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INEQUALITIES FOR DUAL HARMONIC QUERMASSINTEGRALS

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ABSTRACT. In this paper, we study the properties of the dual harmonic quermassintegrals systematically and establish some inequalities for the dual harmonic quermassintegrals, such as the Minkowski inequality, the Brunn-Minkowski inequality, the Blaschke-Santaló inequality and the Bieberbach inequality.

1. Introduction

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) and $\mathcal{K}_o^n$ denote the subset of $\mathcal{K}^n$ that contains the origin in their interiors in $\mathbb{R}^n$. Denote by $\mathrm{vol}_i(K|\xi)$ the i-dimensional volume of the orthogonal projection of K onto an i-dimensional subspace $\xi \subset \mathbb{R}^n$. The important geometric invariants related to the projection of convex body K are the *quermassintegrals* defined by

$$W_{n-i}(K) = k_n \int_{G(n,i)} \frac{\mathrm{vol}_i(K|\xi)}{k_i}\, d\mu_i(\xi),\ 0 \le i \le n, \tag{1.1}$$

where the Grassmann manifold $G(n,i)$ is endowed with the normalized Haar measure, and k_n is the volume of the unit ball B_n in $\mathbb{R}^n$. The quermassintegrals are generalizations of the surface area and the volume. Indeed, $nW_1(K)$ is the surface area of K, and $W_0(K)$ is the volume of K.

The quermassintegrals arise in many areas of Mathematics and have different definitions. If K has a C^2 boundary, they are the integrals of elementary symmetric functions of the principal curvatures over the

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boundary. In the theory of mixed volumes, the quermassintegrals are called simple mixed volumes. They are also called projection measure, intrinsic volumes, etc. The reader should consult [16] and [18] for details.

The *dual quermassintegrals* of a star body L, $\tilde{W}_i(L)$, were introduced by Lutwak [11], which are defined by letting $\tilde{W}_0(L) = V(L)$, $\tilde{W}_n(L) = k_n$ and for $0 < i < n$ by

$$\tilde{W}_{n-i}(L) = k_n \int_{G(n,i)} \frac{\mathrm{vol}_i(L \cap \xi)}{k_i} d\mu_i(\xi), \tag{1.2}$$

where $\mathrm{vol}_i(L \cap \xi)$ denotes the i-dimensional volume of slice of L by an i-dimensional subspace $\xi \subset \mathbb{R}^n$.

While the quermassintegrals are connected with the projections of convex bodies, the dual quermassintegrals are closely related to the cross sections of star bodies. It is shown in [6] that they are the only rotation invariant continuous star valuations with the corresponding homogeneity. Recently, Zhang [19] proved that the dual quermassintegrals share the same kind of kinematic formulas as the quermassintegrals.

Also associated with a convex body K are its *harmonic quermassintegrals.* These quermassintegrals were introduced by Hadwiger [5, sect.6.4.8], and can be defined by letting $\hat{W}_0(K) = V(K)$, $\hat{W}_n(K) = k_n$, and for $0 < i < n$, by

$$\hat{W}_{n-i}(K) = k_n \left(\int_{G(n,i)} \left[\frac{\mathrm{vol}_i(K|\xi)}{k_i} \right]^{-1} d\mu_i(\xi) \right)^{-1}. \tag{1.3}$$

In [12], Lutwak found an inequality for the harmonic quermassintegrals whose form is similar to the classical inequality of quermassintegrals, and established an interesting connection between the harmonic quermassintegrals of a convex body and the power-means of the width function of the body.

Following Hadwiger, we introduce the *dual harmonic quermassintegrals* of a star body L, $\breve{W}_{n-i}(L)$, which can be defined by letting $\breve{W}_0(L) = V(L)$, $\breve{W}_n(L) = k_n$, and for $0 < i < n$ by

$$\breve{W}_{n-i}(L) = k_n \left(\int_{G(n,i)} \left[\frac{\mathrm{vol}_i(L \cap \xi)}{k_i} \right]^{-1} d\mu_i(\xi) \right)^{-1}. \tag{1.4}$$

From the Schwarz or Hölder inequality, it follows that

$$\breve{W}_i(L) \le \tilde{W}_i(L), \quad 0 < i < n, \tag{1.5}$$

with equality if and only if L is of constant $(n-i)$-section.

The aim of this paper is to study the properties of the dual harmonic quermassintegrals systematically. For reader's convenience, we try to make the paper self-contained. This paper, except for the introduction, is divided into four sections. In Section 2 we will recall some basics about convex bodies, star bodies and mixed volumes.

