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A CENTRAL LIMIT THEOREM FOR GENERAL WEIGHTED SUMS OF LPQD RANDOM VARIABLES AND ITS APPLICATION

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ABSTRACT. In this paper we derive the central limit theorem for $\sum_{i=1}^{n} a_{ni}\xi_i$, where $\{a_{ni}, 1 \le i \le n\}$ is a triangular array of nonnegative numbers such that $\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty$, $\max_{1\le i\le n} a_{ni} \to 0$ as $n \to \infty$ and $\xi_i' s$ are a linearly positive quadrant dependent sequence. We also apply this result to consider a central limit theorem for a partial sum of a generalized linear process of the form $X_n = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j$.

1. Introduction and results

Lehmann [7] introduced a simple and natural definition of positive dependence: A sequence $\{\xi_i,\ 1 \le i \le n\}$ of random variables is said to be pairwise positive quadrant dependent (pairwise PQD) if for any real α_i, α_j and $i \ne j$ $P(\xi_i > \alpha_i, \xi_j > \alpha_j) \ge P(\xi_i > \alpha_i)P(\xi_j > \alpha_j)$. Much stronger concepts than PQD was considered by Esary, Proschan and Walkup [4]: A sequence $\{\xi_i,\ 1 \le i \le n\}$ of random variables is said to be associated if for any real coordinatewise increasing functions f, g on $\mathbb{R}^n$, $\mathrm{Cov}(f(\xi_1, \dots, \xi_n), g(\xi_1, \dots, \xi_n)) \ge 0$.

Instead of association, Newman's [9] central limit theorem requires only that positive linear combinations of the random variables are PQD. The definition of positive dependence introduced by Newman [9] is the following: A sequence $\{\xi_i,\ 1 \le i \le n\}$ of random variables is said to be linearly positive quadrant dependent(LPQD) if for every pair of disjoint

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subsets $A, B \subset \{1, 2, \dots, n\}$ and positive $r_j's$

$$\sum_{i\in A} r_i\xi_i \quad \text{and} \quad \sum_{j\in B} r_j\xi_j \quad \text{are} \quad PQD. \tag{1.1}$$

Let us remark that LPQD is between pairwise PQD and association and it is well known(see, for example, Newman [9, p.131] that association implies LPQD and LPQD implies PQD.

Newman [9] established the central limit theorem for a strictly stationary LPQD process and Birkel [2] also obtained a functional central limit theorem for LPQD processes. Kim and Baek [6] extended this result to a stationary linear process of the form $Y_k = \sum_{j=0}^{\infty} a_j\xi_{k-j}$, where $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$ and $\{\xi_k\}$ is a strictly stationary LPQD process with $E\xi_i = 0$ and $0 < E\xi_i^2 < \infty$.

In this paper we derive a central limit theorem for a linearly positive quadrant dependent sequence in a double array, replacing the strictly stationarity assumption with uniform integrability (see Theorem 1.1 below). We apply this result to obtain a central limit theorem for a partial sum of a linear process $X_n = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j$ generated by linearly positive quadrant dependent sequence $\{\xi_j\}$ (see Theorem 1.2 below).

Newman [9] proved that the following central limit theorem for strictly stationary LPQD sequence:

THEOREM A. [9] *Let $\{\xi_i, i \geq 1\}$ be a strictly stationary sequence of LPQD random variables with $E\xi_i = 0$ and $E\xi_i^2 < \infty$. If we assume that*

$$\sigma^2 = \sum_{i=1}^{\infty} \mathrm{Cov}(\xi_1, \xi_i) < \infty,$$

then

$$(\sigma^2 n)^{-\frac{1}{2}} \sum_{i=1}^{n} \xi_i \to^D N(0,1) \text{ as } n \to \infty.$$

The next theorem extends Newman's central limit theorem for a strictly stationary LPQD sequence (Theorem A) from equal weights to general weights, while at the same time weakening the assumption of stationarity.

