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# GLOBAL COUPLING EFFECTS ON A FREE BOUNDARY PROBLEM FOR THREE-COMPONENT REACTION-DIFFUSION SYSTEM

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Abstract. In this paper, we consider three-component reaction-diffusion system. With an integral condition and a global coupling, this system gives us an interesting free boundary problem. We shall examine the occurrence of a Hopf bifurcation and the stability of solutions as the global coupling constant varies. The main result is that a Hopf bifurcation occurs for global coupling and this motion is transferred to the stable motion for strong global coupling.

## 1. Introduction and interface equation of motion

A reaction-diffusion system has been modeled in the study of the pattern formation in biology [13, 15], chemistry [5, 12], and physics [19, 20, 22]. If the diffusion of the activator is small compared to that of the inhibitor in the two-component system, the stationary solution should undergo certain instabilities and the loss of stability results a Hopf bifurcation and produces a kind of periodic oscillation in the location of the internal layers which are called breathers [3, 4, 11, 17, 18, 21]. The occurrences of a Hopf bifurcation for the free boundary problem as a parameter varies were investigated in [7, 8, 9].

We now consider three-component reaction-diffusion system that describe the interaction of two inhibitors $v$ and $w$ and one activator $u$ in

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[23, 25]:

$$(1)\qquad \begin{cases} \varepsilon\sigma u_t = \varepsilon^2 u_{xx} + f(u, v, < w >), \\ v_t = v_{xx} + g(u, v, < w >), \\ 0 = \int_0^L h(u, v, < w >)dx, \quad x \in (0, L), \, t > 0 \end{cases}$$

with the Neumann boundary conditions $v_x(0,t) = 0 = v_x(L,t)$. Here $\varepsilon$ and $\sigma$ are positive constants and the brackets $< >$ means the spatial average.

In the system (1) without the integral condition, the stationary solution, being smooth, exhibits an abrupt but continuously differentiable transition at the location of the limiting discontinuity for sufficiently small $\varepsilon$ ([4, 18]). Moreover, for a sufficiently large $\sigma$, a motionless pulse is stable and stationary solutions should undergo certain instabilities and the loss of stability resulting from a Hopf bifurcation produces a kind of periodic oscillation in the location of the internal layers in [3, 4, 17, 18]. The authors in [25] showed that there are two types of Hopf destabilizations for the system (1): breathing and swinging and the global coupling controls which type occurs. The oscillation for a double front pattern refers to a breathing mode and this breathing mode switch to swinging mode when the coupling strength increases was shown in [23].

The global coupling via the integral condition in (1) describes a global restriction of the resources, for example a load resistor in an electrical setup [2, 20]. In the case of the two layer model, there is a phenomenological description of an electrical system by a reaction-diffusion system [1, 2, 20, 22].

In this paper, the kinetics are taken the piecewise linear case of FitzHugh-Nagumo type [6, 16] which are

$$(2)\quad f = -u + H(u - a_0) - v, \; g = \mu u - v - < w >, \; h = \kappa_1 - \kappa_2 v - w,$$

where $\kappa_2$ is a global coupling constant and $\mu, a_0$ and $\kappa_1$ are all positive constants. A spatial average of $w$ is defined by $< w >= \frac{1}{L}\int_0^L w dx$. We assume that there is a single interface $s$ when $\varepsilon$ tends to 0. We assume that the velocity of the interface $\frac{C(v)}{\sigma}$ is a continuously differentiable function defined on an interval $I := (-a_0, 1 - a_0)$ and thus using the method, [4, 17] the velocity of the interface can be normalized by

$$C(r) := \frac{2r + 2a_0 - 1}{\sqrt{(r + a_0)(1 - a_0 - r)}}.$$

By letting $\varepsilon = 0$ in (1), then $< v >$ satisfies that

$$\begin{aligned} < v >' (t) &= \mu < u > - < v > - < w > \\ &= - (\mu + 1 - \kappa_2) < v > + \mu(1 - \tfrac{s}{L}) - \kappa_1 . \end{aligned}$$

Hence, the single interface problem with the global coupling is obtained by

(3)

$$\begin{cases} v_t = v_{xx} - (\mu + 1)v + \mu H(x - s) + \kappa_2 q - \kappa_1, \\ \qquad\qquad\qquad\qquad\qquad\qquad (x \in (0, L) \setminus \{s\}, t > 0) \\ s'(t) = \frac{1}{\sigma} C(v(s(t), t), q(t)), \quad (t > 0) \\ q'(t) = -(\mu + 1 - \kappa_2)q + \mu(1 - \frac{s}{L}) - \kappa_1, \quad (t > 0) \\ v(x, 0) = v_0(x) , \ s(0) = s_0 , \ q(0) = q_0, \end{cases}$$

where $q =< v >$.

We shall investigate an occurrence of a Hopf bifurcation and the stability of solutions of the free boundary problem (3) when the global coupling constant $\kappa_2$ varies in this paper. The regular setting of (3) is adapted from [9] in the next section. In section 3, the existence of steady states is shown for a global coupling constant. To examine the effects on global coupling, an occurrence of a Hopf bifurcation for the weak and strong global couplings is investigated in section 4.

## 2. Regularized equation of motion

Let $A$ be an operator defined by $Av = -v_{xx} + (\mu + 1)v$ together with Neumann boundary conditions $v_x(0) = v_x(L) = 0$ in system (3). For the purposes of the results in this section, $A$ can also be any other invertible second order operator. For the application of semigroup theory to (3), we choose a space $X := L_2((0, L))$ with norm $\| \cdot \|_2$.

DEFINITION 2.1. We call $(v, s, q)$ a solution of (3), if it satisfies the following natural properties: There exists $T > 0$ such that $v(x, t)$ is defined for $(x, t) \in [0, L] \times [0, T)$, $s(t) \in (0, L)$ and $v(s(t), t) \in I$ for $t \in [0, T)$,

(i) $v(\cdot, t) \in C^1([0, L])$ for $t > 0$ with $v_x(0, t) = v_x(L, t) = 0$,
(ii) $s, q \in C^0([0, T)) \cap C^1((0, T))$ with $s(0) = s_0 \in (0, L)$ and $q(0) = q_0 \in \mathbb{R}$,
(iii) $(Av)(x, t)$ and $v_t(x, t)$ exist for $x \in (0, L) \setminus \{s(t)\}$ and $t \in (0, T)$,
(iv) $t \mapsto v(\cdot, t) \in C^0([0, T), X)$ with $v(\cdot, 0) = v_0 \in X$ and

(v) $v$, $s$ and q solve the differential equation for $t \in (0,T)$ and $x \in (0,L) \setminus \{s(t)\}$.

As a first step we obtain more regularity for the solution by semigroup methods, considering $A$ as a densely defined operator

$$\begin{cases} A : D(A) \subset_{\text{dense}} X \longrightarrow X \\ D(A) := \{v \in H^{2,2}((0,1)) \ : \ v_x(0) = v_x(L) = 0\}\,. \end{cases}$$

For fixed $s$ satisfying Definition 2.1, the map $t \mapsto H(\cdot - s(t))$ is locally Hölder-continuous into $X$ on $(0,T)$, so by standard results for parabolic problems [10] we obtain from the first equation in (3) that the following regularity holds for $v$:

THEOREM 2.2. *If $(v,s,q)$ is a solution of (3) then $v(\cdot,t) \in D(A)$ and the map $t \mapsto v(\cdot,t)$ is in $C^0([0,T),X) \cap C^1((0,T),X)$.*

We decompose $v$ in (3), into a part $u$, which is a solution to a more regular problem, and a part $g$, which is worse, but explicitly known in terms of the Green's function of the operator $A$.

