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THE MONOTONY PROPERTIES OF GENERALIZED PROJECTION BODIES, INTERSECTION BODIES AND CENTROID BODIES

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Abstract. In this paper, we established the monotony properties of generalized projection bodies $\Pi_i K$, intersection bodies $\mathrm{I}_i K$ and centroid bodies $\Gamma_i K$.

1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$, and let $\mathcal{K}^n_c$ denote the subset of $\mathcal{K}^n$ that contains the centered (centrally symmetric with respect to the origin) bodies. For $u \in S^{n-1}$ (n-dimensional unit sphere), let E_u denote the hyperplane, through the origin, that is orthogonal to u, and let $K|E_u$ denote the image of the orthogonal projection of the body $K \in \mathcal{K}^n$ onto E_u. We shall use V_i to denote i-dimensional Lebesgue measure. For V_n and V_{n-1} we shall usually write V and v. If K is a convex body, then its *support function*, $h_K(\cdot) = h(K, \cdot)$, is defined by

$$h(K, u) = \max\{u \cdot x : x \in K\} \text{ for all } u \in S^{n-1},$$

where $u \cdot x$ denotes the standard inner product of u and x. Obviously, for $K,\ L \in \mathcal{K}^n$, $K \subset L$ if and only if $h_K \leq h_L$.

A star body in $\mathbb{R}^n$ is a nonempty compact set K satisfying $[o, x] \subset K$ for all $x \in K$ and such that the *radial function* $\rho_K(\cdot) = \rho(K, \cdot)$, defined by

$$\rho(K, u) = \max\{\lambda > 0 : \lambda u \in K\} \text{ for all } u \in S^{n-1},$$

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is positive and continuous. The set of star bodies is denoted by $\mathcal{L}^n$, and let $\mathcal{L}_c^n$ denote the subset of $\mathcal{L}^n$ that contains the centered bodies. Obviously, for $K,\ L \in \mathcal{L}^n$, $K \subset L$ if and only if $\rho_K \leq \rho_L$.

For spherical Lebesgue measure on the i-dimensional unit sphere S^i we write S_i and for S_{n-1} we shall usually write S. We use ω_i to denote the i-dimensional volume of the unit ball B_i in $\mathbb{R}^i$.

Let $K \in \mathcal{K}^n$. The important geometric invariants related to the projection of convex bodies are the *quermassintegrals* defined by

$$W_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{\xi \in G_{i,n}} V_i(K|\xi) d\xi,\ 0 \leq i \leq n,$$

where the Grassmann manifold $G_{i,n}$ is endowed with the normalized Haar measure ξ. The quermassintegrals are generalizations of the surface area and the volume, and the mean width. Indeed, $nW_1(K)$ is the surface area of K, and $W_0(K)$ is the volume of K, and $(2/\omega_n)W_{n-1}(K)$ is the mean width of K. The *dual quermassintegrals* of a star body K are defined by

$$\widetilde{W}_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{\xi \in G_{i,n}} V_i(K \cap \xi) d\xi,\ 0 < i < n,$$

$\widetilde{W}_0(K) = V(K)$ and $\widetilde{W}_n(K) = \omega_n$. While the quermassintegrals are connected with the projection of convex bodies, the dual quermassintegrals are closely related to cross sections of star bodies [8, 9, 10, 19, 21], they play important roles in the study of the well-known Busemann-Petty problem [2, 3, 22, 24].

The *i-th projection bodies* $\Pi_i K$ were introduced by Lutwak [12] as: Let $K \in \mathcal{K}^n$ *and* $0 \leq i \leq n-1$,

$$h_{\Pi_i K}(u) = w_i(K|E_u) \text{ for all } u \in S^{n-1}. \tag{1.1}$$

The *i-th intersection bodies* $\mathrm{I}_i K$ were introduced by Zhang [21]. Note that here the definition of $\mathrm{I}_i K$ equals to that of $\mathrm{I}_{n-1-i} K$ given by Zhang for the sake of uniformity. Let $K \in \mathcal{L}^n$ and $0 \leq i \leq n-1$,

$$\rho_{\mathrm{I}_i K}(u) = \widetilde{w}_i(K \cap E_u) \text{ for all } u \in S^{n-1}. \tag{1.2}$$

The *i-th centroid bodies* $\Gamma_i K$ are defined here for the first time. Let $K \in \mathcal{K}^n$ and $0 \leq i \leq n-1$,