THEOREM 1.1. *Let $\{a_{ni},\ 1 \leq i \leq n\}$ be a triangular array of nonnegative numbers such that*

$$\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty, \tag{1.2}$$

and

$$\max_{1 \le i \le n} a_{ni} \to 0 \text{ as } n \to \infty. \tag{1.3}$$

Let $\{\xi_i\}$ be a centered sequence of linearly positive quadrant dependent random variables such that

$$\{\xi_i^2\} \text{ is an uniformly integrable family,} \tag{1.4}$$

$$\operatorname{Var}(\sum_{i=1}^{n} a_{ni}\xi_i) = 1 \tag{1.5}$$

and

$$\sum_{j:|k-j| \ge u} \operatorname{Cov}(\xi_k, \xi_j) \to 0 \text{ as } u \to \infty \text{ uniformly in } k \ge 1 \tag{1.6}$$

(see Cox and Grimmet [3]*). Then*

$$\sum_{i=1}^{n} a_{ni}\xi_i \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \text{ as } n \to \infty.$$

COROLLARY 1.1. *Let $\{\xi_i\}$ be a centered sequence of linearly positive quadrant dependent random variables such that $\{\xi_i^2\}$ is an uniformly integrable family and let $\{a_{ni},\ 1 \le i \le n\}$ be a triangular array of nonnegative numbers such that*

$$\sup_n \sum_{i=1}^{n} \frac{a_{ni}^2}{\sigma_n^2} < \infty, \tag{1.7}$$

$$\max_{1 \le i \le n} \frac{a_{ni}}{\sigma_n} \to 0 \text{ as } n \to \infty, \tag{1.8}$$

where $\sigma_n^2 = \operatorname{Var}(\sum_{i=1}^{n} a_{ni}\xi_i)$. *If* (1.6) *holds then, as $n \to \infty$*

$$\frac{1}{\sigma_n} \sum_{i=1}^{n} a_{ni}\xi_i \xrightarrow{\mathcal{D}} \mathrm{N}(0,1). \tag{1.9}$$

THEOREM 1.2. *Let $\{a_j, j \in Z\}$ be a sequence of nonnegative numbers such that $\sum_j a_j < \infty$ and let $\{\xi_j, j \in Z\}$ be a centered sequence of linearly positive quadrant dependent random variables which is uniformly integrable in L_2 and satisfies* (1.6). *Let*

$$X_k = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j \quad \text{and} \quad S_n = \sum_{i=1}^{n} X_i.$$

Assume

$$\inf_{n\geq 1} n^{-1}\sigma_n^2 > 0, \tag{1.10}$$

where $\sigma_n^2 = \mathrm{Var}(S_n)$. *Then*

$$\frac{S_n}{\sigma_n} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \ \ \text{as } n \to \infty. \tag{1.11}$$

This result extends Theorem 18.6.5 in Ibragimov and Linnik [5] from the i.i.d. case to the linearly positive quadrant dependence case by adding condition (1.6) and improves on the central limit theorem of Kim and Baek [6] for linear process generated by LPQD sequence.

2. Proofs

We start with the following lemma.

LEMMA 2.1. [8] *Let* $\{Z_i, 1 \leq i \leq n\}$ *be a sequence of linearly positive quadrant dependent random variables with finite second moments. Then*

$$\left|E\exp(it\sum_{j=1}^{n} Z_j) - \prod_{j=1}^{n} E\exp(itZ_j)\right| \leq Ct^2\left|\mathrm{Var}(\sum_{j=1}^{n} Z_j) - \sum_{j=1}^{n}\mathrm{Var}(Z_j)\right|$$

for all $t \in \mathbb{R}$, *where* $C > 0$ *is an arbitrary constant, not depending on* n.

Proof of Theorem 1.1. Without loss of generality we assume that $a_{ni} = 0$ for all $i > n$. For every $1 \leq a < b \leq n$ and $1 \leq u \leq b-a$ we have, after simple manipulations,

$$\begin{aligned} 0 &\leq \sum_{i=a}^{b-u} a_{ni} \sum_{j=i+u}^{b} a_{nj}\mathrm{Cov}(\xi_i,\xi_j) \\ &\leq \sup_k \Big(\sum_{j:|k-j|\geq u} \mathrm{Cov}(\xi_k,\xi_j)\Big)\Big(\sum_{i=a}^{b} a_{ni}^2\Big). \end{aligned} \tag{2.1}$$

In particular by (1.6), we have

$$\sup_k \Big(\sum_{j:|k-j|\geq 1} \mathrm{Cov}(\xi_k,\xi_j)\Big) < \infty. \tag{2.2}$$

Without restricting the generality we can assume that $\sup_k E\xi_k^2 = 1$.