THEOREM 2.3. *Let $G : [0,L]^2 \to \mathbb{C}$ be a Green's function of the operator $A$. Define $g : [0,L] \times [0,L] \times \mathbb{C} \longrightarrow \mathbb{C}$ by*

$$g(x,s,q) := A^{-1}(\mu H(x-s) + \kappa_2 q - \kappa_1)$$

*and $\gamma : [0,L] \times \mathbb{C} \longrightarrow \mathbb{C}$ by*

$$\gamma(s,q) := g(s,s,q)\,.$$

*Then $g(\cdot,s,q) \in D(A) \times \mathbb{C}$ for all $s$, $q$, $\dfrac{\partial g}{\partial s}(x,s,q) = -\mu G(x,s)$ is in $H^{1,\infty}((0,L) \times (0,L))$ and $\gamma \in C^\infty([0,L]) \times \mathbb{C}$.*

*Proof.* Everything follows from the fact that $G$ is in $H^{1,\infty}$ and $C^\infty$ on either $\{x \le y\}$ or $\{x \ge y\}$, and that $H(\cdot - s) \in L^2$. □

Using these preliminary observations, we decompose a solution $(v, s, q)$ of (3) into two parts by defining

$$u(t)(x) := v(x,t) - g(x,s(t),q(t))\,.$$

The initial value problem for $(u,s,q)$ can then be written as

$$\begin{aligned} &\frac{d}{dt}(u,s,q) + \widetilde{A}(u,s,q) = f(u,s,q) \\ &(u,s,q)(0) = (u(0),s(0),q(0)) = (u_0,s_0,q_0) \end{aligned} \tag{4}$$

of a differential equation in a Banach space $\widetilde{X}$ of the form $\widetilde{X} := X\times\mathbb{R}\times\mathbb{C}$. The operator $\widetilde{A}$, represented in matrix form

$$\widetilde{A} = \begin{pmatrix} A & -\frac{\mu\kappa_2}{L(\mu+1)} & -\frac{\kappa_2(\mu+1-\kappa_2)}{\mu+1} \\ 0 & 0 & 0 \\ 0 & \frac{\mu}{L} & \mu+1-\kappa_2 \end{pmatrix}.$$

The nonlinear forcing term $f$ defined on the set

$$W = \{(u,s,q)\in C^1([0,L])\times\mathbb{R}\times\mathbb{C} \,:\, u(s)+\gamma(s,q)\in I\} \subset_{\text{open}} C^1([0,L])\times\mathbb{R}\times\mathbb{C}$$

is as follows :

$$f(u,s,q) = \begin{pmatrix} \frac{1}{\sigma} f_2(u,s,q)\cdot f_1(s) - \frac{\kappa_2(\mu-\kappa_1)}{\mu+1} \\ \frac{1}{\sigma} f_2(u,s,q) \\ \mu-\kappa_1 \end{pmatrix},$$

where

$$f_1 : (0,L)\to X,\ \ f_1(s)(x) := \mu\, G(x,s)$$

and

$$f_2 : W\to\mathbb{C},\ \ f_2(u,s,q) := C\big(u(s)+\gamma(s,q),s,q\big),$$

$$C(u(s)+\gamma(s,q),s,q) = \frac{2\big(u(s)+\gamma(s,q)\big)+2a_0-1}{\sqrt{\big(a_0+u(s)+\gamma(s,q)\big)\big(1-a_0-u(s)-\gamma(s,q)\big)}}.$$

We have the following lemma whose the proof refers to [9].

LEMMA 2.4. *The functions $f_1 : (0,L)\to X$, $f_2 : W\to\mathbb{C}$ and $f : W\to\widetilde{X}$ are continuously differentiable with derivatives given by*

$$f_1'(s) = \mu\frac{\partial G}{\partial x}(\cdot,s),$$

$$Df_2(u,s,q)(\hat{u},\hat{s},\hat{q}) = C'\left(u(s)+\gamma(s,q)\right)\cdot\left(u'(s)\hat{s}+\frac{\partial\gamma}{\partial s}(s,q)\hat{s}+\frac{\partial\gamma}{\partial q}(s,q)\hat{q}+\hat{u}(s)\right),$$

$$Df(u,s,q)(\hat{u},\hat{s},\hat{q}) = \frac{1}{\sigma} f_2(u,s,q)\cdot(f_1'(s),0,0)\cdot\hat{s}+\frac{1}{\sigma} Df_2(u,s,q)(\hat{u},\hat{s},\hat{q})\cdot(f_1(s),1,0).$$

We can now apply semigroup theory to (4) using domains of fractional powers of $A$ and $\widetilde{A}$:

$$X^\alpha := D(A^\alpha)\,,\quad \widetilde{X}^\alpha := D(\widetilde{A}^\alpha) = X^\alpha \times \mathbb{R} \times \mathbb{C}\,, \alpha \in [0,1].$$

In order for $f : W \cap \widetilde{X}^\alpha \to \widetilde{X}$ to be continuously differentiable, we take $\alpha > 3/4$ so that $X^\alpha \subset C^1([0,L])$. Standard applications of theorems for existence, uniqueness and dependence on initial data [10] together with the starting regularity of solutions to (3) (Theorem 2.2), as well as the regularity of the functions $g$ and $\gamma$ (Theorem 2.3) then give the following result:

THEOREM 2.5. (i) *For any $\alpha$, $3/4 < \alpha < 1$, $(u_0, s_0, q_0) \in W \cap \widetilde{X}^\alpha$ and $\sigma \in \mathbb{R}$ there exists a unique solution*

$$(u,s,q)(t) = (u,s,q)(t; u_0, s_0, q_0, \sigma)$$

*of* (4). *The solution operator*

$$(u_0, s_0, q_0, \sigma) \mapsto (u,s,q)(t; u_0, s_0, q_0, \sigma)$$

*is continuously differentiable from $\widetilde{X}^\alpha \times \mathbb{R}$ into $\widetilde{X}^\alpha$ for $t > 0$. The functions $v(x,t)$,*

$$v(x,t) := u(t)(x) + g(x, s(t), q(t)),$$

*$s$ and $q$ then satisfy* (3) *with $v(\cdot, 0) \in X^\alpha$ and $v(s_0, 0) \in I$.*

(ii) *If $(v,s,q)$ is a solution of* (3) *for some $\mu \in \mathbb{R}$ with initial condition $v_0 \in X^\alpha$, $1 > \alpha > 3/4$, $s_0 \in (0,L)$, $v_0(s_0) \in I$, then $(u_0, s_0, q_0) := (v_0 - g(\cdot, s_0, q_0), s_0, q_0) \in \widetilde{X}^\alpha \cap W$ and*

$$(v(\cdot,t), s(t), q(t)) = (u,s,q)(t; u_0, s_0, q_0, \mu) + (g(\cdot, s(t), q(t)), 0, 0)\,,$$

*where $(u,s,q)(t; u_0, s_0, q_0, \mu)$ is the unique solution of* (4).