In Section 3, we introduce the concept of the mixed p-dual harmonic quermassintegrals and establish the Minkowski inequality for the mixed p-dual harmonic quermassintegrals (Theorem 3.2). As an application, the Brunn-Minkowski inequalities for the dual harmonic quermassintegrals are obtained.

A classical affine isoperimetric inequalities is the Blaschke-Santaló inequality which was proved by Blaschke [1] for $n \leq 3$ and by Santaló [15] for all n. In Section 4, we give the Blaschke-Santaló type inequalities for the dual harmonic quermassintegrals.

Let φ_o^n denote the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. In Section 5, following Lutwak's i-th half-width of a convex body [12], we introduce the concept of i-th half-chord length of a star body L, $P_i(L)$, and show that for $L \in \varphi_o^n$,

$$\breve{W}_i(L) \geq k_n P_{i-n}(L)^{n-i}, \ (0 \leq i < n-1) \tag{1.6}$$

with equality if and only if L is a ball. For $i = n-1$, the both sides of inequality (1.6) are equal for all $L \in \varphi_o^n$.

2. Notations and preliminary works

As usual, S^{n-1} denotes the unit sphere, o the origin in Euclidean n-space $\mathbb{R}^n$.

Let K be a nonempty compact convex body in $\mathbb{R}^n$, the support function h_K of K is defined by

$$h_K(u) = \max\{u \cdot x : x \in K\}, \quad u \in S^{n-1}, \tag{2.1}$$

where $u \cdot x$ denotes the usual inner product of u and x in $\mathbb{R}^n$.

If K is a convex body that contains the origin in its interior, the polar body K^* of K, with respect to the origin, is define by

$$K^* = \{x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K\}. \tag{2.2}$$

For $K_i \in \mathcal{K}^n$, and $\lambda_i \geq 0, 1 \leq i \leq r$, the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r : x_i \in K_i\}. \tag{2.3}$$

Of fundamental importance is the fact that the volume of a linear combination of figures defined by (2.3), can be expressed by a symmetric homogenous n-th degree polynomial in the λ_i, i.e.,

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum V_{i_1 \cdots i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.4}$$

where the sum is taken over all $n-$tuples of positive integers $(i_1, \ldots, i_n)$ with entries not exceeding r. The coefficient $V_{i_1 \cdots i_n}$ depends only on the figures $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (2.4). It is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and written as $V(K_{i_1}, \ldots, K_{i_n})$ [4, p.353]. $V(K_1, i_1; \ldots; K_m, i_m)$ will be used when the convex body K_j appears i_j times.

The following elementary properties of mixed volumes will be used later. For $K, L, K_i \in \mathcal{K}^n (1 \le i \le n)$, and $K \supset L$, then

$$V(K_1, \ldots, K_{n-1}, K_n + L) = V(K_1, \ldots, K_{n-1}, K_n) + V(K_1, \ldots, K_{n-1}, L), \tag{2.5}$$

$$V(K_1, \ldots, K_{n-1}, K) \ge V(K_1, \ldots, K_{n-1}, L). \tag{2.6}$$

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L, \cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$,

$$\rho(L, u) = \max\{\lambda > 0 : \lambda u \in L\}. \tag{2.7}$$

If $\rho(L, \cdot)$ is continuous and positive, L will be called a star body. Two star bodies K and L are said to be dilates if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $x_i \in \mathbb{R}^n$, $1 \le i \le m$, then $x_1 \tilde{+} \cdots \tilde{+} x_m$ is defined to be the usual vector sum of the points x_i, if all of them are contained in a line though o, and 0 otherwise.

Let $L_i \in \varphi_o^n$, and $t_i \ge 0, 1 \le i \le m$, then

$$t_1 L_1 \tilde{+} \cdots \tilde{+} t_m L_m = \{t_1 x_1 \tilde{+} \cdots \tilde{+} t_m x_m : x_i \in L_i\},$$

is called a radial linear combination.

Let $L \in \varphi_o^n$, the chordal symmetral of L will be denoted by $\tilde{\Delta} L$, i.e.,

$$\tilde{\Delta} L = \frac{1}{2}(L \tilde{+} (-L)). \tag{2.8}$$

It is easy to verify that

$$2\rho(\tilde{\Delta} L, u) = \rho(L, u) + \rho(L, -u). \tag{2.9}$$

From the dual Brunn-Minkowski inequality [13], it follows that for $L \in \varphi_o^n$,

$$V(\tilde{\Delta}L) \leq V(L), \tag{2.10}$$

with equality if and only if L is centered.