Hence, there exists a constant $C > 0$ such that for every $1 \leq a \leq b \leq n$,

$$\begin{aligned}
\operatorname{Var}\Big(\sum_{i=a}^{b} a_{ni}\xi_i\Big) &\leq \sum_{i=a}^{b} a_{ni}^2 \operatorname{Var}(\xi_i) + 2\sum_{i=a}^{b-1} a_{ni} \sum_{j=i+1}^{b} a_{nj}\operatorname{Cov}(\xi_i,\xi_j) \\
&\leq \sum_{i=a}^{b} a_{ni}^2 \operatorname{Var}(\xi_i) \\
&\quad + 2\sup_k \Big(\sum_{j:|k-j|\geq 1}^{\infty} \operatorname{Cov}(\xi_k,\xi_j)\Big)\Big(\sum_{i=a}^{b} a_{ni}^2\Big) \qquad (2.3)\\
&\leq C\Big(\sum_{i=a}^{b} a_{ni}^2\Big)
\end{aligned}$$

by (2.1) and (2.2).

We shall construct now a triangular array of random variables $\{Z_{ni}, 1 \leq i \leq n\}$ for which we shall make use of Lemma 2.1. Fix a small positive ϵ and find a positive integer $u = u_\epsilon$ such that, for every $n \geq u+1$

$$0 \leq \Big(\sum_{i=1}^{n-u} a_{ni} \sum_{j=i+u}^{n} a_{nj}\operatorname{Cov}(\xi_i,\xi_j)\Big) \leq \epsilon. \qquad (2.4)$$

This is possible because of (2.1) and (1.6). Denote by $[x]$ the integer part of x and define

$$\begin{aligned}
K &= \Big[\frac{1}{\epsilon}\Big], \\
Y_{nj} &= \sum_{i=uj+1}^{u(j+1)} a_{ni}\xi_i, \ j = 0,1,\ldots, \\
A_j &= \Big\{i : 2Kj \leq i < 2Kj + K, \ \operatorname{Cov}(Y_{ni}, Y_{n,i+1}) \\
&\qquad \leq \frac{2}{K}\sum_{i=2Kj}^{2Kj+K} \operatorname{Var}(Y_{ni})\Big\}.
\end{aligned}$$

Since $2\operatorname{Cov}(Y_{ni}, Y_{n,i+1}) \leq \operatorname{Var}(Y_{ni}) + \operatorname{Var}(Y_{n,i+1})$, we get that for every j the set A_j is not empty. Now we define the integers $m_1, m_2, \ldots, m_n$ recursively. Let $m_0 = 0$ and

$$m_{j+1} = \min\{m : m > m_j, \ m \in A_j\}$$

and define

$$Z_{nj} = \sum_{i=m_j+1}^{m_{j+1}} Y_{ni},\ j = 0, 1, \ldots,$$
$$\Delta_j = \{u(m_j + 1) + 1, \ldots, u(m_{j+1} + 1)\}.$$

We observe that

$$Z_{nj} = \sum_{k \in \Delta_j} a_{nk}\xi_k,\ j = 0, 1, \ldots.$$

By the definition of LPQD the sequence $\{Z_{nj}\}$ is LPQD. From the fact that $m_j \geq 2K(j-1)$ and $m_{j+1} \leq K(2j+1)$ every set Δ_j contains no more than 3 Ku elements and $m_{j+1}/m_j \to 1$ as $j \to \infty$. Hence, for every fixed positive ϵ by (1.2)–(1.5) the array $\{Z_{ni} : i = 0, 1, \ldots, n;\ n \geq 1\}$ satisfies the Lindeberg condition(see Petrov [10], Theorem 22, p.100), that is, $\{Z_{nj}\}$ satisfies

$$\sigma_n^{-1} \sum_{j=1}^{n} EZ_{nj}^2 I(|Z_{nj}| > \epsilon\sigma_n) \to 0 \text{ as } n \to \infty, \tag{2.5}$$

where $\sigma_n^2 = \mathrm{Var}(\sum_{j=1}^{n} Z_{nj})$. We can observe that by Lemma 2.1 and the construction,

