n^{f(n)}$ for no $n \in N$. As l is won by B, there is a $n_0 \in N$ such that $\forall n \geq n_0(K \subseteq b_n)$. For $s = (B_1, b_1; \ldots; B_{f(n_0)}, b_{f(n_0)})$, (2.$s$.2$f(n_0)$) yields $b_{f(n_0)} \subseteq \bigcap_{j=1}^{f(n_0)} a_j^{f(n_0)}$, so $K \subseteq b_{f(n_0)} \subseteq a_{n_0}^{f(n_0)}$, which is impossible.

Hence, σ is not a winning strategy for W. □

PROPOSITION 2.3. *If* $^*GameP(3 - \mathcal{K}_{shr}, \mathcal{K})$ *holds then the player* W *has no winning strategy in the game* $GamePyt(a)$ *played on* $C_k(X)$ *at any* $a \in C_k(X)$.

Proof. Due to the homogeneity of $C_k(X)$ we may suppose that Game Pyt is played at the point o. Let σ be a strategy for the player W.

If $A \subseteq C_k(X)$, $o \in \overline{A} \backslash A$, call A small if $\forall \varepsilon > 0\, \exists f \in A\, (|f|[X] \subseteq [0, \varepsilon)$. For a small A, given any $\delta > 0$ one can find an injective sequence $(f_n(A, \delta) : n \in N)$ of elements of A such that $|f_n(A, \delta)|[X] \subseteq [0, \delta)$ for all $n \in N$.

If $A \subseteq C_k(X)$, $o \in \overline{A} \setminus A$ is not small there is a positive real number $\delta(A) > 0$ such that for each $f \in A$ we have $|f|^{\leftarrow}[0, \delta(A)) \neq X$. Given a $\delta > 0$ use Lemma 1.3 to fix an open 3-k-shrinkable cover $\mathcal{U}_\delta^A$ with $\mathcal{U}_\delta^A \preceq \{|f|^{\leftarrow}[0, \delta) : f \in A\}$, and an injection $g_\delta^A : \mathcal{U}_\delta^A \to A$ such that $U \subseteq g_\delta^A(U)^{\leftarrow}(-\delta, \delta)$ for all $U \in \mathcal{U}_\delta^A$. If $\delta \leq \delta(A)$ then obviously $\mathcal{U}_\delta^A$ is nontrivial.

Now we let players W_X and B_X play the game Game$P(3 - \mathcal{K}_{shr}, \mathcal{K})$ on the space X and we define a strategy θ for W_X (the one that starts first) in this game.

Fix a nontrivial open 3-k-shrinkable cover $\mathcal{V}$. If A_1 is W's first σ-move and if A_1 is not small, put $\delta_1 := \min\{\delta(A_1), 1\}$ and let $\mathcal{U}_{\delta_1}^{A_1}$ be W_X's first θ-move. Otherwise we put $\delta_1 := 1$ and we let W_X start the game by $\mathcal{V}$. In either case denote the cover W_X has thus played by $\mathcal{U}_1$.

Should B_X now respond by a $u_1^1 \in \mathcal{U}_1$ (to be more precise: by a $u^1 = \underline{(u_1^1)}$) we proceed as follows:

If A_1 is not small then $g_{\delta_1}^{A_1}(u_1^1)$ is defined, so pretending B has responded to W by it, we get a new move from W. If A_1 is small we pretend B has responded by $f_1(A_1, 1)$ thus obtaining again a new move from W. In either case denote the W's new move by A_2. We let the W_X's response to the u^1 of B_X be either $\mathcal{U}_{\delta_2}^{A_2}$ if A_2 is not small, where $\delta_2 := \min\{\delta(A_2), 1/2\}$, or $\mathcal{V}$ if A_2 is small, in which case we put $\delta_2 := 1/2$, and denote that response by $\mathcal{U}_2$.

Should B_X now respond by a $u^2 = (u_1^2, u_2^2)$, where $u_i^2 \in \mathcal{U}_i$, we proceed as follows:

We let a_i^2, for $i = \overline{1,2}$, be either $g_{\delta_i}^{A_i}(u_i^2)$ if A_i is not small, or $f_{2-i+1}(A_i, \frac{1}{i})$ if A_i is small. Then we pretend that B has responded to W's move A_2 by $a^2 = (a_1^2, a_2^2)$ and thus get a new move from W, say A_3. We let W_X respond to B_X's u^2 by either $\mathcal{U}_{\delta_3}^{A_3}$ if A_3 is not small, where $\delta_3 := \min\{\delta(A_3), 1/3\}$, or $\mathcal{V}$ if A_3 is small, in which case we put $\delta_3 := 1/3$, and denote that response by $\mathcal{U}_3$ and so on. In this way we have described θ.