Also associated with a star body $L \in \varphi_o^n$ is its star dual L^o, which was introduced by Maria [14]. Let i be the inversion of the one-point compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$, with respect to S^{n-1}:

$$i(x) := \frac{x}{||x||^2} \text{ for } x \in \mathbb{R}^n \backslash \{0\}.$$

Then the star dual L^o of a star body $L \in \varphi_o^n$ is defined by

$$L^o = \mathrm{cl}(\mathbb{R}^n \backslash i(L)).$$

Generally, star dual of a convex body is different from its polar dual. It is easy to verify that for every $u \in S^{n-1}$ [14],

$$\rho(L^o, u) = \frac{1}{\rho(L,u)}. \tag{2.11}$$

We will simply write $\tilde{\Delta}^\circ L$ rather than $(\tilde{\Delta}L)^\circ$.

For $K, L \in \mathcal{K}^n$, the Minkowski inequality for mixed volumes [4, p.369] is

$$V(K, n-1\,; L)^n \geq V(K)^{n-1}V(L), \tag{2.12}$$

with equality if and only if K and L are homothetic.

If L is a star body such that for some i with $1 \leq i \leq n-1$, $\mathrm{vol}_i(L \cap \xi)$ has the same value for each $\xi \in G(n,i)$. We say L is of constant i-section.

The above elementary results (and definitions) are from the theory of convex bodies. The reader may consult the standard works on the subject [3, 4, 7, 16, 18] for reference.

3. The Brunn-Minkowski inequalities for dual harmonic quermassintegrals

In this section, we will prove two Brunn-Minkowski inequalities for the dual harmonic quermassintegrals. At first, let us list some elementary properties of the dual harmonic quermassintegrals.

LEMMA 3.1. *Let $L, L' \in \varphi_o^n$.*

(i) (Positive homogeneity of degree $n-i$) *If $c \geq 0$, then*

$$\breve{W}_i(cL) = c^{n-i}\breve{W}_i(L).$$

(ii) (Invariance against motions) *If* $\phi \in SL(n)$, *then*

$$\breve{W}_i(\phi L) = \breve{W}_i(L).$$

(iii) (Continuity) $\breve{W}_i(L)$ *is a continuous function of* L, *i.e. if* $\{L_m\}$ *is a sequence in* φ_o^n *such that* L_m *converges to* L, *then*

$$\lim_{m\to\infty} \breve{W}_i(L_m) = \breve{W}_i(L).$$

(iv) (Monotonicity) *If* $L \subset L'$, *then*

$$\breve{W}_i(L) \le \breve{W}_i(L').$$

Let $K, L \in \mathcal{K}_o^n$, $\xi \in G(n,i)$ and $0 \le p \le i$. Let $V_{p,i}(K, L; \xi)$ denote $V(K \cap \xi, i-p\ ; L \cap \xi, p)$. Then we define the mixed p-dual harmonic quermassintegrals, $\breve{W}_{p,n-i}(K, L)$ by

$$\breve{W}_{p,n-i}(K,L) = k_n \left(\int_{G(n,i)} \left[\frac{V_{p,i}(K,\ L;\xi)}{k_i} \right]^{-1} d\mu_i(\xi) \right)^{-1} \tag{3.1}$$

If $p = 1$, we shall write $\breve{W}_i(K, L)$, rather than $\breve{W}_{1,i}(K, L)$. It follows that $\breve{W}_{p,i}(K, K) = \breve{W}_i(K)$, for all $0 \le p \le n-i$ and $\breve{W}_{n-i,i}(K, L) = \breve{W}_i(L)$, for all K.

For the mixed p-dual harmonic quermassintegrals, we have the following Minkowski inequality.

THEOREM 3.2. *Let* $K, L \in \mathcal{K}_o^n$ *and* $0 \le i < n$. *If* $0 \le p \le n-i$, *then*

$$\breve{W}_{p,i}(K, L)^{n-i} \ge \breve{W}_i(K)^{n-i-p}\ \breve{W}_i(L)^p, \tag{3.2}$$

with equality if and only if K *and* L *are dilates.*

Proof. By the Minkowski inequality for mixed volumes (2.12) and the definition of $V_{p,i}(K, L; \xi)$, we get

$$\begin{aligned} V_{p,n-i}(K, L; \xi) &= V(K \cap \xi, n-i-p\ ; L \cap \xi, p) \\ &\ge \mathrm{vol}_{n-i}(K \cap \xi)^{\frac{n-i-p}{n-i}} \mathrm{vol}_{n-i}(L \cap \xi)^{\frac{p}{n-i}}. \end{aligned} \tag{3.3}$$