(1.3)

$$h_{\Gamma_i K}(u) = \frac{1}{(n+1)\widetilde{W}_i(K)} \int_{S^{n-1}} |u \cdot v| \rho_K(v)^{n+1-i} dS(v) \text{ for all } u \in S^{n-1}.$$

Here we call $\Pi_i K$, $\mathrm{I}_i K$, and $\Gamma_i K$ as generalized projection bodies, intersection bodies and centroid bodies, respectively. Obviously, for $i =$

0, $\Pi_0 K$, $I_0 K$, and $\Gamma_0 K$ turn into the well known projection bodies ΠK, intersection bodies $\mathrm{I}K$ and centroid bodies ΓK, respectively [4, 10, 11, 15, 16, 19]. We shall use Π^n, I^n, and Γ^n to denote the set of n-dimensional projection bodies, intersection bodies and centroid bodies respectively. Π_*^n denote the class of polar of bodies in Π^n.

Shephard [20] posed the question: If $K, L \in \mathcal{K}_c^n$, and

$$v(K|E_u) < v(L|E_u) \text{ for all } u \in S^{n-1},$$

does it follow that

$$V(K) < V(L)?$$

Petty [16] and Schneider [18], independently, showed that the answer is no, in general, but if L is a projection body, then the answer is yes, and according to the definition of projection bodies, their theorem can be expressed as:

THEOREM 1. *If $K \in \mathcal{K}^n, L \in \Pi^n$, and $\Pi K \subset \Pi L$, then*

$$V(K) \leq V(L),$$

with equality if and only if K and L are translates of each other.

Lutwak showed that intersection body $\mathrm{I}K$ and centroid body ΓK are also have the similar properties.

THEOREM 2. [10] *If $K \in \mathrm{I}^n, L \in \mathcal{L}^n$, and $\mathrm{I}K \subset \mathrm{I}L$, then*

$$V(K) \leq V(L),$$

with equality if and only if $K = L$.

THEOREM 3. [11] *If $K \in \mathcal{L}^n, L \in \Pi_*^n$, and $\Gamma K \subset \Gamma L$, then*

$$V(K) \leq V(L),$$

with equality if and only if $K = L$.

The aim of this paper is to present the similar monotony properties of the generalized projection bodies, intersection bodies and centroid bodies. We shall establish the following theorems that are closely relative to the forementioned theorems.

THEOREM 1*. *Let $K \in \mathcal{K}^n, L \in \Pi^n$. If $0 \leq i < n-1$, and*

$$\Pi_i K \subset \Pi_i L, \tag{1.4}$$

then

$$W_i(K) \leq W_i(L), \tag{1.5}$$

with equality if and only if K and L are translates of each other.

THEOREM 2*. *Let $K \in \mathrm{I}^n, L \in \mathcal{L}^n$. If $0 \leq i < n-1$, and*

$$\mathrm{I}_i K \subset \mathrm{I}_i L, \tag{1.6}$$

then

$$\widetilde{W}_i(K) \leq \widetilde{W}_i(L), \tag{1.7}$$

with equality if and only if $K = L$.

THEOREM 3*. *Let $K \in \mathcal{L}^n, L \in \Pi_*^n$. If $0 \leq i < n-1$, and*

$$\Gamma_i K \subset \Gamma_i L, \tag{1.8}$$

then

$$\widetilde{W}_i(K) \leq \widetilde{W}_i(L), \tag{1.9}$$

with equality if and only if $K = L$.

A similar result to Theorem 2* has been given by Zhang [21], but here we shall propose a stronger result which regards Theorem 2* as its special case.

2. Preliminaries

To prove the theorems in the above section, we first make some preparations.

If $K_1, \ldots, K_r \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then the Minkowski linear combination, $\lambda_1 K_1 + \cdots + \lambda_r K_r$, is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r : x_i \in K_i\}.$$

Of fundamental importance is the fact that the volume of a Minkowski linear combination can be expressed by a symmetric homogenous n-th degree polynomial in the λ_i, i.e.,

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum V_{i_1 \cdots i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples of positive integers $(i_1, \ldots, i_n)$ with entries not exceeding r. The coefficient $V_{i_1 \cdots i_n}$ depends only on the figures $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (2.1). It is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and written as $V(K_{i_1}, \ldots, K_{i_n})$ [9, 19].