(iii) *For any $1 > \alpha > 3/4$, $\mu \in \mathbb{R}$, $(v_0, s_0, q_0) \in U := \{(v,s,q) \in X^\alpha \times (0,L) \times \mathbb{C}' : v(s) \in I\}$ problem* (3) *has a unique solution*

$$(v(x,t), s(t), q(t)) = (v,s,q)(x,t; v_0, s_0, q_0, \sigma)\,.$$

*Additionally, the mapping*

$$(v_0, s_0, q_0, \sigma) \mapsto (v,s,q)(\cdot, t; v_0, s_0, q_0, \sigma)$$

*is continuously differentiable from $\widetilde{X}^\alpha \times \mathbb{R}$ into $\widetilde{X}^\alpha$.*

The proof refers to [9].

## 3. Steady states and Linearized equation of motion

The stationary problem of (4) is given by

$$(5)\qquad \begin{cases} Au^* = \frac{\mu}{\sigma}\, G(\cdot, s^*)\, C(u^*(s^*) + \gamma(s^*, q^*)\,) \\ \qquad -\frac{\kappa_2}{\mu+1}\big( -(\mu+1-\kappa_2)q^* + \mu(1-\frac{s^*}{L}) - \kappa_1\big), \\ 0 = \frac{1}{\sigma}\, C(u^*(s^*) + \gamma(s^*, q^*)\,), \\ 0 = -(\mu+1-\kappa_2)q^* + \mu(1-\frac{s^*}{L}) - \kappa_1, \\ u^{*\prime}(0) = 0 = u^{*\prime}(L), \end{cases}$$

for $(u^*, s^*, q^*) \in W \cap (D(A) \times \mathbb{R} \times \mathbb{C})$. We thus obtain the following theorem.

THEOREM 3.1. (1) *Case without global coupling* ($\kappa_2 = 0$). *Assume that* $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1}$. *For all* $\sigma > 0$, *the problem of* (4) *has the solution* $(0, s^*, q^*)$ *with* $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right) - \kappa_1}{\mu+1}$ *and* $s^* \in (0, L)$.

(2) *Case with global coupling* ($\kappa_2 > 0$).

(a) *Suppose* $\kappa_2 = \mu+1$. *Then for all* $\sigma > 0$ *the stationary problem of* (4) *has the solution* $(0, s^*, q^*)$ *with* $q^* = \frac{1}{2} - a_0 + \frac{\kappa_1}{\mu+1} - \frac{\cosh L\sqrt{\mu+1}(1-\frac{\kappa_1}{\mu})\,\sinh(L\sqrt{\mu+1}\frac{\kappa_1}{\mu})}{(\mu+1)\sinh(L\sqrt{\mu+1})}$ *and* $s^* = L(1-\frac{\kappa_1}{\mu})$.

(b) *Suppose that* $\kappa_2 < \mu+1$ *and* $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1-\kappa_2}$. *For all* $\sigma > 0$, *the problem of* (4) *has the stationary solution* $(0, s^*, q^*)$ *with* $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right) - \kappa_1}{\mu+1-\kappa_2}$ *and* $s^* \in (0, L)$.

(c) *For* $\mu+1 < \kappa_2 < \infty$, *assume that* $\frac{1}{2} - a_0 < \frac{\kappa_1}{\kappa_2-\mu-1}$. *Then there exists the solution* $(0, s^*, q^*)$ *such that* $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right) - \kappa_1}{\mu+1-\kappa_2}$ *and* $s^* \in (s_c, L - s_c)$, *where*

$$s_c = s_c(\kappa_2) = \frac{L}{2} - \frac{1}{2\sqrt{\mu+1}}\ln\left(K + \sqrt{K^2-1}\right),$$
$$K = \frac{\kappa_2\sinh(L\sqrt{\mu+1})}{L\sqrt{\mu+1}(\kappa_2-\mu-1)}.$$

(3) *Case with strong global coupling* ($\kappa_2 \uparrow \infty$). *Suppose that*

$$\frac{1}{2} - a_0 < \frac{\mu}{2L\,(\mu+1)^{3/2}K_\infty}\Big(K_\infty \ln\left(K_\infty + \sqrt{K_\infty^2-1}\right) - \sqrt{K_\infty^2-1}\Big),$$

*where* $K_\infty = \frac{\sinh(L\sqrt{\mu+1})}{L\sqrt{\mu+1}}$. *Then for all* $\sigma > 0$, *the problem of* (4) *has the stationary solution* $(0, s^*, q^*)$ *with* $q^* = 0$ *and* $s^* \in (s_\infty, L - s_\infty)$, *where* $s_\infty = \lim_{\kappa_2\to\infty} s_c(\kappa_2)$.

*For all cases, the linearization of $f$ at $(0,s^*,q^*)$ is*

$$\begin{aligned} &Df(0,s^*,q^*)(\hat{u},\hat{s},\hat{q}) \\ &= \begin{pmatrix} \frac{4}{\sigma}\Big(\hat{u}(s^*)+\gamma_s(s^*,q^*)\hat{s}+\gamma_q(s^*,q^*)\hat{q}\Big)\,\mu G(\cdot,s^*) \\ \frac{4}{\sigma}\Big(\hat{u}(s^*)+\gamma_s(s^*,q^*)\hat{s}+\gamma_q(s^*,q^*)\hat{q}\Big) \\ 0 \end{pmatrix}. \end{aligned}$$

*The pair $(0,s^*,q^*)$ corresponds to a unique steady state $(v^*,s^*,q^*)$ of (3) for $\sigma\neq 0$ with $v^*(x)=g(x,s^*,q^*)$.*

*Proof.* System (5) is equivalent to the pair of equations

$$u^*=0,\ C(\gamma(s^*,q^*))=0 \quad \text{and} \quad (\mu+1-\kappa_2)q^*=\mu(1-\tfrac{s^*}{L})-\kappa_1$$

which implies that

$$\gamma(s^*,q^*)-(\tfrac{1}{2}-a_0)=0 \quad \text{and} \quad (\mu+1-\kappa_2)\,q^*=\mu(1-\tfrac{s^*}{L})-\kappa_1\,.$$

Let $\xi(s):=\int_0^L \mu G(s,y)\ H(y-s)dy$. Then $\gamma(s,q)=\xi(s)+\frac{\kappa_2 q-\kappa_1}{\mu+1}$. We define a function

$$F(s):=\xi(s)+\frac{\kappa_2\,q-\kappa_1}{\mu+1}-(\tfrac{1}{2}-a_0) \tag{6}$$

satisfying

$$(\mu+1-\kappa_2)\,q=\mu(1-\tfrac{s}{L})-\kappa_1\,.$$

(1) For $\kappa_2=0$, $q^*$ is easily obtained. Since $F'(s)<0$ and $F(L)<0$, there exists a unique $s^*$ in $(0,L)$ provided that $F(0)>0$ which is equivalent to $\frac{1}{2}-a_0<\frac{\mu-\kappa_1}{\mu+1}$.