As θ can not be a winning strategy for W_X there is a θ-play $(\mathcal{U}_n, u^n : n \in N)$ of the game Game$P(3 - \mathcal{K}_{shr}, \mathcal{K})$ won by B_X, where $u^n = (u_1^n, \dots, u_n^n)$. By the definition of θ there is a σ-play $(A_n, a^n : n \in N)$ of the game GamePyt played on $C_k(X)$ at o, where $a^n = (a_1^n, \dots, a_n^n)$, and a sequence of real numbers $(\delta_n : n \in N)$, $0 < \delta_n \leq 1/n$, such that $\mathcal{U}_i = \mathcal{U}_{\delta_i}^{A_i}$ and $a_i^n = g_{\delta_i}^{A_i}(u_i^n)$ if A_i is not small, and $\mathcal{U}_i = \mathcal{V}$ and $a_i^n = f_{n-i+1}(A_i, \frac{1}{i})$ if A_i is small, for all $n, i \in N$, $i \leq n$.

We claim that $(A_n, a^n : n \in N)$ is won by B.

First, for each $m \in N$ the sequence $(a_m^n : n \geq m)$ is injective because, whenever they are defined, both functions g_δ^A and sequences $(f_k(A, \delta) : k \in N)$ are injective and the sequence $(u_m^n : n \geq m)$ is injective, too.

It remains to show that the family $\mathcal{F} := \{\{a_k^n : n \geq k\} : k \in N\}$ is a π-network of $C_k(X)$ at o. Let $S := \{n \in N : A_n \text{ is small}\}$. We distinguish two cases:

Case 1: S is infinite. Let $\varepsilon > 0$. Take a $n_0 \in S$ with $n_0 > 1/\varepsilon$. We have that $\forall n \geq n_0$ $(a_{n_0}^n = f_{n-n_0+1}(A_{n_0}, \frac{1}{n_0}))$ so if $n \geq n_0$ then $a_{n_0}^n[X] \subseteq (-1/n_0, 1/n_0) \subset (-\varepsilon, \varepsilon)$. This means that $\{a_{n_0}^n : n \geq n_0\} \subseteq O(K, \varepsilon)$ for each compact subset K of X. Thus $\mathcal{F}$ is a π-network at o.

Case 2: S is finite. We actually need to show that for each $h : N \to N$ with $h(n) \geq n$, the point o is in the closure of the set $\{a_n^{h(n)} : n \in N\}$. Let K be a compact subset of X and let $\varepsilon > 0$. Fix a positive integer $n_0 > \max(S \cup \{1/\varepsilon\})$. As $\mathcal{U}_1, \dots, \mathcal{U}_{n_0}$ are nontrivial and $u_i^{h(i)} \in \mathcal{U}_i$, $i = \overline{1, n_0}$, there is a finite $F \subseteq X$ such that $F \subseteq u_i^{h(i)}$ for no $1 \leq i \leq n_0$. Since $\{u_n^{h(n)} : n \in N\} \in \mathcal{K}$ (remember that B_X has won the play $(\mathcal{U}_k, u^k : k \in N)$), there is a $m \in N$ with $K \cup F \subseteq u_m^{h(m)}$. We must have that $m > n_0$ so $m \notin S$ and hence $a_m^{h(m)} = g_{\delta_m}^{A_m}(u_m^{h(m)})$. Therefore $K \subseteq u_m^{h(m)} \subseteq |a_m^{h(m)}|^{\leftarrow}[0, \delta_m) \subseteq |a_m^{h(m)}|^{\leftarrow}[0, \frac{1}{n_0}) \subseteq |a_m^{h(m)}|^{\leftarrow}[0, \varepsilon)$ (because $\delta_m \leq 1/m$). This just means $a_m^{h(m)} \in O(K, \varepsilon)$. So, o is adherent to $\{a_n^{h(n)} : n \in N\}$.