For $K \in \mathcal{K}^n$, the i-th quermassintegral is denoted by

$$W_i(K) = V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B_n, \ldots, B_n}_{i}). \tag{2.2}$$

LEMMA 2.1 [7] *If $K, L \in \mathcal{K}^n, 0 \le j \le n-1$, then*

$$V(\underbrace{K,\ldots,K}_{n-j-1},\ \underbrace{B_n,\ldots,B_n}_{j},\ L)^{n-j} \ge W_j(K)^{n-j-1}W_j(L), \tag{2.3}$$

with equality if and only if K and L are homothetic .

The inequality (2.3) is a special case of the Aleksandrov-Fenchel inequality, but the equality condition in the Aleksandrov-Fenchel inequality is, in general, unknown [1, 7, 19].

Let $K_1, \ldots, K_{n-1}$ be fixed convex bodies, one can view $V(K_1, \ldots, K_{n-1}, \cdot)$ as a functional defined on $\mathcal{K}^n$. If each $K \in \mathcal{K}^n$ is identified with its support function on S^{n-1}, $h(K,u)$, then $V(K_1, \ldots, K_{n-1}, \cdot)$ can be uniquely extended to a continuous functional on $C(S^{n-1})$, endowed with the maximum norm. According to the Riesz representation theorem, there exists a unique positive Borel measure on S^{n-1}, denoted by $S(K_1, \ldots, K_{n-1}; \cdot)$, such that for any $L \in \mathcal{K}^n$,

$$V(K_1, \ldots, K_{n-1},\ L) = \frac{1}{n}\int_{S^{n-1}} h(L,v)\ dS(K_1, \ldots, K_{n-1}; v). \tag{2.4}$$

The measure $S(K_1, \ldots, K_{n-1}; \cdot)$ is called the mixed area measure of $K_1, \ldots, K_{n-1}$. For $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = B_n$, we call it the i-th surface area function of K, denoted by $S_i(K; \cdot)$. In particular, for $K_1 = \cdots = K_{n-1} = K$, we call it the surface area measure of K, simply denoted by $S(K; \cdot)$.

For $u \in S^{n-1}$, the mixed volume $v(K_1|E_u, \ldots, K_{n-1}|E_u)$ can be denoted by

$$v(K_1|E_u,\ \ldots, K_{n-1}|E_u) = nV(K_1, \ldots, K_{n-1}, \bar{u}), \tag{2.5}$$

where $\bar{u}$ denotes the closed line segment $\{\lambda u : |\lambda| \le \frac{1}{2}\}$ [9].

Since $h(\bar{u}, v) = \frac{1}{2}|u \cdot v|$, from (2.4) and (2.5) one obtains

$$v(K_1|E_u, \ldots, K_{n-1}|E_u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v|\ dS(K_1, \ldots, K_{n-1}; v). \tag{2.6}$$

For $0 \le i < n$, taking $K_1 = \cdots = K_{n-1-i} = K$, and $K_{n-i} = \cdots = K_{n-1} = B_n$ in (2.6), we obtain the following lemma.

LEMMA 2.2 [12] *If $K \in \mathcal{K}^n$ and $0 \le i < n$ then*

$$w_i(K|E_u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v|\ dS_i(K; v), \tag{2.7}$$

where w_i denotes the i-th quermassintegral in $\mathbb{R}^{n-1}$, i.e.,

$$w_i(K|E_u) = v(\underbrace{K|E_u, \ldots, K|E_u}_{n-1-i}, \underbrace{B_{n-1}, \ldots, B_{n-1}}_{i}).$$

For the Brunn-Minkowski theory one can consult Gardner and Schneider [4, 19].

If $x_1, \ldots, x_r \in \mathcal{K}^n$, then $x_1 \tilde{+} \cdots \tilde{+} x_r$ is defined to be the usual vector sum of $x_1, \ldots, x_r$, provided $x_1, \ldots, x_r$ all lie in a 1-dimensional subspace of $\mathbb{R}^n$, and as the zero vector otherwise. If L_i are star bodies in $\mathbb{R}^n$, and $t_i \in \mathbb{R}, (1 \leq i \leq m)$, then the radial Minkowski linear combination, $t_1 L_1 \tilde{+} \cdots \tilde{+} t_m$, is defined by:

$$t_1 L_1 \tilde{+} \cdots \tilde{+} t_m L_m = \{t_1 x_1 \tilde{+} \cdots \tilde{+} t_m x_m : x_i \in L_i\}.$$