(2) (a) If $\kappa_2=\mu+1$, then $s^*=L(1-\frac{\kappa_1}{\mu})$ and $q^*$ is a solution of $F\big(L(1-\frac{\kappa_1}{\mu})\big)=0,;\ q^*=-\xi\big(L(1-\frac{\kappa_1}{\mu})\big)+\frac{\kappa_1}{\mu+1}+\frac{1}{2}-a_0$, where $\xi\big(L(1-\frac{\kappa_1}{\mu})\big)=-\frac{\cosh(L\sqrt{\mu+1}(1-\frac{\kappa_1}{\mu}))\,\sinh(L\sqrt{\mu+1}\,\frac{\kappa_1}{\mu})}{(\mu+1)\sinh(L\sqrt{\mu+1})}$.

(b) Suppose $\kappa_2<\mu+1$. We have $F'(s)=\xi'(s)-\frac{\kappa_2}{\mu+1}\frac{\mu}{L(\mu+1-\kappa_2)}<0$ and $F(L)=-\frac{\kappa_1}{\mu+1-\kappa_2}-(\frac{1}{2}-a_0)<0$. Hence in order to exist a solution $s^*\in(0,L)$, we need a condition $F(0)>0$, where $F(0)=\frac{\mu}{\mu+1}+\frac{1}{\mu+1}\Big(\frac{(\mu-\kappa_1)\kappa_2}{\mu+1-\kappa_2}-\kappa_1\Big)-(\frac{1}{2}-a_0)=\frac{\mu-\kappa_1}{\mu+1-\kappa_2}-(\frac{1}{2}-a_0)$.

(c) Suppose $\mu+1<\kappa_2$. We shall find the critical points and the local maximum of $F(s)$ in $(0,L)$. The derivative of $F(s)$ is given by

$$\begin{aligned} F'(s)=&-\frac{\mu}{\sqrt{\mu+1}\,\sinh(L\sqrt{\mu+1})}\cosh\sqrt{\mu+1}(L-2s) \\ &+\frac{\mu\kappa_2}{L(\mu+1)(\kappa_2-\mu-1)}\,. \end{aligned}$$

The critical point $s_c$ is given by

$$s_c := s_c(\kappa_2) = \frac{L}{2} - \frac{1}{2\sqrt{\mu+1}} \ln\left(K - \sqrt{K^2-1}\right)$$

with

$$K = \frac{\kappa_2 \sinh(L\sqrt{\mu+1})}{L\sqrt{\mu+1}(\kappa_2 - \mu - 1)}.$$

We note that $F'(s) = F'(L-s)$ and $F'(s) > 0$ for $s_c < s < L - s_c$, and $F'(s) < 0$ for $s < s_c$ or $L - s_c < s$. Since $F(0) = \frac{\mu-\kappa_1}{\mu+1-\kappa_2} - (1/2 - a_0) < 0$ and $F'(s) < 0$ for $0 < s < s_c$, we have $F(s_c) < 0$. And since $F'(s) > 0$ for $s \in (s_c, L - s_c)$, we have $F(L - s_c) > F(L)$. If we assume that $F(L) > 0$ then $F(L - s_c) > 0$ and thus there is the only point $s^* \in (s_c, L - s_c)$.

(3) Assume $\kappa_2$ is sufficiently large. In the equation (6),

$$\kappa_2 = \mu + 1 - \frac{\mu(1 - \frac{s}{L}) - \kappa_1}{q}$$

we have $q^* = 0$. We now show the existence of $s^*$. For large $\kappa_2$, the equation (6) implies that

$$\xi(s) - \frac{\mu}{L(\mu+1)}(L - s) - (\tfrac{1}{2} - a_0) = 0$$

since $\lim_{\kappa_2 \uparrow \infty} \kappa_2 q = \mu(\frac{s}{L} - 1) + \kappa_1$. Let $F_\infty(s) := \xi(s) - \frac{\mu}{L(\mu+1)}(L - s) - (\frac{1}{2} - a_0)$ and then $F_\infty(0) = F_\infty(L/2) = F_\infty(L) = a_0 - \frac{1}{2} F'_\infty(0) = F'_\infty(L) < 0$ and $F'_\infty(L/2) > 0$. Thus our claim is that there is a point $s$ in $(L/2, L)$ such that $F_\infty(s) > 0$. To do this we shall find a critical point $s_\infty \in (0, L/2)$ of $F'(s_\infty) = 0$ and find the conditions of $\mu$ and $L$ to satisfy $F(L - s_\infty) > 0$. The critical point $s_\infty$ satisfy that $\cosh(\sqrt{\mu+1}\,(2s_\infty - L)) = \frac{\sinh(L\sqrt{\mu+1})}{L\sqrt{\mu+1}}$, that is,

$$s_\infty = \frac{L}{2} - \frac{1}{2\sqrt{\mu+1}} \ln\left(K_\infty - \sqrt{K_\infty^2 - 1}\right),$$

where $\lim_{\kappa_2 \to \infty} K := K_\infty = \frac{\sinh(L\sqrt{\mu+1})}{L\sqrt{\mu+1}}$. The formula for $Df(0, s^*, q^*)$ follows from Lemma 2.4 in [9] and the relation $C'(\gamma(s^*, q^*)) = 4$. The corresponding steady state $(v^*, s^*, q^*)$ for (3) is obtained using Theorem 2.5 in [9]. □

## 4. Effects of global coupling and the Hopf bifurcation

We shall deal with the linearized eigenvalue problem for (4) which can be obtained at the stationary solutions. The linearized eigenvalue problem is

$$-\widetilde{A}(u,s,q)+\tau Df(0,s^*,q^*)(u,s,q)=\lambda(u,s,q)$$

which is equivalent to

$$(7)\qquad \begin{cases}(A+\lambda)u=\tau\big(u(s^*)+\gamma_s(s^*,q^*)\,s+\gamma_q(s^*,q^*)\,q\big)\,\mu G(x,s^*)\\ \qquad\qquad +\frac{\kappa_2}{\mu+1}\big((\mu+1-\kappa_2)q+\frac{\mu}{L}s\big),\\ \lambda\,s=\tau\big(u(s^*)+\gamma_s(s^*,q^*)\,s+\gamma_q(s^*,q^*)\,q\big),\\ \lambda\,q=-(\mu+1-\kappa_2)q-\frac{\mu}{L}s,\end{cases}$$

where $\tau=\frac{4}{\sigma}$. Then we show that there is a Hopf bifurcation from the curve $\tau\mapsto(0,s^*,q^*)$ of steady states as a global coupling constant $\kappa_2$ varies, and therefore introduce the following definition:

DEFINITION 4.1. Under the assumptions of Theorem 3.1, define (for $1\geq\alpha>3/4$) the operator $B:=Df(0,s^*,q^*)\in L(\widetilde{X}^\alpha,\widetilde{X})$. We then define $(0,s^*,q^*,\tau^*)$ to be a Hopf point for (4) if and only if there exists an $\epsilon_0>0$ and a $C^1$-curve

$$(-\epsilon_0+\tau^*,\tau^*+\epsilon_0)\mapsto(\lambda(\tau),\phi(\tau))\in\mathbb{C}\times\widetilde{X}_C$$

($Y_C$ denotes the complexification of the real space $Y$) of eigendata for $-\widetilde{A}+\tau B$ with

(i) $(-\widetilde{A}+\tau B)(\phi(\tau))=\lambda(\tau)\phi(\tau),\quad(-\widetilde{A}+\tau B)(\overline{\phi(\tau)})=\overline{\lambda(\tau)}\,\overline{\phi(\tau)}$;
(ii) $\lambda(\tau^*)=i\beta$ with $\beta>0$;
(iii) $\mathrm{Re}\,(\lambda)\neq0$ for all $\lambda\in\sigma(-\widetilde{A}+\tau^*B)\setminus\{\pm i\beta\}$;
(iv) $\mathrm{Re}\,\lambda'(\tau^*)\neq0$ (transversality).