According to (3.3) and the Hölder inequality, we have

$$\breve{W}_{p,i}(K,L) = k_n \left(\int_{G(n,n-i)} \left[\frac{V_{p,n-i}(K,\ L;\xi)}{k_{n-i}} \right]^{-1} d\mu_i(\xi) \right)^{-1}$$

$$\geq k_n \left(\int_{G(n,n-i)} \left[\frac{\mathrm{vol}_{n-i}(K\cap\xi)}{k_{n-i}} \right]^{-\frac{n-i-p}{n-i}} \left[\frac{\mathrm{vol}_{n-i}(L\cap\xi)}{k_{n-i}} \right]^{-\frac{p}{n-i}} d\mu_i(\xi) \right)^{-1}$$

$$\geq \breve{W}_i(K)^{\frac{n-i-p}{n-i}}\ \breve{W}_i(L)^{\frac{p}{n-i}}.$$

□

As an application of Theorem 3.2, we have the following Brunn-Minkowski inequality for the dual harmonic quermassintegrals.

THEOREM 3.3. *Let $K, L \in \mathcal{K}_o^n$ and $0 \leq i < n$. Then*

$$\breve{W}_i(K+L)^{\frac{1}{n-i}} \geq \breve{W}_i(K)^{\frac{1}{n-i}} + \breve{W}_i(L)^{\frac{1}{n-i}}, \tag{3.4}$$

with equality if and only if K and L are dilates.

Proof. Let $\xi \in G(n,i)$. Since $K, L \in \mathcal{K}_o^n$, it is easy to prove that

$$(K+L)\cap\xi \supset (K\cap\xi) + (L\cap\xi). \tag{3.5}$$

By (2.5), (2.6) and (3.5), for $M \in \mathcal{K}_o^n$, we have

$$\begin{aligned} V_{1,i}(M,K+L;\xi) &= V(M\cap\xi, i-1\,; (K+L)\cap\xi) \\ &\geq V(M\cap\xi, i-1\,; (K\cap\xi)+(L\cap\xi)) \\ &= V(M\cap\xi, i-1\,; K\cap\xi) + V(M\cap\xi, i-1\,; L\cap\xi) \\ &= V_{1,i}(M,K;\xi) + V_{1,i}(M,L;\xi). \end{aligned}$$

According to (3.1), (3.2) and the Minkowski inequality, we have

$$\breve{W}_i(M,K+L)$$

$$= k_n \left(\int_{G(n,n-i)} \left[\frac{V_{1,n-i}(M,K+L;\xi)}{k_{n-i}} \right]^{-1} d\mu_{n-i}(\xi) \right)^{-1}$$

$$\geq k_n \left(\int_{G(n,n-i)} \left[\frac{V_{1,n-i}(M,K;\xi)+V_{1,n-i}(M,L;\xi)}{k_{n-i}} \right]^{-1} d\mu_{n-i}(\xi) \right)^{-1}$$

$$\geq \breve{W}_i(M,K) + \breve{W}_i(M,L)$$

$$\geq \breve{W}_i(M)^{\frac{n-i-1}{n-i}} \left(\breve{W}_i(K)^{\frac{1}{n-i}} + \breve{W}_i(L)^{\frac{1}{n-i}} \right),$$

with equality if and only if K and L are dilates of M. Now if take $K+L$ for M, and recall that $\breve{W}_i(K,K) = \breve{W}_i(K)$, then Theorem 3.3 follows. □

Let K and L be star bodies in $\mathbb{R}^n$, $p \geq 1$, the p-radial addition $K \tilde{+}_p L$ be a star body whose radial function is given by

$$\rho(K \tilde{+}_p L, u)^p = \rho(K,u)^p + \rho(L,u)^p. \tag{3.6}$$

We will establish the Brunn-Minkowski inequality for the p-radial addition and dual harmonic quermassintegrals.

THEOREM 3.4. *Let $K, L \in \varphi_o^n$, $0 \leq i < n$ and $p > n-i$. Then*

$$\breve{W}_i(K \tilde{+}_p L)^{\frac{p}{n-i}} \geq \breve{W}_i(K)^{\frac{p}{n-i}} + \breve{W}_i(L)^{\frac{p}{n-i}}, \tag{3.7}$$

with equality if and only if L is a dilatate of K.