For arbitrary $u \in S^{n-1}$, there are

$$\rho_{t_1 L_1 \tilde{+} \cdots \tilde{+} t_m L_m}(u) = t_1 \rho_{L_1}(u) + \cdots + t_m \rho_{L_m}(u).$$

Let $L_j \in \mathcal{L}^n (1 \leq j \leq n)$, the dual mixed volume $\widetilde{V}(L_1, \ldots, L_n)$ [8] is defined by

$$\widetilde{V}(L_1, \ldots, L_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{L_1}(u) \cdots \rho_{L_n}(u) dS(u).$$

For $K \in \mathcal{L}^n$, the i-th dual quermassintegral is denoted by

$$\begin{aligned} \widetilde{W}_i(K) &= \widetilde{V}(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B_n, \ldots, B_n}_{i}) \\ &= \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} \, dS(u), \ (0 \leq i \leq n). \end{aligned} \tag{2.8}$$

For the dual Brunn-Minkowski theory one can consult Lutwak, Gardner and Schneider [8, 10, 11, 4, 19].

Two star bodies K and L are said to be dilates (of one another) if $\dfrac{\rho_K(u)}{\rho_L(u)}$ is independent of $u \in S^{n-1}$.

Suppose that f is a Borel function on S^{n-1}, the *spherical Radon transform* Rf [6] of f is defined by

$$(Rf)(u) = \int_{S^{n-1} \cap E_u} f(v) dS_{n-2}(v). \tag{2.9}$$

Using spherical Radon transform, the definition of the generalized intersection body I_iK can be rewrited as

(2.10)
$$\rho_{\mathrm{I}_iK}(u) = \frac{1}{n-1}\int_{S^{n-1}\cap E_u} \rho_K(v)^{n-1-i} dS_{n-2}(v) = R\left(\frac{1}{n-1}\rho_K^{n-1-i}\right)(u).$$

Two important properties of the spherical Radon transform are : (1) The spherical Radon transform is a continuous bijection of $C_e^\infty(S^{n-1})$ to itself; (2) The spherical Radon transform is self-adjoint, that is: if f and g are bounded Borel functions on S^{n-1}, then

$$\int_{S^{n-1}} f(u)Rg(u)dS(u) = \int_{S^{n-1}} Rf(u)g(u)dS(u). \tag{2.11}$$

For $K \in \mathcal{K}_o^n$ (the set of convex bodies containing the origin in their interiors), the polar body of K, K^*, is defined by

$$K^* = \{x \in R^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

It is easy to verify that for $u \in S^{n-1}$,

$$h(K^*, u) = \frac{1}{\rho(K, u)}. \tag{2.12}$$

3. Mixed p-quermassintegrals and mixed p-dual quermassintegrals

Mixed p-quermassintegrals were introduced by Lutwak [13]. For $p \geq 1$, $0 \leq i \leq n-1$, and K, $L \in \mathcal{K}_o^n$, the mixed quermassintegrals $W_{p,i}(K, L)$ have the following integral representation

$$W_{p,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p} dS_i(K; u). \tag{3.1}$$

Obviously, if $K = L$, then the Mixed p-quermassintegrals $W_{p,i}(K, L)$ will turn into the quermassintegrals $W_i(K)$.

In this section we shall introduce the notion of Mixed p-dual quermassintegrals and give some properties. These properties will play important roles in proving Theorem 2* and Theorem 3*.

For $K, L \in \mathcal{L}^n$ and $p > 0$, we define the *mixed p-dual quermassintegrals* as follows

$$\widetilde{W}_{-p,i}(K, L) = \frac{-p}{n-i}\lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K \tilde{+}_{-p}\, \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}, \tag{3.2}$$

where the $p-$ *harmonic radial combination* $K\tilde{+}_{-p}\,\varepsilon\cdot L\in\mathcal{L}^n$ [14] was defined by

$$\rho(K\tilde{+}_{-p}\,\varepsilon\cdot L,\cdot)^{-p}=\rho(K,\cdot)^{-p}+\varepsilon\rho(L,\cdot)^{-p}. \tag{3.3}$$

Meanwhile, we define

$$\widetilde{W}_{0,i}(K,L)=\lim_{p\to 0^+}\widetilde{W}_{-p,i}(K,L). \tag{3.4}$$

From (3.2), (3.3), and (3.4), the mixed p-dual quermassintegrals $\widetilde{W}_{-p,i}(K,L)$ have the following integral representation.