### 4.1. A Hopf bifurcation without global coupling($\kappa_2=0$)

We state our main theorem:

THEOREM 4.2. *Suppose that $\frac{1}{2}-a_0<\frac{\mu-\kappa_1}{\mu+1}$. The problem (4), respectively (3), has stationary solutions $(u^*,s^*,q^*)$ where $u^*=0$ and $q^*=\frac{\mu\left(1-\frac{s^*}{L}\right)-\kappa_1}{\mu+1}$ respectively $(v^*,s^*,q^*)$ for all $\tau>0$. Then there exists a unique $\tau^*$ such that the linearization $-\widetilde{A}+\tau^*B$ has a purely imaginary pair of eigenvalues $\beta>0$. The point $(0,s^*,q^*,\tau^*)$ is then a Hopf point for (4) and there exists a $C^0$-curve of nontrivial periodic orbits for (4), respectively (3), bifurcating from $(0,s^*,q^*,\tau^*)$, respectively $(v^*,s^*,q^*,\tau^*)$.*

### 4.2. A Hopf bifurcation with global coupling ($\kappa_2 > 0$)

THEOREM 4.3. *Suppose that* (i) $\kappa_2 = \mu + 1$ *then the problem* (4), *respectively* (3), *has a unique stationary solution* $(u^*, s^*, q^*)$ *where* $u^* = 0$, $q^* = \frac{1}{2} - a_0 + \frac{\kappa_1}{\mu+1} - \frac{\cosh L\sqrt{\mu+1}(1-\frac{\kappa_1}{\mu})\sinh(L\sqrt{\mu+1}\frac{\kappa_1}{\mu})}{(\mu+1)\sinh(L\sqrt{\mu+1})}$ *and* $s^* = L(1 - \frac{\kappa_1}{\mu})$ *respectively* $(v^*, s^*, q^*)$ *for all* $\tau > 0$. (ii) *If* $0 \leq \kappa_2 < \mu + 1$ *and* $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1-\kappa_2}$, *the problem* (4), *respectively* (3), *has stationary solutions* $(u^*, s^*, q^*)$ *where* $u^* = 0$ *and* $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right) - \kappa_1}{\mu+1-\kappa_2}$ *respectively* $(v^*, s^*, q^*)$ *for all* $\tau > 0$. *Then there exists a unique* $\tau^*$ *such that the linearization* $-\widetilde{A} + \tau^* B$ *has a purely imaginary pair of eigenvalues* $\beta > 0$. *The point* $(0, s^*, q^*, \tau^*)$ *is then a Hopf point for* (4) *and there exists a* $C^0$*-curve of nontrivial periodic orbits for* (4), *respectively* (3), *bifurcating from* $(0, s^*, q^*, \tau^*)$, *respectively* $(v^*, s^*, q^*, \tau^*)$.

In order to prove the Theorems 4.2 and 4.3, we shall show the next two lemmas. The following lemma shows that there is a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (7) such that for some $\tau^*$ $(0, s^*, q^*, \tau^*)$ is a Hopf point.

LEMMA 4.4. *Assume that* $\kappa_2 = \mu + 1$ *or* $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1-\kappa_2}$ *with* $0 \leq \kappa_2 < \mu+1$. *Suppose that for* $\tau^* \in \mathbb{R}\backslash\{0\}$, *the operator* $-\widetilde{A}+\tau^* B$ *has a unique pair* $\{\pm i\beta\}$ *of purely imaginary eigenvalues. Then* $(0, s^*, q^*, \tau^*)$ *is a Hopf point for* (4).

*Proof.* We assume without loss of generality that $\beta > 0$, and $\phi^*$ is the (normalized) eigenfunction of $-\widetilde{A} + \tau^* B$ with eigenvalue $i\beta$. We need to show that $(\phi^*, i\beta)$ can be extended to a $C^1$-curve $\tau \mapsto (\phi(\tau), \lambda(\tau))$ of eigendata for $-\widetilde{A} + \tau B$ with $\mathrm{Re}(\lambda'(\tau^*)) \neq 0$.

For this let $(\psi_0, s_0, q_0) \in D(A) \times \mathbb{R} \times \mathbb{C}$. First, we see that $s_0 \neq 0$ and $q_0 \neq 0$. If $s_0 = 0$ then $q_0 = 0$ from (7) and thus we have $(A + i\beta)\psi_0 = \mu i\beta\,(s_0\,G(x, s^*) + q_0\,G(x, q^*)) = 0$, which is not possible because $A$ is symmetric. So without loss of generality, let $s_0 = 1$. Then $E(\psi_0, q_0, i\beta, \tau^*) = 0$ by (7), where

$$E : D(A)_C \times \mathbb{C} \times \mathbb{C} \times \mathbb{R} \longrightarrow X_C \times \mathbb{C},$$

$$E(u, q, \lambda, \tau) :=$$

$$\begin{pmatrix} (A+\lambda)u - \tau(u(s^*) + \gamma_s(s^*, q^*) + \gamma_q(s^*, q^*)\,q)\mu G(x, s^*) - \frac{\kappa_2}{\mu+1}((\mu+1-\kappa_2)q + \frac{\mu}{L}) \\ \lambda - \tau(u(s^*) + \gamma_s(s^*, q^*) + \gamma_q(s^*, q^*)\,q) \\ \lambda q + (\mu+1-\kappa_2)q + \frac{\mu}{L} \end{pmatrix}.$$

The equation $E(u,q,\lambda,\tau)=0$ is equivalent to $\lambda$ being an eigenvalue of $-\widetilde{A}+\tau B$ with eigenfunction $(u,1,q)$. We shall here apply the implicit function theorem to $E$. For that it necessary to check $E$ is of $C^1$-class and that

$$D_{(u,q,\lambda)}E(\psi_0,q_0,i\beta,\tau^*)\in L(D(A)_C\times\mathbb{C},X_C\times\mathbb{C}) \tag{8}$$

is an isomorphism.

It is easy to see that $E$ is $C^1$ and in addition, the mapping

$$\begin{aligned} &D_{(u,q,\lambda)}E(\psi_0,q_0,i\beta,\tau^*)(\hat u,\hat q,\hat\lambda)\\ &=\begin{pmatrix} (A+i\beta)\hat u+\hat\lambda\,\psi_0-\tau^*\left(\hat u(s^*)+\gamma_q(s^*,q^*)\,\hat q\right)\mu G(x,s^*)-\frac{\kappa_2}{\mu+1}(\mu+1-\kappa_2)\hat q\\ \hat\lambda-\tau^*\left(\hat u(s^*)+\gamma_q(s^*,q^*)\,\hat q\right)\\ \hat\lambda\,q_0+i\beta\,\hat q+(\mu+1-\kappa_2)\hat q \end{pmatrix}\end{aligned}$$

is a compact perturbation of the mapping

$$(\hat u,\hat q,\hat\lambda)\longmapsto\Big((A+i\beta)\hat u,\ \hat q,\ \hat\lambda\Big)$$