To prove the Theorem 3.4, we first introduce the following lemma:

LEMMA 3.5. *Let $K, L \in \varphi_o^n$ and $0 < i < n$. If $\xi \in G(n,i)$ and $p > i$, then*

$$\operatorname{vol}_i[(K \tilde{+}_p L) \cap \xi]^{\frac{p}{i}} \geq \operatorname{vol}_i(K \cap \xi)^{\frac{p}{i}} + \operatorname{vol}_i(L \cap \xi)^{\frac{p}{i}}, \tag{3.8}$$

with equality if and only if K and L are dilates.

Proof. By the polar coordinate formula for volume and (3.6), we have

$$\begin{aligned}
\operatorname{vol}_i[(K \tilde{+}_p L) \cap \xi]^{\frac{p}{i}} &= \left[\frac{1}{i}\int_{S^{n-1}} \rho^i_{[(K \tilde{+}_p L)\cap\xi]}(u)du\right]^{\frac{p}{i}} \\
&= \left[\frac{1}{i}\int_{S^{n-1}\cap\xi} \rho^i_{(K \tilde{+}_p L)}(u)du\right]^{\frac{p}{i}} \\
&= \left[\frac{1}{i}\int_{S^{n-1}\cap\xi} [\rho^p_K(u) + \rho^p_L(u)]^{\frac{i}{p}} du\right]^{\frac{p}{i}} \\
&\geq \left[\frac{1}{i}\int_{S^{n-1}\cap\xi} \rho^i_K(u)du\right]^{\frac{p}{i}} + \left[\frac{1}{i}\int_{S^{n-1}\cap\xi} \rho^i_L(u)du\right]^{\frac{p}{i}} \\
&= \operatorname{vol}_i(K\cap\xi)^{\frac{p}{i}} + \operatorname{vol}_i(L\cap\xi)^{\frac{p}{i}}.
\end{aligned}$$

□

Proof of Theorem 3.4. By (1.4), Lemma 3.5 and Minkowski integral inequality, we have

$$\begin{aligned}
&\left(\frac{k_{n-i}\breve{W}_i(K\widetilde{+}_pL)}{k_n}\right)^{\frac{p}{n-i}} \\
&= \left(\int_{G(n,n-i)} \left[\mathrm{vol}_{n-i}[(K\widetilde{+}_pL)\cap\xi]\right]^{-1} d\mu_{n-i}(\xi)\right)^{-\frac{p}{n-i}} \\
&\geq \left(\int_{G(n,n-i)} \left[\mathrm{vol}_{n-i}(K\cap\xi)^{\frac{p}{n-i}} + \mathrm{vol}_{n-i}(L\cap\xi)^{\frac{p}{n-i}}\right]^{-\frac{n-i}{p}} d\mu_{n-i}(\xi)\right)^{-\frac{p}{n-i}} \\
&\geq \left(\int_{G(n,n-i)} \mathrm{vol}_{n-i}(K\cap\xi)^{-1} d\mu_{n-i}(\xi)\right)^{-\frac{p}{n-i}} \\
&\quad + \left(\int_{G(n,n-i)} \mathrm{vol}_{n-i}(L\cap\xi)^{-1} d\mu_{n-i}(\xi)\right)^{-\frac{p}{n-i}} \\
&= \left(\frac{k_{n-i}\breve{W}_i(K)}{k_n}\right)^{\frac{p}{n-i}} + \left(\frac{k_{n-i}\breve{W}_i(L)}{k_n}\right)^{\frac{p}{n-i}},
\end{aligned}$$

which is just the inequality (3.7). □

4. The Blaschke-Santaló type inequalities

In this section, we shall give the Blaschke-Santaló type inequalities for the dual harmonic quermassintegrals.

To prove the main theorem of this section, we first introduce the following lemma:

LEMMA 4.1. [12] *Let K be a convex body containing the origin in its interior and $\xi \in G(n,i)$. Then*

$$K^* \cap \xi = (K|\xi)^*. \tag{4.1}$$

THEOREM 4.2. *Let K be a centered convex body in $\mathbb{R}^n$ and $0 \leq i < n$. Then*

$$\breve{W}_i(K^*)W_i(K) \leq k_n^2, \tag{4.2}$$

with equality if and only if K is an ellipsoid.