As we see, σ cannot be a winning strategy for W. $\square$

THEOREM 2.1. *For a space X, the implications $(i) \Rightarrow (i+1)$, $i = \overline{1,4}$ between the statements (1)-(5) listed bellow are true:*

(1) *$C_k(X)$ is strictly Fréchet;*
(2) *$^*GameP(\mathcal{K}(X), \mathcal{K}(X))$;*
(3) *$^*GameP(3 - \mathcal{K}_{shr}(X), \mathcal{K}(X))$;*
(4) *the player W has no winning strategy in the game $GamePyt(a)$ played on $C_k(X)$ at any $a \in C_k(X)$;*
(5) *$C_k(X)$ has the selectively Pytkeev property.*

Proof. (1) $\Rightarrow$ (2): If $C_k(X)$ is strictly Fréchet then X satisfies $C(3 - \mathcal{K}_{shr}(X), \mathcal{C}(X))$ and each element of $\mathcal{K}(X)$ has a countable subset which is also in $\mathcal{K}(X)$. Thus we can use Proposition 2.1 to conclude that *Game$C(3 - \mathcal{K}_{shr}(X), \mathcal{C}(X))$ so, by virtue of Proposition 2.2, we have that *Game$P(\mathcal{K}(X), \mathcal{K}(X))$.

As $3 - \mathcal{K}_{shr} \subseteq \mathcal{K}$, the implication (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4) is just Proposition 2.3.

(4) $\Rightarrow$ (5) is trivial. □

THEOREM 2.2. *The statements of Theorem 2.1 are equivalent for a locally compact X.*

Proof. It remains to show (5) $\Rightarrow$ (1). If X is compact then the compact-open topology coincides with the topology of uniform convergence on $C(X)$, so $C_k(X)$ is a first-countable space, thus strictly Fréchet. Let X be non compact and let $(B_n : n \in N)$ be a sequence of subsets of $C(X)$ with $o \in \overline{B_n}$.

Denote by $\mathcal{L}$ the family of all open subsets of X whose closure is compact. For each let and put $A := \{f \in C(X) : f[X \setminus U] \subseteq \{1\}$ for a $U \in \mathcal{L}\}$. As X is locally compact it is easy to see that $o \in \overline{A}$ in the space $C_k(X)$. As X is not compact it must be that $o \notin A$. So we can apply the selectively Pytkeev property of $C_k(X)$ to the sequence $(A_n : n \in N)$, where $A_n = A$ for all $n \in N$ so as to obtain an infinite "matrix" $(f_n^m : n, m \in N)$ of elements of A such that the family $\mathcal{F}{:=}\{\{f_n^m : n \in N\} : m \in N\}$ is a π-network at o. For each $n, m \in N$ we have $f_n^m \in A$ so there is a $U_n^m \in \mathcal{L}$ with $f_n^m[X \setminus U_n^m] \subseteq \{1\}$. Put $\mathcal{U}_n := \{U_{n-k+1}^k : k = \overline{1,n}\}$ and $K_n := \overline{\cup \mathcal{U}_n}$. Each K_n is clearly compact. For each n, $o \in \overline{B_n}$ so there is a $f_n \in B_n$ with $f_n[K_n] \subseteq (-1/n, 1/n)$. We show that $(f_n : n \in N)$ converges to o in $C_k(X)$.

Let K be a compact subset of X and $\varepsilon > 0$. Take a $m_0 \in N \setminus \{1\}$ with $m_0 > 1/\varepsilon$. As $\mathcal{F}$ is a π-network at o, there is a n_0 such that $\forall n(f_n^{n_0}[K] \subseteq R \setminus \{1\})$ (R is the real line). Let $l > \max\{m_0, n_0\}$ be arbitrary. $f_l[K_l] \subseteq (-1/l, 1/l)$ and $K_l \supseteq U^{n_0}_{l-n_0+1}$ imply $f_l[U^{n_0}_{l-n_0+1}] \subseteq (-1/l, 1/l) \subseteq (-1/m_0, 1/m_0) \subseteq (-\varepsilon, \varepsilon)$. $f^{n_0}_{l-n_0+1}[K] \subseteq R \setminus \{1\}$ and $f^{n_0}_{l-n_0+1}[X \setminus U^{n_0}_{l-n_0+1}] \subseteq \{1\}$ imply $K \subseteq U^{n_0}_{l-n_0+1}$. Thus $f_l[K] \subseteq (-\varepsilon, \varepsilon)$, i.e., $\forall l > \max\{m_0, n_0\}$ $(f_l \in O(K, \varepsilon))$. □

THEOREM 2.3. *For a locally compact space X the following are equivalent*:

(1) $C_k(X)$ *is strictly Fréchet*;
(2) $C_k(X)$ *has the selectively Pytkeev property*;
(3) $C_k(X)$ *has the selectively Reznichenko property.*

Proof. By Propositions 1.1 and 1.3 it remains to conclude (3) ⇒ (1). But due to Theorems 1.1 and 1.2 this follows directly from Lemma 2.1. □

NOTE. By the last theorem and the proof of Theorem 2.2 one can conclude that the following holds: *if X is locally compact and $C_k(X)$ has the selectively Reznichenko property, then X is σ-compact.*

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