which is invertible. Thus $D_{(u,q,\lambda)}E(\psi_0,q_0,i\beta,\tau^*)$ is a Fredholm operator of index 0. Therefore in order to verify (8), it suffices to show that the system

$$D_{(u,q,\lambda)}E(\psi_0,q_0,i\beta,\tau^*)(\hat u,\hat q,\hat\lambda)=0$$

which are

$$\begin{cases} (A+i\beta)\hat u+\hat\lambda\psi_0=\tau^*\left(\hat u(s^*)+\gamma_q(s^*,q^*)\,\hat q\right)\mu G(\cdot,s^*)\\ \qquad\qquad\qquad +\frac{\kappa_2}{\mu+1}(\mu+1-\kappa_2)\hat q\\ \hat\lambda=\tau^*\left(\hat u(s^*)+\gamma_q(s^*,q^*)\hat q\right)\\ \hat\lambda\,q_0+i\beta\,\hat q=-(\mu+1-\kappa_2)\hat q \end{cases} \tag{9}$$

necessarily implies that $\hat u=0$, $\hat q=0$ and $\hat\lambda=0$. We define $\phi(x):=\psi_0(x)-\mu G(x,s^*)+\frac{\kappa_2}{\mu+1}q_0$. Then the first equation of (9) is given by

$$(A+i\beta)\hat u+\hat\lambda\phi=-i\beta\frac{\kappa_2}{\mu+1}\,\hat q\,. \tag{10}$$

Since $(u,s,q,\lambda)=(\psi_0,1,q_0,i\beta)$ solves (7), $\phi$ is a solution to the equation

$$(A+i\,\beta)\phi=-\mu\delta_{s^*}+\kappa_2 q_0 \tag{11}$$

and

(12)

$$i\beta=\tau^*\Big(\phi(s^*)+\mu G(s^*,s^*)+\gamma_s(s^*,q^*)+\gamma_q(s^*,q^*)q_0-\frac{\kappa_2}{\mu+1}\,q_0\Big)$$

$$i\beta q_0=-\,(\mu+1-\kappa_2)q_0-\mu/L.$$

Multiply (11) by $\overline{\phi}$ and integrate. Then we have

$$\begin{aligned}
&\int |A^{1/2}\phi|^2 + i\beta|\phi|^2 \\
&= -\mu\overline{\phi(s^*)} + \kappa_2 q_0 \int \overline{\phi} \\
&= \mu\Big(\frac{i\beta}{\tau^*} + \mu G(s^*, s^*) + \gamma_s(s^*, q^*) + \gamma_q(s^*, q^*)q_0 - \tfrac{\kappa_2}{\mu+1}\, q_0\Big) + \kappa_2\, |q_0|^2 L
\end{aligned}$$

since $\int \phi = \frac{-\mu+\kappa_2\, q_0 L}{\mu+1+i\beta} = q_0\, L$ by (12). This implies that

$$\int |\phi|^2 = \frac{\mu}{\tau^*}. \tag{13}$$

Now multiply (11) by $\hat{u}$ and integrate then we have

$$\int (A + i\beta)\phi\, \hat{u} = -\mu\, \hat{u}(s^*) + \kappa_2\, q_0 \int \hat{u}(x)dx. \tag{14}$$

Since

$$\int (A + i\beta)\phi\, \hat{u} = \int \Big( - \hat{\lambda}\phi - i\beta\, \frac{\kappa_2}{\mu+1}\, \hat{q}\, \Big)\phi = -\hat{\lambda} \int \phi^2 - i\beta\, \frac{\kappa_2}{\mu+1}\, \hat{q}\, q_0 L$$

and

$$\int \hat{u}(x)dx = -\frac{1}{\mu+1+i\beta}\Big(\hat{\lambda} q_0 L + \frac{\kappa_2}{\mu+1} i\beta\, L\, \hat{q}\Big) = \frac{\mu+1-\kappa_2}{\mu+1}\, L\, \hat{q},$$

equation (14) implies that

$$-\hat{\lambda} \int \phi^2 - i\beta\, \frac{\kappa_2}{\mu+1}\, \hat{q}\, q_0 L = -\frac{\mu}{\tau^*}\, \hat{\lambda} + \frac{\mu\kappa_2}{\mu+1}\hat{q} + \frac{\mu+1-\kappa_2}{\mu+1}\, \kappa_2\, q_0\, \hat{q} L\, .$$

Using fact (13), we have

$$\begin{aligned}
\hat{\lambda} \int (|\phi|^2 - \phi^2) &= \frac{\mu\kappa_2}{\mu+1}\hat{q} + \frac{i\beta}{\mu+1}\kappa_2\, q_0 L\, \hat{q} + \frac{\mu+1-\kappa_2}{\mu+1}\, \kappa_2\, q_0\, L\hat{q} \\
&= \frac{\kappa_2 \hat{q}}{\mu+1}\Big(\mu + q_0\, L(\mu+1-\kappa_2+i\beta\,)\Big)
\end{aligned}$$

which implies

$$\hat{\lambda} \int (|\phi|^2 - \phi^2) = 0$$

since $\mu + q_0\, L(\mu+1-\kappa_2+i\beta\,) = 0$ by (12). Hence we obtain that $\hat{\lambda} = 0$ and thus $\hat{q} = 0, \hat{u} = 0$ for $0 \le \kappa_2 < \infty$.

We now show the transversality condition $(iv)$ in the Definition 4.1 holds. By the implicit differentiation of $E(\psi_0(\tau), q(\tau), \lambda(\tau), \tau) = 0$,

$$D_{(u,q,\lambda)}E(\psi_0, i\beta, \tau^*)(\psi_0'(\tau^*), q'(\tau), \lambda'(\tau^*)) = \begin{pmatrix} (\psi_0(s^*) + \gamma_s + \gamma_q\, q_0)\mu G(x, s^*) \\ \psi_0(s^*) + \gamma_s + \gamma_q\, q_0 \\ 0 \end{pmatrix}.$$

This means that the functions $\tilde{u} := \psi_0'(\tau^*)$, $\tilde{q} := q'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

(15)
$$\begin{cases} (A + i\beta)\tilde{u} + \tilde{\lambda}\psi_0 - \tau^*\left(\tilde{u}(s^*) + \gamma_q\,\tilde{q}\right)\mu G(x, s^*) - \frac{\kappa_2}{\mu+1}(\mu + 1 - \kappa_2)\tilde{q} \\ \qquad = (\psi_0(s^*) + \gamma_s + \gamma_q\, q_0)\mu G(x, s^*), \\ \tilde{\lambda} - \tau^*(\tilde{u}(s^*) + \gamma_q\,\tilde{q}) = \psi_0(s^*) + \gamma_s + \gamma_q\, q_0, \\ \tilde{\lambda}\, q_0 + i\,\beta\,\tilde{q} = -(\mu + 1 - \kappa_2)\tilde{q}. \end{cases}$$