LEMMA 3.1 *Let $K,L\in\mathcal{L}^n$. Then for $0\le i\le n-1$ and $p\ge 0$, we have*

$$\widetilde{W}_{-p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i+p}\rho_L(u)^{-p}dS(u). \tag{3.5}$$

Proof. When $p>0$, from (2.8) and (3.3), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\tilde{+}_{-p}\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}\\
=&\lim_{\varepsilon\to 0^+}\frac{1}{n\varepsilon}\int_{S^{n-1}}\left[\rho(K\tilde{+}_{-p}\,\varepsilon\cdot L,u)^{n-i}-\rho(K,u)^{n-i}\right]dS(u)\\
=&\lim_{\varepsilon\to 0^+}\frac{1}{n\varepsilon}\int_{S^{n-1}}\left[(\rho(K,u)^{-p}+\varepsilon\rho(L,u)^{-p})^{\frac{n-i}{-p}}-\rho(K,u)^{n-i}\right]dS(u)\\
=&\frac{n-i}{-pn}\int_{S^{n-1}}\rho(K,u)^{n-i+p}\rho(L,u)^{-p}dS(u),
\end{aligned}$$

where the last equality can be got according to the L'Hospital's rule [17]. Combine (3.2) just get (3.5).

From (3.4), it's easy to know that when $p=0$ the representation still hold and $\widetilde{W}_{0,i}(K,L)=\widetilde{W}_i(K)$. □

From Lemma 3.1 we can obtain the following Theorem.

THEOREM 3.2 *Let $K,L\in\mathcal{L}^n$. Then for $0\le i\le n-1$ and $p\ge 0$, we have*

$$\widetilde{W}_{-p,i}(K,L)^{n-i}\ge\widetilde{W}_i(K)^{n-i+p}\widetilde{W}_i(L)^{-p}, \tag{3.6}$$

with equality if and only if $p=0$ or K and L are dilates of each other.

Proof. If $p=0$, from (3.5) it is obviously that (3.6) holds with equality. So, we consider $p>0$. From (3.5) and Hölder inequality [5], we obtain

$$\begin{aligned}&\widetilde{W}_{-p,i}(K,L)\\&=\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i+p}\rho_L(u)^{-p}dS(u)\\&\geq\left[\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i}dS(u)\right]^{\frac{n-i+p}{n-i}}\left[\frac{1}{n}\int_{S^{n-1}}\rho_L(u)^{n-i}dS(u)\right]^{\frac{-p}{n-i}}\\&=\widetilde{W}_i(K)^{\frac{n-i+p}{n-i}}\widetilde{W}_i(L)^{\frac{-p}{n-i}},\end{aligned}$$

that is,

$$\widetilde{W}_{-p,i}(K,L)^{n-i}\geq\widetilde{W}_i(K)^{n-i+p}\widetilde{W}_i(L)^{-p}.$$

From (3.5) and the equality condition of Hölder inequality, we know that the equality holds in (3.6) if and only if $p=0$ or K and L are dilates of each other. □

THEOREM 3.3 *Let $K\in I^n$, and $L,Q\in\mathcal{L}^n$. If for all $u\in S^{n-1}$ and some $0\leq i<n-1$*

$$\widetilde{w}_{-p,i}(K\cap E_u,Q\cap E_u)\leq\widetilde{w}_{-p,i}(L\cap E_u,Q\cap E_u), \tag{3.7}$$

then

$$\widetilde{W}_{-p,i}(K,Q)\leq\widetilde{W}_{-p,i}(L,Q), \tag{3.8}$$

with equality if and only if $K=L$.