Proof. Let $s = n - i$, and $\xi \in G(n,s)$. By the Blaschke-Santaló inequality, for the body $K|\xi$ in ξ, we have

$$\mathrm{vol}_s[(K|\xi)^*]\mathrm{vol}_s(K|\xi) \leq k_s^2.$$

According to the Lemma 4.1, we obtain

$$\frac{\mathrm{vol}_s(K|\xi)}{k_s} \leq \left[\frac{\mathrm{vol}_s(K^* \cap \xi)}{k_s}\right]^{-1}, \tag{4.3}$$

with equality if and only if $K|\xi$ is an ellipsoid in ξ. We integrate both sides of inequality (4.3) over $G(n,s)$ and get

$$\int_{G(n,s)} \frac{\mathrm{vol}_s(K|\xi)}{k_s}\, d\mu_s(\xi) \leq \int_{G(n,s)} \left[\frac{\mathrm{vol}_s(K^* \cap \xi)}{k_s}\right]^{-1} d\mu_s(\xi),$$

that is,

$$\frac{W_i(K)}{k_n} \leq \left(\frac{\breve{W}_i(K^*)}{k_n}\right)^{-1}.$$

This is the desired inequality

$$\breve{W}_i(K^*)W_i(K) \leq k_n^2,$$

with equality if and only if K is an ellipsoid. □

By (1.5) and noticing that $\tilde{W}_i(K) \leq W_i(K)$ [8], we have

COROLLARY 4.3. *Let K be a centered convex body in $\mathbb{R}^n$ and $0 \leq i < n$. Then*

$$\breve{W}_i(K^*)\breve{W}_i(K) \leq k_n^2, \tag{4.4}$$

equality holds when $i \neq 0$ if and only if K is an ball.

The case $i = 0$ of (4.4) is the well-known Blaschke-Santaló inequality.

5. The Bieberbach inequality for dual harmonic quermass-integrals

For $K \in \mathcal{K}^n$ and $u \in S^{n-1}$, let $b(K,u)$ denote the width of K in the direction u. For a real number $i \neq 0$, Lutwak [12] defined the i-th half-width of a convex body K, $B_i(K)$, by

$$B_i(K) = \left[\frac{1}{nk_n}\int_{S^{n-1}} [b(K,u)/2]^i dS(u)\right]^{1/i}$$

For $i = -\infty$, 0, ∞, define $B_i(K)$ by

$$B_i(K) = \lim_{s \to i} B_s(K).$$

It thus follows that $B_{-\infty}(K)$ is half the minimum width of K, while $B_{\infty}(K)$ is half the diameter of K, and $B_1(K)$ is half the mean width of K.

Consider the general Bieberbach inequality:

$$V(K) \leq k_n B_i(K)^n, \ (K \in \mathcal{K}^n)$$

with equality if and only if K is a ball.

Bieberbach [2] established this general inequality, whose special form, when $i = \infty$, is the famous Bieberbach inequality. This was later improved by Urysohn [17] when he proved the Urysohn inequality thereby establishing the general Bieberbach inequality for $i = 1$. In [10], Lutwak obtain a further improvement by proving the harmonic Urysohn inequality which established the general Bieberbach inequality for $i = -1$. And in [9], he showed that the Bieberbach inequality holds for i if and only if $-n < i \leq \infty$, that is for $K \in \mathcal{K}^n$, then

$$V(K) \leq k_n B_i(K)^n, \ (-n < i \leq \infty) \tag{5.1}$$

with equality if and only if K is a ball.

In [12], Lutwak established an extension of (5.1), which states that for $K \in \mathcal{K}^n$, one has the inequality:

$$\hat{W}_i(K) \leq k_n B_{i-n}(K)^{n-i}, \ (0 \leq i < n-1) \tag{5.2}$$

with equality if and only if K is an ellipsoid. For $i = n-1$, both sides of inequality (5.2) are equal for all $K \in \mathcal{K}^n$.

Following Lutwak's i-th half-width of a convex body, in this section, we introduce the concept of i-th half-chord length of a star body.

Suppose $L \in \varphi_o^n$. For $u \in S^{n-1}$, let $p(L,u) = \rho(L,u) + \rho(L,-u)$ denote the length of the chord of L in the direction u which through origin. For $i \neq 0$, we define the i-th half-chord length of L, $P_i(L)$, by

$$P_i(L) = \left[\frac{1}{nk_n}\int_{S^{n-1}} [p(L,u)/2]^i dS(u)\right]^{1/i}. \tag{5.3}$$

For $i = -\infty$, 0, ∞, define $P_i(L)$ by

$$P_i(L) = \lim_{s \to i} P_s(L).$$

It thus follows that $P_{-\infty}(L), P_{\infty}(L)$ is the maximum and minimum chord length of L which through origin, respectively.