The equations (11) and (15) imply that

$$(A + i\beta)\,\tilde{u} + \tilde{\lambda}\,\phi = -\tfrac{\kappa_2}{\mu+1}\, i\,\beta\,\tilde{q}, \tag{16}$$

$$\tilde{u}(s^*) = \frac{\tilde{\lambda}}{\tau^*} - \frac{i\beta}{\tau^{*2}} - \tfrac{\kappa_2}{\mu+1}\tilde{q}, \tag{17}$$

where $\phi := \psi_0 - \mu G(\cdot, s^*) + \frac{\kappa_2}{\mu+1} q_0$. Multiplying $\overline{\phi}$ by (16) and integrating, we obtain

$$\int (A + i\beta)\,\tilde{u}\,\overline{\phi} + \tilde{\lambda}\int |\phi|^2 = -\frac{\kappa_2}{\mu+1} i\,\beta\,\tilde{q}\int \overline{\phi}$$

which implies that

$$-\mu\,\tilde{u}(s^*) + \kappa_2\,\overline{q_0}\int \tilde{u} + 2i\beta\int \tilde{u}\,\overline{\phi} + \tilde{\lambda}\int |\phi|^2 = -\frac{\kappa_2}{\mu+1} i\,\beta\,\tilde{q}\, q_0\, L\,. \tag{18}$$

Multiplying $\tilde{u}$ by the complex conjugate of (16) and integrating, we obtain

$$\int |A^{1/2}\,\tilde{u}|^2 - i\beta\int |\tilde{u}|^2 + \overline{\tilde{\lambda}}\int \overline{\phi}\,\tilde{u} = \frac{\kappa_2}{\mu+1}\, i\,\beta\,\overline{\tilde{q}}\int \tilde{u}.$$

Compare this equation with (18), then we have

(19)
$$\mu\bar{\tilde{\lambda}}\Big(\frac{i\beta}{(\tau^*)^2}+\frac{\kappa_2\,\tilde{q}}{\mu+1}\Big)+\Big(\kappa_2\,\overline{\tilde{\lambda}\,q_0}-2\frac{\kappa_2}{\mu+1}\,\beta^2\,\bar{\tilde{q}}\Big)\int\tilde{u}+\frac{\kappa_2}{\mu+1}i\,\beta\,L\tilde{q}\,\overline{\tilde{\lambda}\,q_0}$$
$$=2i\beta\int|A^{1/2}\tilde{u}|^2+2\beta^2\int|\tilde{u}|^2$$

Since $\int\tilde{u}=\frac{\mu+1-\kappa_2}{\mu+1}\,\tilde{q}\,L$, the second and third terms of the left hand side of (19) is

$$\Big(\kappa_2\,\overline{\tilde{\lambda}\,q_0}-\frac{2\kappa_2}{\mu+1}\,\beta^2\,\bar{\tilde{q}}\Big)\int\tilde{u}+\frac{\kappa_2}{\mu+1}i\,\beta\,L\,\tilde{q}\,\overline{\tilde{\lambda}\,q_0}$$
$$=\frac{\kappa_2\,L}{\mu+1}\Big(\tilde{q}\,(\mu+1-\kappa_2+i\beta)\,\overline{\tilde{\lambda}\,q_0}-2\beta^2|\tilde{q}|^2\,\frac{\mu+1-\kappa_2}{\mu+1}\Big)$$
$$=-\,\frac{\kappa_2\,L}{\mu+1}\,|\tilde{q}|^2\Big((\mu+1-\kappa_2)^2+\beta^2+2\,\beta^2\,\frac{\mu+1-\kappa_2}{\mu+1}\Big)$$

which has a real value. Therefore, the imaginary part of (19) is given by

$$\beta\frac{\mu}{(\tau^*)^2}\mathrm{Re}\tilde{\lambda}-2\beta\,\frac{\kappa_2\,L}{\mu+1}\,(\mu+1-\kappa_2)\,|\tilde{q}|^2=2\beta\int|A^{1/2}\tilde{u}|^2$$

and thus, the real part of $\tilde{\lambda}$ is

$$\frac{\mu}{(\tau^*)^2}\,\mathrm{Re}\tilde{\lambda}=\frac{2\kappa_2\,L}{\mu+1}(\mu+1-\kappa_2)|\tilde{q}|^2+2\int|A^{1/2}\tilde{u}|^2.$$

Hence the transversality condition $\mathrm{Re}\lambda'(\tau^*)>0$ holds for $0\le\kappa_2\le\mu+1$. □

There is a unique $\tau^*>0$ such that $(0,s^*,q^*,\tau^*)$ is a Hopf point, thus $\tau^*$ is the origin of a branch of nontrivial periodic orbits in the following lemma.

LEMMA 4.5. (i) *Suppose that $\kappa_2=0$. There exists a unique, purely imaginary eigenvalue $\lambda=i\beta$ of (7) with $\beta>0$ for a unique critical point $\tau^*>0$ in order for $(0,s^*,q^*,\tau^*)$ to be a Hopf point.*

(ii) *Suppose that $\kappa_2<\mu+1$ and $\frac{1}{2}-a_0<\frac{\mu-\kappa_1}{\mu+1-\kappa_2}$. There exists a unique, purely imaginary eigenvalue $\lambda=i\beta$ of (7) satisfying $\beta<\beta_c$ for a unique critical point $\tau^*>0$ in order for $(0,s^*,q^*,\tau^*)$ to be a Hopf point, where*

$$\beta_c^2=(\mu+1)(\mu+1-\kappa_2)+\sqrt{(\mu+1)(\mu+1-\kappa_2)}\,(2(\mu+1)-\kappa_2).$$

*Proof.* Suppose that $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1-\kappa_2}$ $(\kappa_2 < \mu+1)$. We need to show that the function $(u,q,\beta,\tau) \mapsto E(u,q,i\beta,\tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving system (7) with $\lambda = i\beta$ and $v(x) := u(x) - \mu G(x,s^*) + \frac{\kappa_2}{\mu+1} q_0$ which is

$$(20) \qquad \begin{cases} (A+i\beta)v &= -\mu\delta_{s^*} + \kappa_2\, q_0 \\ \quad i\beta &= \tau^* \Big(v(s^*) + \mu G(s^*,s^*) + \gamma_s(s^*,q^*)\Big) \\ \quad i\beta\, q_0 &= -(\mu+1-\kappa_2)q_0 - \mu/L. \end{cases}$$

Thus, we have

$$\begin{aligned} \tfrac{i\beta}{\tau^*} &= -\mu G_\beta(s^*,s^*) + \mu G(s^*,s^*) + \xi'(s^*) + \tfrac{\kappa_2}{\mu+1+i\beta} q_0 \\ &= -\mu G_\beta(s^*,s^*) + \mu G(s^*,s^*) + \xi'(s^*) - \tfrac{\mu\kappa_2}{L}\tfrac{1}{(\mu+1+i\beta)(\mu+1-\kappa_2+i\beta)}, \end{aligned}$$

where $G_\beta$ is a Green's function of the differential operator $A+i\beta$. The real and imaginary part of the above equation are given by

$$\begin{cases} \frac{\beta}{\tau^*} = -\mu\,\mathrm{Im}G_\beta(s^*,s^*) + \frac{\mu\kappa_2\,\beta}{L((\mu+1)^2+\beta^2)((\mu+1-\kappa_2)^2+\beta^2)}(2(\mu+1)-\kappa_2) \\ 0 = -\mu\,\mathrm{Re}G_\beta(s^*,s^*) + \mu G(s^*,s^*) + \xi'(s^*) - \frac{\mu\kappa_2\big((\mu+1)(\mu+1-\kappa_2)-\beta^2\big)}{L((\mu+1)^2+\beta^2)((\mu+1-\kappa_2)^2+\beta^2)}. \end{cases}$$