(2.6)

$$\begin{aligned}
&\left|E\exp(\mathrm{it}\sum_{j=1}^{n} Z_{nj}) - \prod_{j=1}^{n} E\exp(itZ_{nj})\right| \\
&\leq Ct^2\left|\left\{\mathrm{Var}(\sum_{j=1}^{n} Z_{nj}) - \sum_{j=1}^{n}\mathrm{Var}(Z_{nj})\right|\right\} \\
&\leq Ct^2\left\{2\left(\sum_{i=1}^{n}\mathrm{Cov}(Z_{ni}, Z_{n,i+1})\right) + 2\left(\sum_{i=1}^{n-2}\sum_{j=i+2}^{n}\mathrm{Cov}(Z_{ni}, Z_{nj})\right)\right\} \\
&\leq Ct^2\left[\left\{2\sum_{j=1}^{n}\mathrm{Cov}(Y_{n,m_j}, Y_{n,m_j+1}) + 2\sum_{i=1}^{n-u} a_{ni}\sum_{j=i+u}^{n} a_{nj}\mathrm{Cov}(\xi_i, \xi_j)\right\}\right. \\
&\quad \left. + \left\{2\sum_{i=1}^{n-u} a_{ni}\sum_{j=i+u}^{n} a_{nj}\mathrm{Cov}(\xi_i, \xi_j)\right\}\right] \\
&= Ct^2\left\{4\sum_{i=1}^{n-u} a_{ni}\sum_{j=i+u}^{n} a_{nj}\mathrm{Cov}(\xi_i, \xi_j) + 2\sum_{j=1}^{n}\mathrm{Cov}(Y_{n,m_j}, Y_{n,m_j+1})\right\}
\end{aligned}$$

$$\le Ct^2\Big\{4\epsilon + \frac{8}{K}\sum_{i=1}^{n}\mathrm{Var}(Y_{ni})\Big\}$$

$$= Ct^2\Big\{4\epsilon + \frac{8}{K}\sum_{j=1}^{n}\mathrm{Var}(\sum_{i=uj+1}^{u(j+1)} a_{n_i}\xi_i)\Big\}$$

$$\le Ct^2\Big\{4\epsilon + \frac{8M}{K}\sum_{j=1}^{n}\sum_{i=uj+1}^{u(j+1)} a_{n_i}^2\Big\} \text{ (by (2.3))}$$

$$\le C_1t^2\epsilon\Big\{1 + \sup_n \sum_{i=1}^{n} a_{ni}^2\Big\}$$

$\le C_2t^2\epsilon$ for every positive ϵ.

Therefore the problem is now reduced to the study of the central limit theorem of a decoupled sequence $\{\tilde{Z}_{nj}\}$ of independent random variables such that for each n and j, the variable $\tilde{Z}_{nj}$ is distributed as Z_{nj}.

By (2.5), $\{\tilde{Z}_{nj}\}$ also satisfies the Lindeberg condition, that is, $\{\tilde{Z}_{nj}\}$ satisfies $\tilde{\sigma}_n^{-1}\sum_{j=1}^n E\tilde{Z}_{nj}^2 I(|\tilde{Z}_{nj}| > \epsilon\tilde{\sigma}_n) \to 0$ as n $\to \infty$, where $\tilde{\sigma}_n^2 = \mathrm{Var}(\sum_{j=1}^n \tilde{Z}_{nj})$, and hence by Theorem 7.2 of Billingsley [1]

$$\tilde{\sigma}_n^{-1}\sum_{j=1}^{n}\tilde{Z}_{nj} \xrightarrow{\mathcal{D}} N(0,1) \text{ as } n \to \infty, \tag{2.7}$$

where $\tilde{\sigma}_n^2 = \mathrm{Var}(\sum_{j=1}^n \tilde{Z}_{nj})$. It follows from (2.5), (2.6), and (2.7) that

$$\sigma_n^{-1}\sum_{j=1}^{n} Z_{nj} \xrightarrow{\mathcal{D}} N(0,1) \text{ as } n \to \infty, \tag{2.8}$$

where $\sigma_n^2 = \mathrm{Var}(\sum_{j=1}^n Z_{nj})$, and now the proof is complete by (2.7), (2.8), and Theorem 4.2 in Billingsley [1]. □

Proof of Corollary 1.1. Let $A_{ni} = \frac{a_{ni}}{\sigma_n}$. Then we have

$$\max_{1\le i\le n} A_{ni} \to 0 \text{ as } n \to \infty,$$

$$\sup_n \sum_{i=1}^{n} A_{ni}^2 < \infty,$$

$$\mathrm{Var}(\sum_{i=1}^{n} A_{ni}\xi_i) = 1.$$

Hence, by Theorem 1.1 the desired result (1.11) follows. □

Proof of Theorem 1.2. First note that $\sum_j a_j^2 < \infty$ and, without loss of generality, we can assume that $\sup E\xi_k^2 = 1$. Let

$$S_n = \sum_{k=1}^{n} X_k = \sum_{j=-\infty}^{\infty} \left(\sum_{k=1}^{n} a_{k+j} \right) \xi_j.$$