For a fixed i, the i-th half-chord length is a map

$$P_i : \varphi_o^n \to \mathbb{R}.$$

We list some of its elementary properties .

(i) (Positively homogeneous) If $c \geq 0$, then

$$P_i(cL) = cP_i(L).$$

(ii) (Subadditive) If $L, L' \in \varphi_o^n$, then

$$P_i(L \tilde{+} L') \leq P_i(L) + P_i(L').$$

(iii) (Continuity) $P_i(L)$ is a continuous function of L.

(iv) (Monotonicity for bodies) If $L \subset L'$, then

$$P_i(L) \leq P_i(L').$$

(v) (Monotonicity for power) If $i \leq j$, then

$$P_i(L) \leq P_j(L).$$

Let $L \in \varphi_o^n$ and $\xi \in G(n,1)$. Then $\mathrm{vol}_1(L \cap \xi)$ is just the chord length of L along ξ. We can, for $i \neq 0$, rewrite $P_i(L)$ as:

$$P_i(L) = \left[\int_{G(n,1)} \left[\frac{\mathrm{vol}_1(K \cap \xi)}{2}\right]^i d\mu_1(\xi)\right]^{\frac{1}{i}}. \tag{5.4}$$

Applying the concept of i-th half-chord length, we give a dual inequality of (5.2).

THEOREM 5.1. *Let $L \in \varphi_o^n$. Then*

$$\breve{W}_i(L) \geq k_n P_{i-n}(L)^{n-i}, \quad (0 \leq i < n-1) \tag{5.5}$$

with equality if and only if L is a ball. For $i = n-1$, the both sides of inequality (5.5) are equal for all $L \in \varphi_o^n$.

To prove the inequality (5.5), we shall use the following two lemmas.

LEMMA 5.2. (Dual Blaschke-Santaló Inequality) *Let $L \in \varphi_o^n$. Then*

$$V(L)V(L^\circ) \geq k_n^2, \tag{5.6}$$

with equality if and only if L is a ball.

Proof. By the polar coordinate formulate for volume and (2.11), we have

$$V(L^\circ) = \frac{1}{n}\int_{S^{n-1}} \rho_{L^\circ}(u)^n du = \frac{1}{n}\int_{S^{n-1}} \rho_L(u)^{-n} du.$$

Then by the Cauchy-Schwartz inequality, we get

$$\begin{aligned} V(L)V(L^\circ) &= \left(\frac{1}{n}\int_{S^{n-1}} \rho_L(u)^n du\right)\left(\frac{1}{n}\int_{S^{n-1}} \rho_L(u)^{-n} du\right) \\ &\geq \left(\frac{1}{n}\int_{S^{n-1}} du\right)^2 = k_n^2. \end{aligned}$$

By the equality conditions of Cauchy-Schwartz inequality, the equality of (5.6) holds if and only if L is a ball. □

LEMMA 5.3. (Dual Bieberbach Inequality) *Let $L \in \varphi_o^n$. Then*

$$V(L) \geq k_n P_{-n}(L)^n, \tag{5.7}$$

with equality if and only if L is a ball.

Proof. By (2.9) and (5.3), we have

$$\begin{aligned} P_{-n}(L) &= \left[\frac{1}{nk_n}\int_{S^{n-1}}[p(L,u)/2]^{-n}dS(u)\right]^{-\frac{1}{n}} \\ &= \left[\frac{1}{nk_n}\int_{S^{n-1}}\rho(\tilde{\Delta}L,u)^{-n}dS(u)\right]^{-\frac{1}{n}} \\ &= \left[\frac{1}{k_n}V(\tilde{\Delta}^{\circ}L)\right]^{-\frac{1}{n}}. \end{aligned}$$

It thus follows that

$$P_{-n}(L)^n = \frac{k_n}{V(\tilde{\Delta}^{\circ}L)}.$$

So (5.7) holds if and only if

$$V(L)V(\tilde{\Delta}^{\circ}L) \geq k_n^2. \tag{5.8}$$

By (2.10) and Lemma 5.2, we have

$$V(L)V(\tilde{\Delta}^{\circ}L) \geq V(\tilde{\Delta}L)V(\tilde{\Delta}^{\circ}L) \geq k_n^2.$$

Hence (5.8) holds, then the lemma follows.