Proof. Let $M\in\mathcal{L}^n$, define a body $\widetilde{M}\in\mathcal{L}^n$ by

(3.9)

$$\rho(\widetilde{M},u)=\frac{1}{n-1}\int_{S^{n-1}\cap E_u}\rho_M(v)^{n-1}dS_{n-2}(v)=\left(\frac{1}{n-1}\mathrm{R}(\rho_M^{n-1})(u)\right).$$

It follows from (2.10) that $\widetilde{M}$ is the intersection body of M. Then from (2.10), the self-adjointness of Radon transform (2.11) and (3.5), we can get

$$\begin{aligned}&\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i+p-1}\rho_Q(u)^{-p}\rho_{\widetilde{M}}(u)dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i+p-1}\rho_Q(u)^{-p}\left(\frac{1}{n-1}\mathrm{R}(\rho_M)^{n-1}(u)\right)dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\frac{1}{n-1}\left(\mathrm{R}(\rho_K^{n-i+p-1}\rho_Q^{-p})(u)\right)\rho_M(u)^{n-1}dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\widetilde{w}_{-p,i}(K\cap E_u,Q\cap E_u)\rho(M,u)^{n-1}dS(u).\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{1}{n}\int_{S^{n-1}}\rho_L(u)^{n-i+p-1}\rho_Q(u)^{-p}\rho_{\widetilde{M}}(u)dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\widetilde{w}_{-p,i}(L\cap E_u,Q\cap E_u)\rho_M(u)^{n-1}dS(u).\end{aligned}$$

So from condition (3.7), it follows that

$$(*)\qquad\begin{aligned}&\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i+p-1}\rho_Q(u)^{-p}\rho_{\widetilde{M}}(u)dS(u)\\&\le\frac{1}{n}\int_{S^{n-1}}\rho_L(u)^{n-i+p-1}\rho_Q(u)^{-p}\rho_{\widetilde{M}}(u)dS(u).\end{aligned}$$

It's true that the inequality $(*)$ holds for any $\widetilde{M}$ since the M is an arbitrary star body.

Taking $\widetilde{M}=K$ in $(*)$, by (3.5) and Hölder inequality, we have

$$\begin{aligned}&\widetilde{W}_{-p,i}(K,Q)\\&=\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i+p}\rho_Q(u)^{-p}dS(u)\\&\le\frac{1}{n}\int_{S^{n-1}}\rho_L(u)^{n-i+p-1}\rho_Q(u)^{-p}\rho_K(u)dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\rho_L(u)^{n-i+p-1}\rho_Q(u)^{\frac{-p(n-i+p-1)}{n-i+p}}\rho_Q(u)^{\frac{-p}{n-i+p}}\rho_K(u)dS(u)\\&\le\left(\frac{1}{n}\int_{S^{n-1}}\rho_L(u)^{n-i+p}\rho_Q(u)^{-p}dS(u)\right)^{\frac{n-i+p-1}{n-i+p}}\\&\quad\times\left(\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i+p}\rho_Q(u)^{-p}dS(u)\right)^{\frac{1}{n-i+p}}\\&=\widetilde{W}_{-p,i}(L,Q)^{\frac{n-i+p-1}{n-i+p}}\widetilde{W}_{-p,i}(K,Q)^{\frac{1}{n-i+p}}.\end{aligned}$$

Thus we obtain

$$\widetilde{W}_{-p,i}(K,Q)\le\widetilde{W}_{-p,i}(L,Q).$$

From the equality condition of Hölder inequality and condition (3.7), we know that the equality holds in (3.8) if and only if $K=L$. □

4. Proofs of the theorems

Proof of Theorem 1*. Suppose that $u_1, \dots, u_m \in S^{n-1}$ and $\lambda_1, \dots, \lambda_m$ are positive numbers. Put

$$Z = \sum_{j=1}^{m} \lambda_j \bar{u}_j,$$

where $\bar{u}_j$ denotes the line segment $[-u_j, u_j], j = 1, \dots, m$. Then Z is a zonotope with the support function

$$h(Z, \cdot) = \sum_{j=1}^{m} \lambda_j |u_j, \cdot|.$$

Thus by (2.7) and (2.4), we obtain

$$\begin{aligned}\sum_{j=1}^{m} \lambda_j w_i(K|E_{u_j}) &= \frac{1}{2} \int_{S^{n-1}} \{\sum_{j=1}^{m} \lambda_j |u_j \cdot v|\} \, dS_i(K; v) \\ &= \frac{1}{2} \int_{S^{n-1}} h(Z, v) \, dS_i(K; v) \\ &= \frac{1}{2} \, nV(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B_n, \dots, B_n}_{i}, Z).\end{aligned}$$

Similarly,

$$\sum_{j=1}^{m} \lambda_j w_i(L|E_{u_j}) = \frac{1}{2} \, nV(\underbrace{L, \dots, L}_{n-i-1}, \underbrace{B_n, \dots, B_n}_{i}, Z).$$

So from condition (1.4) and (1.1), it follows that

$$V(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B_n, \dots, B_n}_{i}, Z) \le V(\underbrace{L, \dots, L}_{n-i-1}, \underbrace{B_n, \dots, B_n}_{i}, Z), \tag{4.1}$$

for any zonotope Z.