In order to apply Theorem 1.1, fix W_n such that $\sum_{|j|>W_n} a_j^2 < n^{-3}$ and take $k_n = W_n + n$. Then

$$\frac{S_n}{\sigma_n} = \sum_{|j|\le k_n} \left(\sum_{k=1}^{n} a_{k+j} \right) \frac{\xi_j}{\sigma_n} + \sum_{|j|>k_n} \left(\sum_{k=1}^{n} a_{k+j} \right) \frac{\xi_j}{\sigma_n} = T_n + U_n.$$

By the Cauchy Schwarz inequality and the assumptions we have the following estimate

$$\begin{aligned}
\mathrm{Var}(U_n) &\le C \sum_{|j|>k_n} \mathrm{Var}\left(\sum_{k=1}^{n} a_{k+j} \frac{\xi_j}{\sigma_n} \right) \\
&\le C \sum_{|j|>k_n} \left(\sum_{k=1}^{n} a_{k+j}/\sigma_n \right)^2 E\xi_j^2 \\
&\le C n \sigma_n^{-2} \sum_{|j|>k_n} \left(\sum_{k=1}^{n} a_{k+j}^2 \right) \\
&\le C n^2 \sigma_n^{-2} \sum_{|j|>k_n - n} a_j^2 \\
&\le C n^2 \sigma_n^{-2} \sum_{|j|>W_n} a_j^2 \\
&\le C n^{-1} \sigma_n^{-2} \to 0 \text{ as } n \to \infty
\end{aligned}$$

which yields

$$U_n \to 0 \text{ in probability as n} \to \infty. \tag{2.9}$$

By Theorem 4.1 of Billingsley [1], it remains to prove that $T_n \xrightarrow{\mathcal{D}} \mathrm{N}(0,1)$. Put

$$a_{nk} = \frac{\sum_{j=1}^{n} a_{k+j}}{\sigma_n}. \tag{2.10}$$

From the assumption $\sum_j a_j < \infty$ $(a_j > 0)$, (1.10), and (2.10) we obtain

$$\frac{\sup_{-\infty<k<\infty} \sum_{j=1}^{n} a_{k+j}}{\sigma_n} \to 0 \text{ as } n \to \infty,$$

$$\max_{1\le k\le n} a_{nk} \to 0 \text{ as } n \to \infty,$$

$$\sup_n \sum_{k=1}^{n} a_{nk}^2 < \infty. \text{ (see Remark below.)}$$

Hence, by Theorem 1.1

$$T_n \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \tag{2.11}$$

and from (2.9) and (2.11) the desired result (1.10) follows. □

REMARK. In the proof of Theorem 1.2, let us suppose on the contrary that for some $\epsilon > 0$ there exists a subsequence $(j^{'}, n^{'}), n^{'} \to \infty$ such that $\sum_{k=1}^{n} a_{k+j'} > \epsilon\sigma_n^{'}$. Denote by $A = \sup_{-\infty<k<\infty} a_k$ and notice that for $r > j^{'}$

$$\sum_{k=1}^{n^{'}} a_{k+r} > \epsilon\sigma_n^{'} - 2A(r-j).$$

Hence

$$\frac{\sigma_{n'}^2}{b} \ge \sum_{i=j'}^{j^{'}+w} \left(\sum_{k=1}^{n^{'}} a_{k+j}\right)^2 \ge W\epsilon^2\sigma_{n'}^2 - 4A\sigma_n^{'}\epsilon\left(\sum_{i=j'}^{j^{'}+W}(i-j^{'})\right)$$
$$\ge W\epsilon^2\sigma_{n'}^2 - 4A\sigma_{n'}\epsilon W^2.$$

Taking W to be the least integer greater than or equal to $\frac{3}{b\epsilon^2}$ and because $\sigma_{n'} \to \infty$, we obtain for $n^{'}$ sufficiently large,

$$\frac{\sigma_{n'}^2}{b} \ge \frac{3\sigma_{n'}^2}{b} - \sigma_{n'}\frac{36A}{b^2\epsilon^3} \ge \frac{2\sigma_{n'}^2}{b}$$

which is a contradiction. □

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