By the equality conditions of (2.10) and Lemma 5.2, the equality of (5.7) holds if and only if L is a ball. □

Proof of Theorem 5.1. If ξ is an i-dimensional subspace $\mathbb{R}^n$, then for $j < i$, let $G(\xi, j)$ denote the Grassmann manifold of j-dimensional subspaces of $\mathbb{R}^n$ which are contained in ξ. For the Haar measure on $G(\xi, j)$ we shall write $\mu_j(\xi;\cdot)$, and we assume that it is normalized so that $\mu_j(\xi; G(\xi, j)) = 1$.

Let $s = n-i$. If $\xi \in G(n, s)$, then for $\zeta \in G(\xi, 1)$, one has $(L\cap\xi)\cap\zeta = L\cap\zeta$. Hence, from (5.4) we see that inequality (5.7), for the body $L\cap\xi$ in ξ, can be written as

$$\int_{G(\xi,1)}\left[\frac{\mathrm{vol}_1(L\cap\zeta)}{2}\right]^{-s} d\mu_1(\xi;\zeta) \geq \left[\frac{\mathrm{vol}_s(L\cap\xi)}{k_s}\right]^{-1}. \tag{5.9}$$

By Lemma 5.3, we know the equality holds if and only if $L \cap \xi$ is a ball in ξ. We integrate both sides of the inequality (5.9) over $G(n, s)$ and get

$$\int_{G(n,s)} \int_{G(\xi,1)} \left[\frac{\mathrm{vol}_1(L \cap \zeta)}{2}\right]^{-s} d\mu_1(\xi; \zeta) d\mu_s(\xi)$$
$$\geq \int_{G(n,s)} \left[\frac{\mathrm{vol}_s(L \cap \xi)}{k_s}\right]^{-1} d\mu_s(\xi),$$

with equality if and only if L is a ball.

The quantity on the left of the last inequality is equal to (see [16, (12.53)])

$$\int_{G(n,1)} \left[\frac{\mathrm{vol}_1(L \cap \xi)}{2}\right]^{-s} d\mu_1(\xi) = P_{-s}(L)^{-s},$$

while the quantity on the right is just $[\breve{W}_{n-s}(L)/k_n]^{-1}$. Thus, our last inequality is the desired inequality

$$\breve{W}_{n-s}(L) \geq k_n P_{-s}(L)^s,$$

with equality if and only if L is a ball. □

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References

[1] W. Blaschke, *Über affine Geometrie IX: Verschiedene Bemerkungen und Aufgaben.* Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl **69** (1917), 412–420.

[2] L. Bieberbach, *Über eine extremaleigenschaft des kreises.* Jber. Deutsch. Math.-Verein. **24** (1915), 247–250.

[3] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934.

[4] R. J. Gardner, *Geometric tomography,* Cambridge University Press, New York, 1995.

[5] H. Hadwiger and H. Sagan, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Amer. Math. Monthly **65** (1958), no. 4, 300.

[6] D. Klain, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), no. 1, 80–101.

[7] K. Leichtweiss. *Konvexe Mengen*, Springer-Verlag, Berlin-New York, 1980.

[8] E. Lutwak, *Dual mixed volumes,* Pacific J. Math. **58** (1975), no. 2, 531–538.

[9] ———, *A general Bieberbach inequality,* Math. Proc. Cambrigde Philos. Soc. **78** (1975), 493–495.

[10] ———, *A general isepiphanic inequality,* Proc. Amer. Math. Soc. **90** (1984), no. 3, 415–421.

[11] ———, *Mean dual and harmonic cross-sectional measures*, Ann. Mat. Pura Appl. (4) **119** (1979), 139–148.

[12] ———, *Inequalities for Hadwiger's harmonic quermassintegrals*, Math. Ann. **280** (1988), no. 1, 165–175.

[13] ———, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), no. 2, 232–261.

[14] M. Maria, *Quotient star bodies, intersection bodies and star duality.* J. Math. Anal. Appl. **232** (1999), no. 1, 45–60.

[15] L. A. Santaló, *Un invariante afin para los cuerpos del espacio de n dimensiones.* Portugal. Math. **8** (1949), 155–161.

[16] ———, *Integral geometry and geometric probability.* Reading, MA: Addison-Wesley, 1976.

[17] P. Urysohn, *Mean width and volume of bodies in n dimensional space,* Rec. Math. Soc. Math. Moscow **8** (1924), 477–480.(Russian)

[18] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 1993.

[19] G. Y. Zhang, *Dual kinematic formulas*, Tran. Amer. Math. **351** (1999), no. 3, 985–995.

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