Since any projection body is the limit (with respect to the Hausdorff metric) of zonotopes, by (4.1), this implies for any $M \in \Pi^n$

$$V(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B_n, \dots, B_n}_{i}, M) \le V(\underbrace{L, \dots, L}_{n-i-1}, \underbrace{B_n, \dots, B_n}_{i}, M). \tag{4.2}$$

Taking $M = L$ in (4.2), we get

$$V(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B_n, \dots, B_n}_{i}, L) \le W_i(L). \tag{4.3}$$

Applying Lemma 2.1 to the left hand of (4.3), it yields that

$$W_i(K)^{(n-i-1)/(n-i)}W_i(L)^{1/(n-i)} \le W_i(L).$$

Hence we obtain the desired result (1.5), with equality if and only if K and L are homothetic. Furthermore, if $W_i(K) = W_i(L)$, then we easily derive that K and L are translates of each other. □

Proof of Theorem 2*. It is the special case of $p = 0$ in Theorem 3.3. □

Proof of Theorem 3*. From (3.1), (1.3), Fubini theorem, (1.1), (2.12), and (3.5), it follows that

$$\begin{aligned}
&W_{1,i}(L,\Gamma_i K)\\
&= \frac{1}{n}\int_{S^{n-1}} h_{\Gamma_i K}(u)dS_i(L,u)\\
&= \frac{1}{n(n+1)\widetilde{W}_i(K)}\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|\rho_K(v)^{n+1-i}dS(v)dS_i(L,u)\\
&= \frac{1}{n(n+1)\widetilde{W}_i(K)}\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|dS_i(L,u)\rho_K(v)^{n+1-i}dS(v)\\
&= \frac{2}{n(n+1)\widetilde{W}_i(K)}\int_{S^{n-1}} h_{\Pi_i L}(v)\rho_K(v)^{n+1-i}dS(v)\\
&= \frac{2}{(n+1)\widetilde{W}_i(K)}\frac{1}{n}\int_{S^{n-1}} \rho_{\Pi_i^* L}(v)^{-1}\rho_K(v)^{n+1-i}dS(v)\\
&= \frac{2}{(n+1)\widetilde{W}_i(K)}\widetilde{W}_{-1,i}(K,\Pi_i^* L).
\end{aligned}$$

Since $M \in \Pi_*^n$, choosing $M_0 \in \mathcal{K}^n$ such that $M = \Pi_i^* M_0$, it yields that

$$W_{1,i}(M_0,\Gamma_i K) = \frac{2}{(n+1)\widetilde{W}_i(K)}\widetilde{W}_{-1,i}(K,M). \tag{4.4}$$

Taking $K = L$ in (4.4), we get

$$W_{1,i}(M_0,\Gamma_i L) = \frac{2}{(n+1)\widetilde{W}_i(L)}\widetilde{W}_{-1,i}(L,M). \tag{4.5}$$

From condition (1.8) and the monotony property of $W_{1,i}(M_0,\cdot)$, it follows that

$$\frac{\widetilde{W}_{-1,i}(K,M)}{\widetilde{W}_i(K)} \le \frac{\widetilde{W}_{-1,i}(L,M)}{\widetilde{W}_i(L)}. \tag{4.6}$$

Taking L for M in (4.6) and from (3.5) we get

$$\widetilde{W}_{-1,i}(K,L) \le \widetilde{W}_i(K)\frac{\widetilde{W}_{-1,i}(L,L)}{\widetilde{W}_i(L)} = \widetilde{W}_i(K). \tag{4.7}$$

Applying Theorem 3.2 to (4.7), we obtain that

$$\widetilde{W}_{-1,i}(K,L)^{n-i} \ge \widetilde{W}_i(K)^{n-i+1}\widetilde{W}_i(L)^{-1}. \tag{4.8}$$

The inequality (1.9) is now an immediate consequence when we combine (4.7) and (4.8) with equality if and only if $K = L$. □

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