If $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1-\kappa_2}$, there is an unique $\tau$ from the first equation since $\mathrm{Im}G_\beta(s^*,s^*) < 0$ from [9]. We now define

$$(21) \qquad \begin{aligned} T(\beta) \;:=\; & -\mu\mathrm{Re}G_\beta(s^*,s^*) + \mu G(s^*,s^*) + \xi'(s^*) \\ & -\tfrac{\mu\kappa_2}{L}\tfrac{(\mu+1)(\mu+1-\kappa_2)-\beta^2}{((\mu+1)^2+\beta^2)((\mu+1-\kappa_2)^2+\beta^2)}. \end{aligned}$$

Then $T(\infty) = \mu G(s^*,s^*) + \xi'(s^*) > 0$ and $T(0) = \xi'(s^*) - \frac{\mu\kappa_2}{L(\mu+1)(\mu+1-\kappa_2)} < 0$. The derivative of $T$ with respect to $\beta$ is

$$T'(\beta) = (-\mu\,\mathrm{Re}G_\beta(s^*,s^*))' - \frac{\mu\kappa_2}{L}\,t'(\beta),$$

where

$$t(\beta) = \frac{(\mu+1)(\mu+1-\kappa_2)-\beta^2}{((\mu+1)^2+\beta^2)((\mu+1-\kappa_2)^2+\beta^2)}.$$

We let $a = (\mu+1)$ and $b = (\mu+1-\kappa_2)$. Then

$$\begin{aligned} t'(\beta) &= -\frac{2\beta}{(a^2+\beta^2)^2(b^2+\beta^2)^2} \\ &\quad \times \left((a^2+\beta^2)(b^2+\beta^2) + (ab-\beta^2)\,(a^2+b^2+2\beta^2)\right) \\ &= -\frac{2\beta}{(a^2+\beta^2)^2(b^2+\beta^2)^2}\Big(ab(a+b)^2 - (\beta^2-ab)^2\Big). \end{aligned}$$

If $\beta^2 < \beta_c^2$, where $\beta_c^2 = (\mu+1)(\mu+1-\kappa_2)+\sqrt{(\mu+1)(\mu+1-\kappa_2)}\,(2(\mu+1)-\kappa_2)$, then we have $t'(\beta) < 0$ and thus $T'(\beta) > 0$. There is the only one critical Hopf point $\tau > 0$ and $\beta < \beta_c$.

For the case of without global coupling $\kappa_2 = 0$, there is the only one critical Hopf point $\tau > 0$ and $\beta > 0$ since $T(\infty) > 0, T(0) < 0$ and $T'(\beta) > 0$ in the equation (21). □

Hence we proved Theorem 4.2 and 4.3 from the above two lemmas. In equation (21), the pure imaginary eigenvalues may or may not exist for $\kappa_2 \geq \mu + 1$ and hence we shall examine the stability of the solutions for $\kappa_2 \geq \mu + 1$ in the future.

## References

[1] D. Battogtokh, M. Hildebrand, K. Krischer, and A.S. Mikhailov, *Nucleation kinetics and global coupling in reaction-diffusion systems,* Phys. Rep. **288** (1997), 435–456.

[2] M. Bode and H. -G. Purwins, *Pattern formation in reaction-diffusion systems-dissipative solitons in physical systems,* Phys. D **86** (1995), no. 1-2, 53–63.

[3] X. -F. Chen, *Generation and propagation of interfaces in reaction-diffusion systems,* Trans. Amer. Math. Soc. **334** (1992), no. 2, 877–913.

[4] P. Fife, *Dynamics of internal layers and diffusive interfaces*, CBMS-NSF Regional Conference Series in Applied Mathematics Vol 53, SIAM, 1988.

[5] P. Fife and J. Tyson, *Target patterns in a realistic model of the Belousov-Zhabotinskii reaction,* J. Chem. Phys. **73** (1980), no. 5, 2224–2237.

[6] R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane,* Biophys. J. **1** (1961), 445–466.

[7] Y. M. Ham, *Internal layer oscillations in FitzHugh-Nagumo equation,* J. Comput. Appl. Math. **103** (1999), no. 2, 287–295.

[8] ———, *A Hopf bifurcation in a free boundary problem depending on the spatial average of an activator,* International Journal of Bifurcation and Chaos **13** (2003), no. 10, 3135–3145.

[9] Y. M. Lee, R. Schaaf, and R. Thompson, *A Hopf bifurcation in a parabolic free boundary problem,* J. Comput. Appl. Math. **52** (1994), no. 1-3, 305–324.

[10] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, Vol. 840, Springer, New York-Berlin, 1981.

[11] S. Koga, Y. Kuramoto, *Localized Patterns in Reaction-Diffusion Systems,* Prog. Theor. Phys. **63** (1980), no. 1, 106–121.

[12] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Springer, Berlin, 1984.

[13] M. Meinhardt, *Models of Biological Pattern Formation*, Academic Press, London, 1982.

[14] M. Meixner, P. Rodin, and E. Schöll, *Global control of front propagation in gate-driven multilayered structures,* Phys. Status Solidi (B) **204** (1997), no. 1, 493–496.

[15] J. D. Murray, *Mathematical Biology,* Springer, Berlin, 1985.

[16] J. Nagumo, S. Arimoto, and S. Yoshisawa, *An active pulse transmission line simulating nerve axon,* Proc. IRE. **50** (1962), 2061–2070.

[17] Y. Nishiura and H. Fujii, *Stability of singularly perturbed solutions to systems of reaction-diffusion equations,* SIAM J. Math. Anal. **18** (1987), no. 6, 1726–1770.

[18] Y. Nishiura and M. Mimura, *Layer oscillations in reaction-diffusion systems,* SIAM J. Appl. Math. **49** (1989), no. 2, 481–514.

[19] T. Ohta, M. Mimura, and R. Kobayashi, *Higher-dimensional localized patterns in excitable media,* Phys. D **34** (1989), no. 1-2, 115–144.

[20] C. Radehaus, R. Dohmen, H. Willebrand, and F. J. Niedernostheide, *Model for current patterns in physical systems with two charge carriers,* Phys. Rev. A. **42** (1990), 7426–7446.

[21] K. Sakamoto, *Spatial homogenization and internal layers in a reaction-diffusion system,* Hiroshima Math. J. **30** (2000), no. 3, 377–402.

[22] I. Schebesch and H. Engel, *Self-Organization in Activator-Inhibitor Systems: Semiconductors, Gas Discharges and Chemical Active Media,* Wissenschaft and Technik-Verlag, Berlin, 1996.

[23] M. Suzuki, T. Ohta, M. Mimura, and H. Sakaguchi, *Breathing and wiggling motions in three-species laterally inhibitory systems,* Phys. Rev. E(3) **52** (1995), no. 4, part A, 3654–3655.

[24] H. Willebrand, M. Or-Guil, M. Schilke, H. -G. Purwins, and Yu. A. Astrov, *Experimental and numerical observation of quasiparticle like structures in a distributed dissipative system,* Phys. Lett. A **177** (1993), no. 3, 220–224.

[25] R. Woesler, P. Schütz, M. Bode, M. Or-Guil, and H. -G. Purwins, *Oscillations of fronts and front pairs in two- and three-component reaction-diffusion systems,* Phys. D **91** (1996), no. 4, 376–405.

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